A 2D integral EV problem v3 (29 pages) PhL 1.1.11

Summary of v3: See Contents and Overview below for more details. This doc has tangential "comment" sections which I skip here. Then in Sec 5 I consider both Ku=μu and the Kσ=f Dirichlet problem. Expanding σ/u and f on sin/cos(mθ) partial wave atoms leads to a 1D integral equation for the coefficients (first is EV, second is Dirichlet)

$$2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} * A_m(r') = \mu A_m(r) \quad \text{or} \quad = C_m(r) \qquad (1)$$

I cannot find a solution of this integral equation in Polyanin. Even if I could, that would solve the Dirichlet problem but not the EV problem. I note that Copson 1947 did solve the Dirichlet using the above. I am still confused about my what BC's should be for Ku=μu.

Then I am back to asking about solving the problem $L_{\mathbf{r}}\,(1/r) = 4\pi\,\delta^{(2)}(\mathbf{r})$ which even at this late date I fail to distinguish from $L_{\mathbf{r}}\,(1/R) = 4\pi\,\delta^{(2)}(\mathbf{r-r'})$. I try the Rothwell expansion and can only come up with non-linear L (using the Shankar idea that must have equality at every point in k-space, which misses the Jim solutions found later). I then *pretend* that my non-linear solution $L[f] = 2\sqrt{-f\,\nabla^2 f}$ is linear and that ∇^2 means the 3D thing, all garbage assumptions. This gives the PDE $(\nabla_{3D}^2 + [\pi/\mu]^2)\,u(r,\theta) = 0$ and we are then once again back to those $J_m(kr)\sin(m\theta)$ atoms, but now we have a different k, namely $k = \pi/\mu$.

I am of course allowed to try these atoms in Ku=uμ as I have done twice before in v1 and v2. But this time the Helmholtz k^2 is a little different and is dimensionally correct. Then Ku=μu (now $k = \pi/\mu$ so think of k as EV) becomes this:

$$\int_0^a dr'\, r'\, J_m(kr')\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} = (\pi/k)\,[\,J_m(kr)] \qquad r \le a \qquad (13.1)$$

which is something you could try to solve for successful EV's of k. I show that $k = k_n$ of $J_m(k_n a) = 0$ do not work. In Sec 13 I show that when $a = \infty$, the above equation is valid for all real $k > 0$, so by taking a stab in the dark as to an expansion form, I think I have solved Ku=μu for $a = \infty$!

There are various other flailings in this doc: In Sections 14 and 15 I take separate stabs at the case of finite a and get nowhere. In Sec 16 I try to find an L that inverts equation (1) above and get nowhere. In Sec 17 I try cylindrical expansion for 1/R and just duplicate earlier results. Then in Sec 18 I list my **conclusions**, go read these! In Sec 19 I flail once more with a δ(0) idea and then give up.

Overview (3.5 pages, written 1.4.11) **2**

1. How this problem arose. **5**

2. Comment on the Copson 1947 Disk Dirichlet Problem Method: where are the duals? **5**

3. Comment on diagonalizing integral operator K. **6**

4. Comment on the non-connection of the above to convolution diagonalization with groups. **7**

5. Statement of the disk Dirichlet problem as an integral equation in 1 variable. **7**

6. Searching Polyanin for a solution to this integral equation 8
7. Continuing search for a solution to our Dirichlet Equation (*) 10
8. Comment on Copson's ansatz regarding his partial wave form. 12
9. Comment concerning (u = 0 on disk boundary) as our disk EV-problem BC. 12
10. What differential EV problem can we associate with our integral EV problem? 13
11. The Rothwell/Cloud approach to finding L 14
12. Use this L to find the PDE EV equation that is associated with our integral EV equation. 16
13. See if our candidate form for u(r,θ) in fact solves the integral EV equation 17
14. What about the case a < ∞ ? 18
15. Another approach for a < ∞ using different assumptions. 20
16. What about working with the single variable eigenvalue integral equation? 22
17. What happens if we use the cylindrical 1/R expansion for the kernel? 25
18. Conclusions 26
19. One More Try. 27

Overview (3.5 pages, written 1.4.11)

In **Section 1** I state how this Ku = μu problem arose from reading page 203 of Sneddon.

In **Section 2** I comment on the fact that the Copson 1947 Dirichlet disk solution makes no mention of dual equations. This solution does not use a Smythian form for the potential, it just uses $V = \int dA\sigma/R$, and for a point on the disk we get $f = \int dA\sigma/R$ and he shows a very clever pathway to inverting this problem. I show in my long comment how "the second dual equation" $\partial_z V = \sigma$ works out, and this involves the very subtle work of Stakgold concerning "layers".

In **Section 3** I comment about diagonalizing a general integral operator K in QM bra-ket notation.

In **Section 4** I comment that group theory convolution diagonalization has nothing to do with this.

In **Section 5** I do the usual θ partial wave analysis and express the Dirichlet problem as a single variable integral equation one needs to solve: $2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} * A_m(r') = C_m(r)$. The corresponding single variable EV problem is $2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} * A_m(r') = \mu A_m(r)$.

In **Section 6** I try to find this integral equation in Polyanin's book but with no success. As part of this effort, I write out "the Q object" in two different ways using F(a,b,c,z), but neither of these appears in Polyanin either.

In **Section 7** I just comment on the Copson path for solving the Dirichlet problem. I then try to construct an alternate "JJ path" to solving this same problem, but it does not pan out. That is to say, I use my JJ form for the Q object and try to obtain inversion using Hankel transforms but I get nowhere with this.

In **Section 8** I comment that Copson did not use a fully general partial wave form in his partial wave analysis for θ, but since he found a solution, his simplified form is justified.

In **Section 9** I discuss what **boundary conditions** one might use with the disk integral EV problem. I note that if I require u=0 on the disk boundary, it is hard to see how you would ever get the known σ for the simple charged disk problem. This remains mysterious to me. The other alternative is u (r>a) = 0 as a boundary condition, but all this stuff is pretty hazy.

In **Section 10** I ask the big question: what is the PDE EV problem that corresponds to our integral EV problem? This raises the question of solving L (1/R) = $4\pi\,\delta^{(2)}(\mathbf{r}-\mathbf{r}')$ => L (1/r) = $4\pi\,\delta^{(2)}(\mathbf{r})$ for PDE operator L. I know that the solution is NOT that L = $-\nabla^2_{2D}$ since *that* L gives the above equation for ~ ln(r) and not 1/r. I also consider the possibility that L = [- $(1/\delta(0))\,\nabla^2_{3D}|^{z=0}$] but that makes no sense at all.

In **Section 11** I expand 1/r and $\delta^{(2)}(\mathbf{r})$ as suggested by Rothwell/Cloud. This suggests a possible solution of find an L that satisfies L [$\exp(i\mathbf{k_t}\bullet\mathbf{r})$] = $2k_t \exp(i\mathbf{k_t}\bullet\mathbf{r})$, but I could find no linear L that satisfies that equation. I ended up with L[f] = $2\sqrt{-f\,\nabla^2 f}$ which is non-linear, and which therefore invalidates my whole method. But I continued more Sections with this erroneous thing to see what would happen. I was hoping to have some "grain of truth" . It is not necessary that L satisfy the relation just stated, but I still don't know the solution to L (1/r) = $4\pi\,\delta^{(2)}(\mathbf{r})$. I asked Jim today (1.4.11), and he said he would think about it.

In **Section 12** I pretended that my non-linear L[f] = $2\sqrt{-f\,\nabla^2 f}$ was valid (this is the 3D ∇^2), though I now know it is not. Applying this L to both sides of my integral EV equation then resulted in $(\nabla^2 + [\pi/\mu]^2)\,u = 0$ which is Helmholtz in cylindrical 3D but with u = u(r,θ) only, no z dependence. Now $k = \pi/\mu$ and at least things are dimensionally correct (as opposed to my first doc in this series). I know from elsewhere the solutions of this Helmholtz equation, and this led me to "try out"

$$u(r,\theta) = J_m(kr)[\,A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad \text{where } k = \pi/\mu$$

as a candidate solution to my integral EV problem. Even if my L is completely bogus, I am still allowed to try out any candidate I want and see what happens! At this point I give a pretty good review of what has happened to this point.

In **Section 13** I insert the above candidate solution into both sides of my Ku = μu integral EV problem. I do the dθ' integrals in my usual manner and I end up with this single-variable integral equation which would have to be true if my candidate solution was valid

$$\int_0^a dr'\, r'\, J_m(kr')\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \;=\; (\pi/k)\,[\,J_m(kr)] \qquad r \le a \qquad (13.1)$$

If I regard my original Ku = μu as "existing only on the disk" I should include r ≤ a as on the right. Somewhat to my amazement, if I take a = ∞, I find that the above equation IS true, and therefore my candidate is valid and it is valid for any positive real value of k, hence of μ = π/k. So with a = ∞, we have a full spectrum. Again, this conclusion seems valid even though my L above is bogus.

In **Section 14** I ask about what happens if a < ∞. To answer this, I have to say something about my assumed boundary conditions. So I just assume this

$$u(r,\theta) = \Sigma_{m=0}^{\infty} [\, J_m(kr)] \, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad r \le a$$
$$= 0 \qquad r > a.$$

I have the same equation (13.1) shown above and I arrive at no definite conclusions for a < ∞. There might be some k_n discrete values for which (13.1) is true, and therefore for which we then have found a solution to the Ku = μu problem. If there are some solution k_n, I know they are *not* those that make $J_m(k_n a) = 0$, something I show somewhat indirectly. Alternatively, my entire candidate form for u might be invalid for a < ∞. Or maybe it is valid and you can only get a numerical solution for the k_n and the EFs.

