Math Problem L(1/r)=4 π δ(x)δ(y) PhL 1.4.11

Summary: This doc comes right after the v1,v2,v3 series. It is the first doc where I really try to solve the problem $L_{\mathbf{r}}(1/r) = \delta(\mathbf{r})$ in 2D as an isolated problem. I wrongly assume this is the same as