In **Section 15** I take another shot at the a < ∞ problem. This time I use this boundary condition and form

$$u(r,\theta) = J_m(k_n r)] \, [\, A_m(k_n)\sin(m\theta) + B_m(k_n)\cos(m\theta)] \qquad \text{all r}$$

and k_n ARE those that make $J_m(k_n a) = 0$, so we then have u = 0 on the disk boundary (but not inside or outside it). I still have equation (13.1) shown above but now it is for "all r". I then launch off into a complicated solution chain involving a mixture of Hankel and Fourier-Bessel transforms, but I arrive at the conclusion that for these k_n, things do not work, which adds support to my similar conclusion in Section 14 with a different boundary condition.

In **Section 16** I start over working this time with my single-variable integral equation found earlier

$$2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \; * A_m(r') = \mu\, A_m(r) \qquad (*)$$

Looking for the corresponding ODE EV problem then requires finding L such that

$$L_{(r)} \; \{2r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \; \} = 4\pi\, \delta(r-r')$$

or

$$L_{(r)} \; \{\, 2\pi\, r' \int_0^{\infty} J_m(rx) J_m(r'x)\, dx \,\} = 4\pi\, \delta(r-r')$$

or

$$\int_0^{\infty} \{L_{(r)}\, J_m(rx)\}\, J_m(r'x)\, dx = 2\, \delta(r-r')/r'$$

If I could find $L_{(r)}$ such that $L_{(r)} J_m(rx) = 2x\, J_m(rx)$, that would work! But I cannot find such an L that works here (it cannot depend on x). But I blindly proceed, assuming an L exists, though I don't know what it is. If I apply this L to both sides of (*) above, I then obtain this equation: $L_{(r)} A_m(r) = (4\pi/\mu) A_m(r)$. Our "fact" just stated then suggests that a candidate solution is $A_m(r) = J_m(\alpha r)$ where $\alpha = 2\pi/\mu$. But this is the same candidate solution I have had all along, and I end up with the same conclusion: there may perhaps be some μ value for which (8) is satisfied with $A_m(r) = J_m(\alpha r)$, or there may not be.

In **Section 17** I use my cylindrical coordinates expansion for 1/R and I show that this just replicates what I have already done.

In **Section 18** I state a set of conclusions that I still feel are all valid. Go read them.

In **Section 19** I try to consider my integral EV problem as the limit of a 3D integral EV problem where I replace u(x,y) by u(x,y,z) [I call it ρ instead of u] and then I try $\rho(x,y,z) = \sigma(x,y)\delta(z-z_0)$. I get nowhere.

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1. How this problem arose.

The problem arises on page 203 of Sneddon where we have a tie-in to the Stakgold world. If we write the usual potential-as-sum-q/r expansion $V = \int dA\sigma/R$ for a flat localized piece of metal -- let's assume it's a disk -- then the Dirichlet equation is $\int dx'dy'\ \sigma(x',y')/R = f(x,y)$ where R is the distance between two points on the z=0 plane and we regard this 1/R a kernel, so of course

$$k(x,y;\, x',y') = 1/R = [\,(x-x')^2+(y-y')^2\,]^{-1/2} = k(x,x')$$

So let's associate this kernel with an integral operator K.

Here is the problem: what are the eigenvalues and the eigenfunctions of this 2D operator K ? If we knew these things, we might solve our Dirichlet equation above by expanding σ and f on these eigenfunctions. In fact this is what Sneddon does on page 203 (he uses p for σ). In the eigenfunction expansions of σ and f the coefficients are p_i and f_i and things become diagonal so that $p_i = f_i/\lambda_i$ and the solution to the problem is as shown in 7.2.11 (a small Sneddon typo I have corrected, circled e).

2. Comment on the Copson 1947 Disk Dirichlet Problem Method: where are the duals?

Go back to the general $V = \int dA\sigma/R$ equation. Notice that this is not a "Smythian form". When we assert such a form, we have V = Form. Then for our disk problem we usually come up with dual integral (or series) equations by saying Form(on disk) = f(x,y), and ∂_zForm(outside disk) = 0. For an azisym problem, the duals for the disk appear p 64 3.1.3 for f = 1, and then as p 65 3.2.1 for f = f(r), these being the problems of Weber and Beltrami.

A typical approach to the duals is shown on page 74 in Sneddon's own solution of the Beltrami problem: you assume a Form for the solution (in this case 3.5.1) which automatically solves the second dual, so you then only deal with the first dual after that.

For the more general Copson case f = f(r,θ), it seems that there is no mention of dual equations at all! On page 70 Copson just uses V = $\int$ dAσ/R and on the disk this becomes f(r,θ) = $\int$ dAσ/R. He first expands f and σ in partial waves on θ to get 3.4.5. He then replaces the θ integral with the double-Abel integral to get p 70A which he rewrites as 3.4.7 which shows the double-Abel nature of things. He then does a double-Abel inversion and gets the solution σ. Dual integral equations are never written or even mentioned!

One wonders then how it is that Copson's solution satisfies the two equations $\partial_z V = -\sigma/2$ on the disk, (1/2 because σ on one side only) and $\partial_z V = 0$ outside the disk. We only "made" the solution satisfy V = f(r,θ) on the disk. The answer must be this: somehow, the solution of $\int$ dAσ/R = f(r,θ) on the disk is unique for σ. Thinking of charges moving around on the disk, we can sense that this solution will be unique. The way we get $\partial_z V = -\sigma/2$ on the disk and $\partial_z V = 0$ outside the disk is explained by Stak in all that "layer stuff" in his Section 6.4. In his simple layer analysis, he starts with $u(\mathbf{x}) = \int_\sigma dS_\xi\ a(\boldsymbol{\xi})\ E(x|\xi)$ where a = σ, E = 1/4πR and u = V. As we take the limit that x approaches the disk, nothing unusual happens and we get $u(\mathbf{s}) = \int_\sigma dS_\xi\ a(\boldsymbol{\xi})\ E(\mathbf{s}|\xi)$ which is our f(r,θ) = $\int$ dAσ/R. However, when we consider the limit of the normal derivative, $\partial_\nu u(\mathbf{x})$, we find that $[\partial_\nu u(\mathbf{x})]^{\mathbf{x=s}} = \int_\sigma dS_\xi\ a(\boldsymbol{\xi})\ \partial_\nu E(x|\xi) - a(\mathbf{s})/2$ and that magic second term "appears" as we take the limit. It must be that for our disk case, the first integral then vanishes (I think this happens for a flat surface), and we get just $[\partial_\nu u(\mathbf{x})]^{\mathbf{x=s}} = -a(\mathbf{s})/2$ which here we write as $\partial_z V = -\sigma/2$ on the disk. Then off the disk the integral also vanishes, the extra term is not present, and we get $\partial_z V = 0$. I have not done the details, but I know this is how it all works out. I must congratulate Stakgold for his section of this subject which is quite subtle and which I have seen in no other book.

3. Comment on diagonalizing integral operator K.

If we think of the eigenvalue equation Ku = μu as a matrix equation where K is a symmetric (read Hermitian) matrix, we expect there to exist a basis in which K is diagonal. Recall Matrix binder Theorem 19C: A Hermitian matrix can be diagonalized by a unitary similarity. So if we call this basis φ_i then we have $K\varphi_i = \lambda_i\varphi_i$ where λ_i are the eigenvalues and φ_i are the eigenvectors of K. If we use orthonormal eigenvectors and QM notation, here is what happens:

$K|i\rangle = \lambda_i|i\rangle \quad \Rightarrow \quad \langle j|K|i\rangle = \lambda_i\langle j|i\rangle = \lambda_i\delta_{ij} \quad \Rightarrow \quad K_{ij} = \lambda_i\delta_{ij} \quad K_{ii}=\lambda_i$

so of course the diagonal elements of diagonalized matrix K are the eigenvalues. Then to solve Kp=f:

$K|p\rangle = |f\rangle \quad \Rightarrow \quad \Sigma_j\ \langle i|Kj\rangle\langle j|p\rangle = \langle i|f\rangle \quad \Rightarrow \quad K_{ii}\langle i|p\rangle = \langle i|f\rangle$

$\Rightarrow\ \lambda_i\, p_i = f_i \quad \Rightarrow \quad p_i = \lambda_i^{-1} f_i \quad \Rightarrow \quad |p\rangle = \Sigma_i\ |i\rangle\langle i|p\rangle = \Sigma_i\ \lambda_i^{-1} f_i|i\rangle$

4. Comment on the non-connection of the above to convolution diagonalization with groups.

Due to my "history", I am always trying to associate "diagonalization" with some underlying group theory. I just added Section 6 to "diagonalization of convolution equations.doc" and convinced myself that the diagonalization of K seen above is totally unconnected to the notion of diagonalization of a convolution equation. There is no group theory involved in this problem.

5. Statement of the disk Dirichlet problem as an integral equation in 1 variable.

The Dirichlet problem and the eigenvalue problem for K can be written this way in polar coordinates

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, \sigma(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; f(r,\theta) \qquad K\sigma = f \qquad r \le a$$

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\, u(r,\theta) \qquad Ku = \mu u$$

The first can be regarded as "an integral equation" we want to solve for σ, given f. The second is of course an "eigenvalue problem" which we would like to solve for μ_i and $u_i(r,\theta)$. Of course both equations involve *double* integrals which makes them unpleasant.

An approach one can take is to expand σ, f, and u in θ partial waves, for example: [so that C_m and D_m are known quantities]

$$\sigma(r',\theta') = \Sigma_{m=0}^{\infty} [\, A_m(r')\sin(m\theta') + B_m(r')\cos(m\theta')]. \qquad r \le a$$

$$f(r,\theta) \;= \Sigma_{m=0}^{\infty} [\, C_m(r)\, \sin(m\theta) \; + D_m(r)\, \cos(m\theta)] \qquad r \le a$$

The resulting dθ' integrals are ones I have studied quite a bit and know how to do. If one inserts these partial wave expansions into the Dirichlet equation above and does the dθ' integrals, one obtains,

$$2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \;*\; A_m(r') = \; C_m(r) \qquad (*)$$

and an identical equation with B on the left and D on the right. This then is our integral equation in a single variable.

If we were doing the eigenvalue equation instead of the Dirichlet equation, we would think of A and B as the coefficients of u and we would then have

$$u(r',\theta') = \Sigma_{m=0}^{\infty} [\, A_m(r')\sin(m\theta') + B_m(r')\cos(m\theta')]$$

$$2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \;*\; A_m(r') = \; \mu\, A_m(r)$$

with a similar equation with both A→B. We would wonder if there were any $A_m(r)$ and μ which make this equation be true? Those would be the things we seek! But let's hold off on the EV problem for now and look instead at the Dirichlet equation.

6. Searching Polyanin for a solution to this integral equation

One might wonder whether this little integral equation (*) appears in Polyanin's book! This equation is a Fred 1 ("first kind") with fixed integration endpoints. It would have to appear near page 272 of his book which deals with "kernels containing Legendre functions".

Miss #1. Although his integral 72 is tantalizingly close, it is not quite right:

$$\textbf{72.}\quad \int_0^\infty \left[(x+t-1)^2-4xt\right]^{-1/2} Q^1_{\nu-\frac{1}{2}}\left(\frac{x+t-1}{2\sqrt{xt}}\right) y(t)\,dt = f(x), \qquad \operatorname{Re}\nu > -1.$$

Here $Q_\nu^\mu(x)$ is the associated Legendre function of the second kind (see Supplement 11.11-3).

Solution:

$$y(t) = \frac{1}{4\pi^2}\int_0^\infty (xt)^{-1/2}\left[(x+t-1)^2-4xt\right]^{-1/2} Q^1_{\nu-\frac{1}{2}}\left(\frac{x+t-1}{2\sqrt{xt}}\right) f(x)\,dx.$$

◉ Reference: A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992, p. 520).

So no, my integral equation does not appear in his book as such. The above close form has several problems: (1) it is Q^1 that appears instead of Q^0; (2) the arg of Q is hard to make agree; (3) the factor has to be reconciled; (4) the endpoint is ∞ instead of a.

So, although we have reduced our problem to a single-variable integral equation, we have pretty much arrived at a dead end in this approach to the Dirichlet problem.

Miss #2: *However*, we do know other ways to express the Q object above, [all verified externally!]

$$Q_{m-1/2}[\,(r^2+r'^2)/(2rr')\,] / \sqrt{rr'} \;=\; \pi \int_0^\infty J_m(rx)J_m(r'x)\,dx$$

$$= \; 2\,(rr')^{-m} \int_0^{\min(r,r')} ds\, s^{2m} / \left[\sqrt{r^2-s^2}\,\sqrt{r'^2-s^2}\,\right]$$

From GR7 we also have this:

$$1.^{11}\quad \int_0^\infty J_\mu(ax)\,J_\nu(bx)\,dx = b^\nu a^{-\nu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2},\frac{\nu-\mu+1}{2};\nu+1;\frac{b^2}{a^2}\right)$$

$[a>0, \quad b>0, \quad \operatorname{Re}(\mu+\nu)>-1, \quad b<a.$

For $a>b$, the positions of μ and ν should be reversed.]

ET II 48(6)

So if we set μ = ν = m we find that

$$\int_0^\infty J_m(ax)J_m(bx)\,dx \;= (1/\sqrt{\pi}\,)\,(b/a)^m\,(1/a)\,\Gamma(m+1/2)/m!\;*\;F[m+1/2,1/2;\,m+1;\,(b/a)^2]$$

$$\int_0^\infty J_m(rx)J_m(r'x)\,dx = (1/\sqrt{\pi})\,(r'/r)^m\,(1/r)\,\Gamma(m+1/2)/m!\ast F[m+1/2,1/2;\ m+1;\ (r'/r)^2]$$

so our alternate form is then

$$Q_{m-1/2}[\,(r^2+r'^2)/(2rr')]\,/\,\sqrt{rr'} = \sqrt{\pi}\;(r'/r)^m\,(1/r)\,\Gamma(m+1/2)/m!\ast F[m+1/2,1/2;\ m+1;\ (r'/r)^2]$$

and you see how 1/a on the right provides the required dimension of L^1. We could then write our integral equation (*) in this manner

$$2\int_0^a dr'\,r'\,Q_{m-1/2}[(r^2+r'^2)/(2rr')]\,/\,\sqrt{rr'}\ast A_m(r') = C_m(r) \qquad (*)$$

$$2\int_0^a dr'\,r'\,\sqrt{\pi}\;(r'/r)^m\,(1/r)\,\Gamma(m+1/2)/m!\ast F[m+1/2,1/2;\ m+1;\ (r'/r)^2]\ast A_m(r') = C_m(r)$$

$$2\sqrt{\pi}\;\Gamma(m+1/2)/m!\ast\int_0^a dr'\,(r'/r)^{m+1}\,F[m+1/2,1/2;\ m+1;\ (r'/r)^2]\ast A_m(r') = C_m(r)$$

$$\int_0^a dr'\,(r'/r)^{m+1}\,F[m+1/2,1/2;\ m+1;\ (r'/r)^2]\,A_m(r') = C_m(r)\,m!\,/[\,2\sqrt{\pi}\;\Gamma(m+1/2)\,]$$

I only do this because this gives us another opportunity to look up this integral equation in Polyanin in his hypergeometric section, which is on page 276. But this form does not appear there.

Miss #3: He does show another form that might be relevant. Let's go back to our starting Q object and use Bateman p 134 (41) which says

$$Q_\nu(z) = 2^{-1-\nu}\,\sqrt{\pi}\;\Gamma(\nu+1)\,z^{-1-\nu}\,/\Gamma(\nu+3/2)\ast F(1+\nu/2,\ 1/2+1/2\nu;\ \nu+3/2;\ 1/z^2)$$

First, set $\nu = m-1/2$:

$$Q_{m-1/2}(z) = 2^{-m-1/2}\,\sqrt{\pi}\;\Gamma(m+1/2)\,z^{-m-1/2}\,/\Gamma(m+1)\ast F(m/2+3/4,m/2+1/4;m+1;\ 1/z^2)$$

and then set $z = (r^2+r'^2)/(2rr')$

$$Q_{m-1/2}((r^2+r'^2)/(2rr'))$$

$$= 2^{-m-1/2}\,\sqrt{\pi}\;\Gamma(m+1/2)\,[(r^2+r'^2)/(2rr')]^{-m-1/2}\,/\Gamma(m+1)\ast F(m/2+3/4,m/2+1/4;m+1;\ 4r^2r'^2/(r^2+r'^2)^2)$$

Well, I was hoping to get this into a form to match the following Polyanin page 276 form

$$\int_0^a F\left(\frac{\beta}{2},\ \frac{\beta+1}{2},\ \mu;\ \frac{4x^2t^2}{(x^2+t^2)^2}\right)\frac{y(t)\,dt}{(x^2+t^2)^\beta} = f(x).$$

To get a match, I need to have $\beta = m+1/2$ and $\mu = m+1$. Then my Q form becomes

$$Q_{m-1/2}((r^2+r'^2)/(2rr'))$$

$$= 2^{-\beta}\sqrt{\pi}\ \Gamma(\beta)\,[(r^2+r'^2)/(2rr')]^{-\beta}/\Gamma(\mu) * F(\ (\beta+1)/2,\ \ \beta/2\ \ ;\mu\ \ ;\ 4r^2r'^2/(r^2+r'^2)^2)$$

$$= 2^{-\beta}\sqrt{\pi}\ \ (\Gamma(\beta)/\Gamma(\mu))\,(2rr')^{\beta}\,F(\ \ (\beta+1)/2,\ \ \beta/2\ \ ;\mu\ \ ;\ 4r^2r'^2/(r^2+r'^2)^2)\ /(r^2+r'^2)^{\beta}$$

Let's plug this into my integral equation (*) above:

$$2\ \int_0^a dr'\ r'\ Q_{m-1/2}[(r^2+r'^2)/(2rr')]\ /\ \sqrt{rr'}\ \ * A_m(r') = \ C_m(r) \qquad (*)$$

$$2\ \int_0^a dr'\ \sqrt{r'/r}\ Q_{m-1/2}[(r^2+r'^2)/(2rr')]\ \ A_m(r') = \ C_m(r)$$

$$2\ \int_0^a dr'\ \sqrt{r'/r}\ \{2^{-\beta}\sqrt{\pi}\ \ (\Gamma(\beta)/\Gamma(\mu))\,(2rr')^{\beta}\,F(\ \ (\beta+1)/2,\beta/2\ ;\mu\ ;\ 4r^2r'^2/(r^2+r'^2)^2)\ /(r^2+r'^2)^{\beta}\ \}\ A_m(r') = \ C_m(r)$$

$$2^{-\beta+1}\sqrt{\pi}\ \ (\Gamma(\beta)/\Gamma(\mu))\ \ \int_0^a dr'\ \sqrt{r'/r}\,(2rr')^{\beta}\,F(\ \ \beta/2\ ,(\beta+1)/2\ ;\mu\ ;\ 4r^2r'^2/(r^2+r'^2)^2)\ /(r^2+r'^2)^{\beta} * A_m(r') = C_m(r)$$

The integral here is very close to Polyanin's, but the factor $\sqrt{r'/r}\,(2rr')^{\beta}$ makes it not work! Still, Polyanin's solution does look like Copson's double Abel gizmo.

7. Continuing search for a solution to our Dirichlet Equation (*)

The Copson Path. If we use the second form shown for the Q object, our equation (*) above becomes

$$\int_0^a dr'\ r'\ 4\ (rr')^{-m}\ \ \int_0^{\min(r,r')} ds\ s^{2m}\ /\ [\ \sqrt{r^2-s^2}\ \sqrt{r'^2-s^2}\ ]\ * A_m(r') = \ C_m(r) \qquad r \le a$$

or

$$4r^{-m}\ \int_0^a dr'\ r'^{1-m}\ A_m(r')\ \ \int_0^{\min(r,r')} ds\ s^{2m}\ /\ [\ \sqrt{r^2-s^2}\ \sqrt{r'^2-s^2}\ ]\ * = \ C_m(r) \qquad r \le a$$

This last result appears at the very bottom of Sneddon page 70. On page 71 Sneddon shows that you can invert this thing by doing two Abel transforms, and this then give the "famous" Copson 1947 solution to this Dirichlet problem.

Attempt using the JJ path. If we try instead the JJ approach implied by the above, our (*) becomes instead

$$2\ \int_0^a dr'\ r'\ \pi\ \int_0^{\infty} J_m(rx)J_m(r'x)\ dx\ * A_m(r') = \ C_m(r) \qquad r \le a$$

or

$$2\pi \int_0^\infty dx\, J_m(rx) \int_0^a dr'\, r'\, J_m(r'x)\, A_m(r') = C_m(r) \qquad r \le a$$

Since $A_m(r')$ is related to σ, we might at this point assume that $A_m(r')$ vanishes for $r > a$ with some justification, then we can rewrite the above line as

$$2\pi \int_0^\infty dx\, J_m(rx) \int_0^\infty dr'\, r'\, J_m(r'x)\, A_m(r') = C_m(r) \qquad r \le a$$

We might then define $a_m(x)$ to be the Hankel transform of $A_m(r)$ and we then have

$$2\pi \int_0^\infty dx\, J_m(rx)\, a_m(x) = C_m(r) \qquad r \le a$$

or

$$2\pi \int_0^\infty dx\, x\, J_m(rx)\, [a_m(x)/x] = C_m(r) \qquad r \le a$$

Then we can define $\alpha_m(r)$ to be the Hankel transform of $[a_m(x)/x]$ and we then get

$$2\pi\, \alpha_m(r) = C_m(r) \qquad r \le a$$

In this manner, I think I have found a variation of the Copson method! The reverse chain then looks like the following

$$\alpha_m(r) = (1/2\pi)\, C_m(r)$$
$$[a_m(x)/x] = \int_0^\infty dr\, r\, J_m(rx)\, \alpha_m(r)$$
$$A_m(r) = \int_0^\infty dx\, x\, J_m(rx)\, [a_m(x)/x] = \int_0^\infty dx\, J_m(rx) a_m(x)$$

That is to say, since we know $f(r,\theta)$, we know $C_m(r)$. That tells us $\alpha_m(r)$. Then the second line tells us $a_m(x)$ and the last term tells us $A_m(r)$ which in turn gives us $\sigma(r,\theta)$ which is what we seek. (We have a similar set of equations for the other pair of coefficients.)

BUT, there is a flaw in the ointment. We know that $\alpha_m(r) = (1/2\pi)\, C_m(r)$ only for $r \le a$. Thus, we cannot do the integral shown to get $[a_m(x)/x]$, and our plan falls apart. My method does work for $a = \infty$. This is typical of the problems you run into! The Copson method somehow avoids this problem.

Below I comment that there may be some discrete JJ sum representation of Q that might fix our problem.

Last gasp: But suppose we assume (without justification) that our equation (*) is valid for all r. In this case, we don't know what $C_m(r)$ is for $r > a$, but we can assume "it is something". Then the reverse chain above might be viable, but since we don't know $C_m(r)$ for $r > a$, we can't compute $a_m(x)$! This is the usual problem you have trying to mess only with one equation of a pair of dual equations.

So I cannot make my JJ method work for Dirichlet.

8. Comment on Copson's ansatz regarding his partial wave form.

In the Copson method as outlined by Sneddon on page 70, Copson in fact assumes expansions of the following form

$$\sigma(r',\theta') = \Sigma_{m=0}^{\infty} a_m(r')\cos(m\theta' - \theta_m)$$

where the θ_m are *constant* phase shifts in each partial wave which are not dependent on r. Copson's assumed form is not as general as the form I used which was this

$$\sigma(r',\theta') = \Sigma_{m=0}^{\infty} [\, A_m(r')\sin(m\theta') + B_m(r')\cos(m\theta')] \qquad \text{// my assumed form}$$

$$\sigma(r',\theta') = \Sigma_{m=0}^{\infty} a_m(r') [\, \cos(m\theta')\cos(\theta_m) + \sin(m\theta')\sin(\theta_m)] \qquad \text{// expanding Copson above}$$

$$\Rightarrow\ B_m(r') = a_m(r')\cos(\theta_m) \qquad A_m(r') = a_m(r')\sin(\theta_m) \qquad \text{// connections}$$

You can see that if we select arbitrary functions for A and B, there will in general be no solution of the Copson form. So basically Copson has made an ansatz that the solution in fact has his specialized form, and since he eventually finds a solution that works, his assumption is justified. Somehow I think my approach is a little better, since I don't make any assumption at all. The reason his form works is that the angle integral has the same form for sin(mθ) and cos(mθ).

9. Comment concerning (u = 0 on disk boundary) as our disk EV-problem BC.

What can we say about our eigenvalue problem stated above, which is the second of the following pair (the first being the Dirichlet problem which Copson has solved)

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, \sigma(r',\theta') \,/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = f(r,\theta) \qquad K\sigma = f \qquad r \le a$$

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u_i(r',\theta') \,/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = \mu_i\, u_i(r,\theta) \qquad Ku_i = \mu_i u_i$$

In all of Stak's work, whenever we talked about an eigenvalue problem in 1D, we always used inhomo BC's at the ends of our interval (a,b). When we talked about nD problems, we always assumed that our functions had to vanish on the boundary. I suppose you could talk about a related eigenvalue problem where the normal derivative of the functions vanish, or even you could consider a mixed situation (Sneddon's "radiation" situation).

The question is: if we were trying to solve the Dirichlet problem shown above using an eigenfunction method, would it be reasonable to assert the boundary condition that the eigenfunctions $u_i(r,\theta)$ vanish on the boundary of our disk? I am ambivalent about the answer to this question.

On the one hand, we know that in the simple case f(r,θ) = 1, the resulting σ blows up as r → a from inside the disk, so how on earth would you obtain that result if you assumed that σ was an expansion in eigenfunctions u_i all of which vanish on the boundary! I suspect that σ blows up at the edge for any reasonable f(r,θ) which you assume.

On the other hand, I know that $\sigma = 0$ if you approach the boundary from $r > a$, so in this sense there is some justification. But our eigenvalue equation as shown above is really defined "on the disk" just as a 1D problem is defined "on the interval (a,b)", so I guess I am just not happy about assuming that the eigenfunctions vanish on the boundary of the disk. But if I *don't* make this assumption, I don't know what it means to find eigenvalues for $Ku = \mu u$.

So I am not very motivated to spend 10 days trying to solve the eigenvalue problem assuming the eigenfunctions vanish on the disk boundary!

10. What differential EV problem can we associate with our integral EV problem?

Maybe that would suggest what the EF's might be? So how, given a multi-variable integral Fred 1 equation, do you obtain the corresponding differential equation?

One possible pathway is to note that the kernel is supposed to be the Green's Function of that PDE. In our case, in polar coordinates, we seem to have

$$K(r,\theta; r',\theta') = 1 / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

So what PDE operator acting on this would give a 2D delta function?

Perhaps it is better to use Cartesian coordinates for this question. Then we have

$$K(x,y;x',y') = 1 / \sqrt{(x-x')^2+(y-y')^2}$$

$$K(x,y;0,0) = 1 / \sqrt{x^2+ y^2}$$

The latter is then the Green's Function with source at the origin, and in polars it is just 1/r. So then we seek an operator L such that (using the Stak norm for defining a Green's function to be 1/r).

$$L\,(1/r) = 4\pi\,\delta(x)\delta(y) = 4\pi\,\delta^{(2)}(\mathbf{r})$$

When $r > 0$, we must have $L(1/r) = 0$.

Miss #1: Me might venture to guess that L has some connection to the 2D Laplacian which is this

$$\nabla^2_{2D} = r^{-1}\partial_r\,(\,r\,\partial_r) + r^{-2}\partial^2_\theta$$

Now it happens that

$$\nabla^2_{2D}(1/r) = r^{-1}\partial_r\,(\,r\,\partial_r[r^{-1}]) = r^{-1}\partial_r\,(\,-r\,[r^{-2}]) = -\,r^{-1}\partial_r(r^{-1}) = +r^{-1}(r^{-2}) = r^{-3}$$

so this is not a viable candidate. We know anyway that its Green's Function is ~ ln(1/r).

Miss #2: So OK, let's try the 3D Laplacian. But we know the answer there. We will get

$$\nabla^2_{3D}\,(1/r) = -4\pi\,\delta^{(2)}(\mathbf{r})\delta(0)$$

where the $\delta(0)$ arises from $\delta(z)$, but we set $z = 0$ inside our expression for r. So here somehow is a candidate for our operator L

$$L = [- (1/\delta(0)) \nabla^2_{3D}]$$

Suppose we "formally" apply this L to our equation of interest,

$$\int_0^a dr' r' \int_0^{2\pi} d\theta' u(r',\theta') / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = \mu u(r,\theta) \qquad Ku = \mu u$$

This would seem to give

$$\int_0^a dr' r' \int_0^{2\pi} d\theta' u(r',\theta') [- (1/\delta(0)) \nabla^2_{3D}] / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = [- (1/\delta(0)) \nabla^2_{3D}] \mu u(r,\theta)$$

Presumably we then have

$$[- (1/\delta(0)) \nabla^2_{3D}] / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = [- (1/\delta(0)) \nabla^2_{3D}] (1/R) = 4\pi \delta^{(2)}(\mathbf{r}-\mathbf{r}')$$

and we end up with

$$4\pi u(r,\theta) = [- (1/\delta(0)) \nabla^2_{3D}] \mu u(r,\theta)$$

and this is then our differential form of the EV equation. We could install the cylindrical ∇^2_{3D}, but then we have $\delta(0)$ sitting in our equation, <u>it makes no sense at all!</u>

So this is a dead end I have hit before!

11. The Rothwell/Cloud approach to finding L

Does my work with the Rothwell/Cloud parallel plate problem assist in any way? This is located in curvilinear/Cartesian. One thing I show there is this (where $z = z'$ in R)

$$1/R = 1/\sqrt{(x-x')^2 + (y-y')^2} = (1/\pi) \int d^2k_t \exp[\pm i\mathbf{k_t}\bullet\mathbf{(r-r')}] /(2k_t)$$

where both the k_t integrals range $(-\infty,\infty)$, and I am still looking for L such that

$$L (1/R) = 4\pi \delta(x-x')\delta(y-y') \qquad (1)$$

So Roth/Cloud has provided an integral representation of 1/R, and I should be able to simplify by setting $r' = 0$, so we have

$$1/R = / \sqrt{x^2+y^2} = (1/\pi) \int d^2k_t \exp(\pm i\mathbf{k_t}\bullet \mathbf{r}) /(2k_t)$$

$$L\,(1/R) = 4\pi\,\delta(x)\delta(y) \qquad (2)$$

[This was a misleading step! Equation (2) => (1) only if L has constant coefficients! But it is true that going the other way (1) => (2) so if one has a solution for L, it must satisfy (2), so I continue OK:]

Well, this would require that

$$L\,(1/R) = (1/\pi)\int d^2k_t\,(1/2k_t)\,L\,[\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})] = 4\pi\,\delta(x)\delta(y) \qquad (*)$$

But I know a nice integral representation for the RHS:

$$\int d^2k_t\,\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r}) = 4\pi^2\,\delta(x)\delta(y) \qquad \text{// Spectral Theory page 14}$$

So we seek L then such that

$$(1/\pi)\int d^2k_t\,(1/2k_t)\,L\,[\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})] = (1/\pi)\,4\pi^2\,\delta(x)\delta(y) = (1/\pi)\int d^2k_t\,\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})$$

which is to say we seek L such that

$$\int d^2k_t\,(1/2k_t)\,L\,[\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})] = \int d^2k_t\,\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})$$

Can I come up with an L such that this is true:

$$L\,[\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r})] = 2k_t\,\exp(\pm i\mathbf{k_t}\bullet\,\mathbf{r}) \qquad ?$$

L cannot depend on the dummy integration variable k_t. All I can think of is this

$$L\,[f] = 2\,|\,\nabla f\,|\,f \qquad \text{or} \qquad L\,[f] = 2\sqrt{-f\,\nabla^2 f} \quad \text{or} \quad 2\sqrt{-\nabla^2}\,f \qquad ???$$

where we have 2D operators here. These are <u>not linear operators</u> (but the second is at least "scalar linear" in that $L[\alpha f] = \alpha\,L[f]$), but fine. Let's try using the L shown on the right above. [I don't think that the operator $(\partial_x^2 + \partial_y^2)^{1/2}$ means anything because if you do a binomial expansion of this, you bring in negative powers like ∂_x^{-2} which have no meaning at least to me.]

Note added 1.4.11. As noted, neither of these L operators is linear, $L[f+g] \neq L[f] + L[g]$. Therefore we know that $L(\Sigma_i f_i) \neq \Sigma_i L(f_i)$ and similarly $L(\int dk\ f_k(r)) \neq \int dk\ L(f_k(r))$. Thus, equation (*) above is invalid for our non-linear L, so everything is garbage above. Also, we know that linear K must have a corresponding linear L, so all I can hope is "grain of truth" buried in this somewhere. <u>In any event, the L I obtain here is invalid, and all work below based on it is invalid.</u>

12. Use this L to find the PDE EV equation that is associated with our integral EV equation.

Suppose we take the one on the right. Then here would be our differential version of the EV equation

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u_i(r',\theta')\ (1/R) \;=\; \mu_i\, u_i(r,\theta) \qquad\qquad Ku_i = \mu_i u_i$$

Apply L to both sides (very sloppy indeed I realize, but maybe somehow there will be a tiny grain of truth in all this)

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta')\ L(1/R) \;=\; \mu\, L\, u(r,\theta)$$

$4\pi\, u = \mu\, 2\sqrt{-u\, \nabla^2 u}$ // non linear PDE

$(4\pi)^2\, u^2 = -\,4\,\mu^2\, u\, \nabla^2 u$

$(\pi)^2 u = -\,\mu^2\, \nabla^2 u \quad \Rightarrow \quad (\pi)^2 u + \mu^2\, \nabla^2 u = 0 \quad \Rightarrow \quad (\pi/\mu)^2 u + \nabla^2 u = 0$

$(\nabla^2 + [\pi/\mu]^2)\, u = 0$ // linear PDE with $k = \pi/\mu$ (formerly had $k^2 = 1/\mu$!)

and we end up then with some kind of 2D Helmholtz equation where $k^2 = [\pi/\mu]^2$. The solution to this equation in cylindricals with no z dependence is known to be (Moon Spencer p 16) something like

$$u(r,\theta) \sim [\, J_m(kr)]\, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad\qquad k = \pi/\mu \;\; m = 0,1,2....$$

where m just be an integer. But this is just the cylindrical atomic form for this kind of disk problem when you are seeking a solution of the 3D Laplace equation, but where we have set z = 0.

Let's review what just happened in this section. I wanted to convert the integral eigenvalue equation (having 2D kernel 1/R) into a differential eigenvalue equation in hopes of then being able to find some kind of solution. The problem then was to find a differential operator L which satisfies $L(1/R) = 4\pi\, \delta(x)\delta(y)$, because I knew that the kernel is supposed to be the Green's function of the differential equation, and the kernel is 1/R. I did Rothwell/Cloud expansions for 1/R and $\delta(x)\delta(y)$ and I came up with the following non-linear operator for L : $Lf = 2\sqrt{-f\, \nabla^2 f}$. I then applied this operator (with very weak justification) to both sides of my integral eigenvalue equation, and the result was $4\pi\, u = \mu\, 2\sqrt{-u\, \nabla^2 u}$. But when both sides are squared, this becomes a *linear* PDE which is just $(\nabla^2 + [\pi/\mu]^2)\, u = 0$ and this is just the 2D Helmholtz equation with $k = \pi/\mu$ and ∇^2 is the 2D Laplacian. Rather indirectly, I looked up the solution to this equation in Moon & Spencer in the case of no z dependence, but I could have done this directly, separating my own variables. I found that the solutions u must have a form which matches my usual atomic form for the solution of the cylindrical 3D Laplace equation with z = 0. In that solution k is an unquantized separation constant, and here we identify this with the eigenvalue $k = \pi/\mu$.

So, if we are on the right track here, we can conclude that $u(r,\theta)$ can have the general form shown above where m is any integer and k can have any positive real value, and so μ also has any positive real value, the spectrum is continuous. In general then

$$u(r,\theta) = \Sigma_{m=0}^{\infty} [\, J_m(kr)] \, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad \text{where } k = \pi/\mu$$

is a candidate solution for our integral eigenvalue equation

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;\; = \mu\, u(r,\theta) \qquad\qquad Ku = \mu u$$

13. See if our candidate form for u(r,θ) in fact solves the integral EV equation

This thing must balance in each partial wave, so we are eager now to insert our solution and see what happens. Doing this for the mth partial wave we have

$$LHS = \int_0^a dr'\, r'\, J_m(kr') \int_0^{2\pi} d\theta' \; [\, A_{mk}\sin(m\theta') + B_{mk}\cos(m\theta')] / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \quad r \le a$$

We know how to do these angular integrals and we get

$$LHS = \int_0^a dr'\, r'\, J_m(kr') \, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \, 2\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'}$$

$$= [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \int_0^a dr'\, r'\, J_m(kr')\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \qquad r \le a$$

Meanwhile, we have

$$RHS = \mu\, u(r,\theta) = (\pi/k) \, [\, J_m(kr)] \, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad r \le a$$

So we want to somehow show that

$$\int_0^a dr'\, r'\, J_m(kr')\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \;\; = (\pi/k) \, [\, J_m(kr)] \qquad r \le a \qquad (13.1)$$

From our experience with a simpler comparison of integral EV problem to differential EV problem, we might expect this only to be true if certain boundary conditions are met, but let's proceed anyway. The LHS here *might* be best written using the JJ form of the Q object, so we have

$$LHS = \int_0^a dr'\, r'\, J_m(kr') \, [\pi \int_0^{\infty} dx\, J_m(rx) J_m(r'x)] \qquad r \le a$$

All I can do is change integration order, so we get

$$\text{LHS} = \pi \int_0^\infty dx\, J_m(rx) \int_0^a dr'\, r'\, J_m(kr')\, J_m(r'x) \qquad r \le a$$

Let's first consider the special case that $a = \infty$. Then we would have

$$\text{LHS} = \pi \int_0^\infty dx\, J_m(rx) \left[\int_0^\infty dr'\, r'\, J_m(kr')\, J_m(r'x) \right] \qquad r \le a = \infty$$

$$= \pi \int_0^\infty dx\, J_m(rx)\, [\delta(x-k)/k] \;=\; (\pi/k)\, J_m(rk)$$

and we have it! I have successfully found (for the case $a = \infty$) the eigenfunctions and eigenvalues of our integral EV equation. There are two separation constants. The constant m must be an integer, while the constant k can be any positive real number and is not quantized. The eigenvalues are $\mu = \pi/k$ and so our problem has a continuous spectrum which is the entire positive real axis. The eigenfunctions are

$$J_m(kr)\, [\sin(m\theta), \cos(m\theta)]$$

which are well behaved at least at r=0. (1) I have shown that these eigenfunctions solve the 2D Helmholtz equation which I found to be associated with the integral EV equation; (2) I have shown explicitly that these eigenfunctions solve the integral EV equation!

14. What about the case $a < \infty$?

Can we somehow get a result for finite a? Let's go back to our integral eigenvalue equation:

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\, u(r,\theta) \qquad Ku = \mu u \qquad r \le a$$

Suppose we make the *ansatz* that u vanishes outside the disk. Will this somehow help? This lets us replace a by ∞ on the LHS integral upper endpoint. Our *ansatz* solution for u must be this for all r :

$$u(r,\theta) = \Sigma_{m=0}^{\infty} [\, J_m(kr)] \, [\, A_{mk}\sin(m\theta) + B_{mk}\cos(m\theta)] \qquad r \le a$$
$$= 0 \qquad r > a.$$

If we want the solution to be continuous at r = a, we have to quantize k so that $J_m(ka) = 0$ and then we have a discrete spectrum for k and μ.

Now let's go through the steps and see what happens. For $r \le a$, our LHS is this:

$$\text{LHS} = \int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, \{\, \Sigma_{m=0}^{\infty} [\, J_m(kr')] \, [\, A_{mk}\sin(m\theta') + B_{mk}\cos(m\theta')] \,\} \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

We *cannot* now replace endpoint a by endpoint ∞ because the form shown for u is only valid for $r \le a$, we are not allowed to integrate over it for $r > a$! So basically the steps are the same and we get to this point as we did above: this is what our eigenvalue equation now looks like (I cancel π's)

$$\int_0^\infty dx\, J_m(rx) \int_0^a dr'\, r'\, J_m(kr')\, J_m(xr') \quad = (1/k)\,[\, J_m(kr)] \qquad r \le a$$

Is it possible that this equation is true for certain eigenvalues of k ? It is certainly tempting to propose that maybe the k_n that satisfy $J_m(k_n a) = 0$ would work. This would make our u be continuous at the disk boundary, as noted above. But I don't know how to prove or disprove this equation

$$\int_0^\infty dx\, J_m(rx) \int_0^a dr'\, r'\, J_m(k_n r')\, J_m(xr') \quad =?= \quad (1/k_n)\,[\, J_m(k_n r)] \qquad r \le a$$

Since x is a continuous variable, we cannot use our usual Fourier-Bessel orthogonality. The second integral has one J leg in each camp! Somehow we would need to have this fact be true:

$$\int_0^a dr'\, r'\, J_m(k_n r')\, J_m(xr') = \delta(k_n - x)/k_n$$

Then our =?= equation would be true! But I know the "fact" above cannot be true for the following reason: suppose $x = k_{n'}$, some zero. Then we know that (from transforms.doc)

$$\int_0^a dr'\, r'\, J_m(k_n r')\, J_m(k_{n'} r') = [a\, J_{m+1}(k_n a)/\sqrt{2}\,]^2 \delta_{n,n'}$$

We would then have to have

$$\delta(k_n - k_{n'})/k_n = [a\, J_{m+1}(k_n a)/\sqrt{2}\,]^2 \delta_{n,n}$$

For n = n' this says

$$\delta(0)/\, k_n = [a\, J_{m+1}(k_n a)/\sqrt{2}\,]$$

which is nonsense.

So my conclusions are these:

(1) We have rewritten the integral EV equation in this manner, where we have assumed a certain form for the solution u :

$$\int_0^\infty dx\, J_m(rx) \int_0^a dy\, y\, J_m(ky)\, J_m(xy) \quad = (1/k)\,[\, J_m(kr)] \qquad r \le a$$

(2) There may or may not be values of k which make this equation be valid. If such values of k exist, then those values are the eigenvalues (actually $\mu_n = \pi/k_n$) for our problem, and $J_m(k_n r)\sin/\cos(m\theta)$ are the corresponding eigenfunctions.

(3) The values k_n are NOT the zeros of $J_m(k_n a)$.

(4) For a = ∞, our equation is true for all k, so we have a continuous spectrum of the entire real + axis.

15. Another approach for a < ∞ using different assumptions.

Suppose we take our form based on the results of section 12 above,

$$u(r,\theta) = J_m(k_n r)]\ [\ A_m(k_n)\sin(m\theta) + B_m(k_n)\cos(m\theta)] \qquad \text{all r}$$

to be valid both on the disk, outside the disk, and on the disk boundary. I have no justification for this, it is just something I want to "try". Our boundary condition now shall be only that $u(a,\theta) = 0$, we vanish just on the boundary, so now $k_n a$ *are* the zeros of J_m. We are well behaved at r = 0, and we will ignore the vagueness concerning r = ∞ behavior. We now have a discrete spectrum for μ which is $\mu_n = \pi/k_n$. What now happens if we insert this solution for u into our single variable integral eigenvalue equation? We now have (this is equation 13.1 from Section 13 above)

$$\int_0^a dr'\ r'\ J_m(k_n r')\ Q_{m-1/2}[\ (r^2+r'^2)/(2rr')] / \sqrt{rr'} \quad =?= \quad (\pi/k_n)\ [\ J_m(k_n r)] \qquad \text{all r}$$

Again, in the previous section we assumed this equation was true for r ≤ a, but *now* we assume it is true for all r. We still have finite a as our integration endpoint, however. Remember that we are merely *conjecturing* that this equation is true with our candidate solution, hence the =?= notation.

If we then use our JJ expansion for Q we get

$$\text{LHS} = \int_0^a dr'\ r'\ J_m(k_n r')\ [\pi \int_0^\infty dx\ J_m(rx) J_m(r'x)] \qquad \text{all r}$$

$$= \pi \int_0^\infty dx\ J_m(rx)\ [\ \int_0^a dr'\ r'\ J_m(k_n r')\ J_m(x\ r')\ ] \qquad \text{all r}$$

In our F-B transform.doc section we have this transform,

$$F_{n,m} = 2/[a^2 J_{m+1}(ak_n)^2]\ \int_0^a dr'\ r'\ J_m(k_n r')\ f(r') \qquad \text{// projection}$$

$$f(r) = \Sigma_{n=1}^{\infty} F_{n,m}\ J_m(k_n r) \qquad \text{// expansion}$$

So in our application here we have $f(r') = J_m(x\ r')$. So let's make a definition:

$$j_{m,n}(x) \equiv 2/[a^2 J_{m+1}(ak_n)^2]\ \int_0^a dr'\ r'\ J_m(k_n r')\ J_m(x\ r') \qquad \text{// projection}$$

$$J_m(x\ r) = \Sigma_{n=1}^{\infty} j_{m,n}(x)\ J_m(k_n r) \qquad \text{// expansion}$$

so that $j_{m,n}(x)$ is the *Fourier-Bessel* transform of the Bessel function $J_m(x\ r')$, and $j_{m,n}(x)$ is therefore defined over the full range of x in (0,∞).

Question: is $j_{m,n}(x)$ something we might have a closed form for? This kind of integral is not going to appear in GR I am afraid, this is a highly specialized integral that you would have to go figure out at great pain.

In any event, then we have

$$\text{LHS} = \pi \int_0^\infty dx\, J_m(rx) \left[\int_0^a dr'\, r'\, J_m(k_n r')\, J_m(x\, r') \right] \qquad \text{// from above}$$

$$= \pi \int_0^\infty dx\, J_m(rx) \left[j_{m,n}(x) \left[a^2 J_{m+1}(a k_n)^2 \right]/2 \right] \qquad \text{// insert}$$

$$= (\pi/2) \left[a^2 J_{m+1}(a k_n)^2 \right] \int_0^\infty dx\, J_m(rx)\, j_{m,n}(x) \qquad \text{all r}$$

Now let's define the remaining integral to be the *Hankel* Transform of $j_{m,n}(x)/x$

$$h_{m,n}(r) \equiv \int_0^\infty dx\, x\, J_m(rx) \left[j_{m,n}(x)/x \right] \qquad \text{// inverse: } \left[j_{m,n}(x)/x \right] = \int_0^\infty dr\, r\, J_m(rx)\, h_{m,n}(r)$$

We end up then with

$$\text{LHS} = (\pi/2) \left[a^2 J_{m+1}(a k_n)^2 \right] h_{m,n}(r) \qquad \text{all r}$$

$$\text{RHS} = (\pi/k_n) \left[J_m(k_n r) \right] \qquad \text{all r}$$

or

$$(\pi/2) \left[a^2 J_{m+1}(a k_n)^2 \right] h_{m,n}(r) \;=?=\; (\pi/k_n) \left[J_m(k_n r) \right] \qquad \text{all r}$$

If we assume our equation is true, we then have a candidate for $h_{m,n}(r)$,

$$h_{m,n}(r) = (\pi/k_n) \left[J_m(k_n r) \right] (2/\pi) \left[a^2 J_{m+1}(a k_n)^2 \right]^{-1} = (2/k_n) \left[J_m(k_n r) \right] \left[a^2 J_{m+1}(a k_n)^2 \right]^{-1}$$

and we are claiming now this is valid for all r. We then compute $[j_{m,n}(x)/x]$ using the inversion already shown above

$$j_{m,n}(x) = x \int_0^\infty dr\, r\, J_m(rx)\, h_{m,n}(r) = x \int_0^\infty dr\, r\, J_m(rx)\, (2/k_n) \left[J_m(k_n r) \right] \left[a^2 J_{m+1}(a k_n)^2 \right]^{-1}$$

$$= x (2/k_n) \left[a^2 J_{m+1}(a k_n)^2 \right]^{-1} \int_0^\infty dr\, r\, J_m(rx)\, J_m(k_n r)$$

Now the *big test* would be to see if we get this (which is the inversion of our Fourier Bessel see above)

$$J_m(x\, r) = \Sigma_{n=1}^{\infty}\, j_{m,n}(x)\; J_m(k_n r) \qquad ?$$

$$J_m(x\, r) = \Sigma_{n=1}^{\infty} \{ x (2/k_n) \left[a^2 J_{m+1}(a k_n)^2 \right]^{-1} \int_0^\infty dr'\, r'\, J_m(r'x)\, J_m(k_n r') \}\; J_m(k_n r) \quad ?$$

If we could show this to be true, then we have really showed that u is a solution with these k_n. But sadly I don't know how to evaluate the RHS. We would like to know this sum

$$\text{sum}(r,r';a) = 2 \Sigma_{n=1}^{\infty} (1/k_n) J_m(k_n r') J_m(k_n r) / [a^2 J_{m+1}(ak_n)^2]$$

then we would want to show that

$$J_m(x\, r) = x \int_0^{\infty} dr'\, r'\, J_m(r'x)\, \text{sum}(r,r';a)\} \qquad ?$$

Our sum is very close to the Fourier Bessel completeness sum, but it has an extra factor k_n in the denominator which ruins it. But *if* that factor were not there, we could use (transforms.doc)

$$2 \Sigma_{n=1}^{\infty} [a J_{m+1}(ak_n)]^{-2} J_m(k_n r) J_m(k_n r') = \delta(r-r')/r' = \text{sum}(r,r';a) \qquad \text{//completeness}$$

to conclude that

$$J_m(x\, r) = x \int_0^{\infty} dr'\, r'\, \text{sum}(r,r';a)\}\ J_m(r'x) = \int_0^{\infty} dr'\, r'\, \delta(r-r')/r'\ J_m(r'x) = J_m(xr)$$

That would all be nice, but we cannot just throw out our $1/k_n$ factor in the sum. My conclusion then is that we do NOT get a solution using eigenvalues k_n !!! So the conclusion of this entire section is negatory. This is similar to the conclusion reached in the previous section : the k_n zeros don't work, although there we had a different boundary condition.

So I guess I now give up on the finite-a situation. I still don't know how to solve the integral eigenvalue problem except for infinite a.

But I do know how to solve the Dirichlet problem for finite a using the Copson method outlined above. The approach here allows us to solve the Dirichlet integral equation, but if we tried that same approach for the eigenvalue problem, we just convert it from one integral equation into another integral equation! Maybe there is some subtle way to go half way, but this is WAY beyond what I should be doing, so I think I will let this problem now come to rest.

16. What about working with the single variable eigenvalue integral equation?

From far above we wrote this as

$$2 \int_0^{a} dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}\ * A_m(r') = \mu\, A_m(r) \qquad (*)$$

We could ask: what ODE is associated with this integral equation? We would then seek an L such that

$$L_{(r)} \{ 2\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}\ \} = 4\pi\, \delta(r-r')$$

If we could find such an operator L, then we would apply L to both sides of (*) to get

$$4\pi\, A_m(r) = \mu\, L\, A_m(r)$$

and that would be our ODE of interest. The L here is some differential operator that involves only the variables r and m.

But of course I have no idea how to find out what L is! I am used to doing problems the other way around: given L, find the Green's function. But here we are given the Green's function and we are supposed to find L!

I can *try* to insert the JJ form ,

$$2r'\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] \,/\, \sqrt{rr'} \;\; = \;\; 2\pi\, r' \int_0^\infty J_m(rx) J_m(r'x)\, dx$$

so the problem is then this: [in this next section I note that the JJ integral is "a piece of 1/R"]

$$L_{(r)}\; \{2\,\pi\, r' \int_0^\infty J_m(rx) J_m(r'x)\, dx\;\} \;= 4\pi\, \delta(r-r')$$

$$\int_0^\infty \; \{L_{(r)}\, J_m(rx)\}\; J_m(r'x)\, dx \;= 2\, \delta(r-r')/r'$$

I do have this helpful fact available (Hankel in transforms.doc)

$$\int_0^\infty dx\; x\, J_m(rx)\, J_m(r'x) \;\; = \;\; \delta(r-r')/r'$$

So this suggests that we might want

$$L_{(r)}\, J_m(rx) = \; 2x\, J_m(rx)$$

which is at least something I can grab hold of.

Miss #1: Somewhere I have a discussion of the Bessel equation written in various forms. I will now attempt to find that doc. I get 39 doc hits in my math folder. It is "Jackson on cylindricals.doc". I show there in section (b) that

$$K_{(r)} \;=\; r^2\partial_r^2 + r\partial_r + (x^2r^2-m^2) \;\; = r\partial_r(r\partial_r) + (x^2r^2-m^2) \qquad\qquad K_{(r)}\; J_m(rx) = 0$$

So this is just how you write the Bessel ODE with a constant before the variable.

So this at least gives me something to start with. I know that

$$[\; r^2\partial_r^2 + r\partial_r + (x^2r^2-m^2)]\; J_m(rx) = 0$$

Suppose I just add $2x\, J_m(rx)$ to both sides. I get

$$[\, r^2\partial_r^2 + r\partial_r + (x^2r^2 - m^2) + 2x]\, J_m(rx) = 2x\, J_m(rx)$$

and this then gives me a candidate for my desired object $L_{(r)}$:

$$L_{(r)} = [\, r^2\partial_r^2 + r\partial_r + (x^2r^2 - m^2) + 2x]$$

But this L is no good because it is only allowed to contain r and m, as noted above! It cannot contain x.

Miss #2. Can I construct an $L_{(r)}$ just using Bessel function recursion relations and maybe the ODE? Or some other way? I fiddled a bit and nothing jumps out at me. I have this sinking suspicion that the correct $L_{(r)}$ will involve some "fractional derivatives" of some sort.

Lets go back to our requirement

$$L_{(r)}\, J_m(\alpha r) = 2\alpha\, J_m(\alpha r)$$

This says that $J_m(\alpha r)$ must be an eigenfunction of $L_{(r)}$ with $\lambda = 2\alpha$.

Maybe I can avoid having to write down $L_{(r)}$ and just use the above fact. Our ODE is supposed to be this:

$$4\pi\, A_m(r) = \mu\, L_{(r)}\, A_m(r)$$

$$L_{(r)}\, A_m(r) = (4\pi/\mu)\, A_m(r)$$

Suppose we try this candidate solution, where α is unknown,

$$A_m(r) = J_m(\alpha r)$$

Then we have

$$L_{(r)}\, A_m(r) = L_{(r)}\, J_m(\alpha r) = 2\alpha\, J_m(ar) = 2\alpha\, A_m(r) = (4\pi/\mu)\, A_m(r) \qquad \alpha = 2\pi/\mu$$

This tells us that a candidate solution for our ODE eigenvalue equation is this:

$$A_m(r) = J_m(\alpha r) \qquad \alpha = 2\pi/\mu$$

so we get a continuous eigenvalue μ and α. Let's now look back at our starting point:

$$2 \int_0^a dr'\, r'\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}\ * A_m(r') = \mu\, A_m(r) \qquad (*)$$

Is it possible that this equation has solution $A_m(r) = J_m(2\pi r/\mu)$ for certain values of μ ?

Well, I think I have just replicated my earlier question which was this: does the following equation have a solution for some values of k

$$\int_0^\infty dx\, J_m(rx) \int_0^a dy\, y\, J_m(ky)\, J_m(xy) = (1/k)\,[\, J_m(kr)] \qquad r \le a$$

No matter how I approach this problem, I always seem to end up in the same place which is that I don't know how to find the eigenvalues and eigenfunctions of my original integral EV problem (unless a = ∞).

17. What happens if we use the cylindrical 1/R expansion for the kernel?

Our kernel is "separable" only in the following sense (see study of..) (integral instead of sum)

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] \int_0^\infty dk \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho') \qquad \text{// general}$$

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\theta-\theta')] \int_0^\infty dk\, J_m(kr)\, J_m(kr') \qquad \text{// } z = 0 \text{ and } z' = 0$$

But of course we recognize that

$$(1/\pi)\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] / \sqrt{rr'} = \int_0^\infty dk\, J_m(rk) J_m(r'k)$$

so we are really just saying that

$$1/R = (1/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\theta-\theta')]\; Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] / \sqrt{rr'} \qquad \text{// } z = 0 \text{ and } z' = 0$$

We can easily show what happens if we use this in our starting integral EV equation:

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/ \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = \mu\, u(r,\theta) \qquad Ku = \mu u$$

$$(1/\pi) \int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \;\{ \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\theta-\theta')]\; Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] / \sqrt{rr'} \} = \mu\, u(r,\theta)$$

$$(1/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^a dr'\, r'\, Q_{m-1/2}[\, (r^2+r'^2)/(2rr')] / \sqrt{rr'} * \int_0^{2\pi} d\theta'\, u(r',\theta') \cos[m(\theta-\theta')] = \mu\, u(r,\theta)$$

If we now do a partial wave expansion of u as we did above

$$u(r',\theta') = \Sigma_{m'=0}^{\infty} [\, A_{m'}(r')\sin(m'\theta') + B_{m'}(r')\cos(m'\theta')]$$

we will encounter these integrals (Schaum p 96)

$$\int_0^{2\pi} d\theta' \cos[m(\theta-\theta')] \sin(m'\theta') = \sin(m\theta) \int_0^{2\pi} d\theta' \sin(m\theta')\sin(m'\theta') = \sin(m\theta)\, \pi\, \delta_{m,m'}$$

$$\int_0^{2\pi} d\theta' \cos[m(\theta-\theta')] \cos(m'\theta') \quad = \cos(m\theta) \int_0^{2\pi} d\theta' \cos(m\theta')\cos(m'\theta') \; = \; \cos(m\theta)\, 2\pi\, \delta_{m,m'}/\varepsilon_m$$

Our LHS will then be

$$(1/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\int_0^{2\pi} d\theta' \,\{ \Sigma_{m'=0}^{\infty} [\, A_{m'}(r')\sin(m'\theta') + B_{m'}(r')\cos(m'\theta')]\} \cos[m(\theta-\theta')]$$

$$= (1/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\Sigma_{m'=0}^{\infty} \{\, A_{m'}(r') \sin(m\theta)\, \pi\, \delta_{m,m'} + B_{m'}(r') \cos(m\theta)\, 2\pi\, \delta_{m,m'}/\varepsilon_m \,\}$$

$$= \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\{\, A_m(r') \sin(m\theta) \; + B_m(r') \cos(m\theta)\, 2 /\varepsilon_m \,\}$$

$$= \Sigma_{m=1}^{\infty} 2 \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\{\, A_m(r') \sin(m\theta) \; + B_{m'}(r') \cos(m\theta) \,\}$$

$$+ 2 \int_0^a dr'\, r'\, Q_{0-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*\, B_0(r') \qquad \text{// the m=0 term}$$

$$= \Sigma_{m=0}^{\infty} 2 \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\{\, A_m(r') \sin(m\theta) \; + B_{m'}(r') \cos(m\theta) \,\}$$

So we end up then with this as our integral equation

$$\Sigma_{m=0}^{\infty} 2 \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*$$

$$\{\, A_m(r') \sin(m\theta) \; + B_{m'}(r') \cos(m\theta) \,\} \quad = \mu\, \Sigma_{m=0}^{\infty} [\, A_m(r)\sin(m\theta) + B_m(r)\cos(m\theta)]$$

Then we must have balance in each partial wave, so we get

$$\Sigma_{m=0}^{\infty} 2 \int_0^a dr'\, r'\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')] / \sqrt{rr'} \,*\, A_m(r') \; = \mu\, A_m(r)$$

and similarly for B, and this merely duplicates our equation at the end of section 5 above. So we learn nothing new using this "separable" 1/R expansion.

18. Conclusions

(1) I know how to solve the disk Dirichlet problem using the Copson method. That problem is to solve this integral equation for σ

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, \sigma(r',\theta') \,/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; f(r,\theta) \qquad K\sigma = f \qquad r \le a$$

(2) I do *not* know how to solve this related integral eigenvalue problem for a disk of radius $a < \infty$:

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\, u(r,\theta) \qquad Ku = \mu u$$

(3) I don't even know what the proper boundary conditions should be for problem (2). In the above, I tried two different boundary condition ideas (u vanish $r \ge a$, and u vanish r = a), but I could not get a solution either way.

(4) For $a = \infty$ I *do* know how to solve the integral eigenvalue problem. In this case, we get to sort of ignore what the boundary conditions might be. Our disk is the entire x-y plane. Our eigenvalues are μ = the continuous positive real axis. The eigenfunctions are $J_m(kr)$ $[\sin(m\theta),\cos(m\theta)]$ where $k = \pi/\mu$, and these are well behaved at least at r=0.

(a) I have shown that these eigenfunctions solve the 2D Helmholtz equation which I found to be associated with the integral EV equation. [even though this may not be the correct PDE]

(b) I have shown *explicitly* that these eigenfunctions solve the integral EV equation!!!!

(5) It is conceivable that the $a=\infty$ solution form works for finite a, provided you find the right k_n eigenvalues. If there are in fact such solution eigenvalues, I do know that they are *not* simply the zeros of the $J_m(ak_n)$. (Note that if solution k_n exist, k_n depends on m of the partial wave.)

(6) It is ironic that Sneddon mentions solving the eigenvalue problem as a *tool* to use in solving the Dirichlet equation. Even in this simple case of the disk, it seems to me that solving the eigenvalue problem is a much harder problem than doing the Copson solution to the Dirichlet problem! On page 203 Sneddon cavalierly states that finding the eigenfunctions is a "standard problem" that is treated in Mikhlin's book Chapter II, but as I look now at the contents of that chapter, I don't believe there is any simple solution. There is only a theoretical solution. If there were some magic bullet, I think Stak would have said something in his Chapter 2. Our kernel in this problem is symmetric, it is close to Hilbert-Schmidt, so most of the general theory applies. It is separable in the sense of Section 17 above, but this buys us nothing new. I can imagine using one of the many approximation methods to get an approximate set of EF's and EV's, about which Stak has a lot to say. But I am looking for exactly analytic solutions in my efforts here. Maybe this eigenvalue problem has no analytic solution! I will keep a look-out in case I stumble onto a treatment of this problem somewhere. I might expect to see this problem as an example in some book on integral equations.

19. One More Try. I am thinking maybe the δ(0) approach combined with ρ(x,y,z) = σ(x,y)δ(z) might yield something combined with my answer that $L = -(1/\delta(0))\,\nabla^2_{3D}$. That is to say, we might start with this full 3D eigenvalue equation

$$\int dx'dy'dz'\ \rho(x',y',z')/ R(x,y,z; x',y',z') = \mu\ \rho(x,y,z)$$

If we then take $\rho(x,y,z) = \sigma(x,y)\delta(z-z_0)$ on both sides we get

$$\int dx'dy'dz'\ \sigma(x',y')\delta(z'-z_0) / R(x,y,z; x',y',z') = \mu\ \sigma(x,y)\delta(z-z_0)$$

We can use up the LHS delta function to get

$$\int dx'dy'\sigma(x',y')/ R(x,y,z; x',y',z_0) = \mu\ \sigma(x,y)\delta(z-z_0)$$

Now set $z_0 = 0$

$$\int dx'dy'\sigma(x',y')/ R(x,y,z; x',y',0) = \mu\ \sigma(x,y)\delta(z)$$

and finally set $z = 0$

$$\int dx'dy'\sigma(x',y')/ R(x,y,0; x',y',0) = \mu\ \sigma(x,y)\delta(0) = \{\ \delta(0)\ \mu\ \}\ \sigma(x,y)$$

I think all these steps are valid at least in the distribution sense. When I say $\rho(x,y,z) = \sigma(x,y)\delta(z-z_0)$, I am specifying a planar surface charge, I think that is OK. I end up with a 2D integral equation for σ which is similar to the one I have been trying to solve, and we just have this extra $\delta(0)$ constant sitting on the RHS. I guess that should be a clue that something is not going well. I could just define $\mu_1 = \delta(0)\ \mu$.

Nevertheless, let's go back to the first equation and apply $-\nabla^2_{3D}$ to both sides. As long as we avoid singular situations (perhaps this is some kind of Stak surface layer situation!) , I think I can move this operator onto the R on the LHS to get

$$\int dx'dy'dz'\ \rho(x',y',z')/ R(x,y,z; x',y',z') = \mu\ \rho(x,y,z) \qquad \dim(\mu) = L^2$$

$$\int dx'dy'dz'\ \rho(x',y',z')[\ -\nabla^2_{3D}]\ [1/ R(x,y,z; x',y',z')] = \mu\ [\ -\nabla^2_{3D}]\rho(x,y,z)$$

I then replace [][] by $4\pi\ \delta(\mathbf{r}-\mathbf{r}')$ and obtain

$$4\pi\ \rho(x,y,z) = \mu\ [\ -\nabla^2_{3D}]\rho(x,y,z)$$

and I end up with Helmholtz 3D

$$(\nabla^2_{3D}\ + [4\pi/\mu])\ \rho(x,y,z) = 0$$

Now let's try taking our limit after obtaining this equation. We then have

$$(\nabla^2_{3D} + [4\pi/\mu])\,[\sigma(x,y)\delta(z-z_0)] = 0$$

I then set $z_0 = 0$ and get

$$(\nabla^2_{3D} + [4\pi/\mu])\,[\sigma(x,y)\delta(z)] = 0$$

This then is "something new" because $\delta(z)$ is now present. I think I can write this in cylindricals as

$$(\nabla^2_{3D} + [4\pi/\mu])\,[\sigma(r,\theta)\delta(z)] = 0$$

What are the solutions of this equation? Are there solutions? Let's try to separate out the z part:

$$\nabla^2 u = r^{-1}\partial_r(r\,\partial_r u) + r^{-2}\partial_\theta^2 u + \partial_z^2 u \qquad \text{// cylindrical}$$

$$(r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\theta^2 + \partial_z^2 + [4\pi/\mu])\,[\sigma(r,\theta)\delta(z)] = 0$$

$$(r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\theta^2 + [4\pi/\mu])\,[\sigma(r,\theta)\delta(z)] = -\sigma(r,\theta)\delta''(z)$$

$$\delta(z)\,(r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\theta^2 + [4\pi/\mu])\,\sigma(r,\theta) = -\sigma(r,\theta)\delta''(z)$$

$$(r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\theta^2 + [4\pi/\mu])\,\sigma(r,\theta) = -\sigma(r,\theta)\delta''(z)/\delta(z)$$

$$[\,(r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\theta^2 + [4\pi/\mu])\,\sigma(r,\theta)] / \sigma(r,\theta) = -\delta''(z)/\delta(z)$$

What is this telling me? If I think of δ as some approximate Gaussian to a δ, then RHS = f(z). But the LHS is NOT a function of z. I think that is a contradiction:

something not a function of z = function of z

So I conclude that there is NO solution to $(\nabla^2_{3D} + [4\pi/\mu])\,[\sigma(x,y)\delta(z)] = 0$. Is this saying that my integral EV problem has no solution? Well, all it is saying that this little attempt has failed.

Recall the Stak situation

$$[\partial_\nu u(\mathbf{x})]^{\mathbf{x}=\mathbf{s}} = \int_\sigma dS_\xi\ \sigma(\boldsymbol{\xi})\,\partial_\nu E(x|\xi) - \sigma(\mathbf{s})/2$$

Something strange happens with a single derivative when you jam it inside and then go to a surface. So you might imagine something odd might happen if you have ∇^2 going inside and then taking a surface limit.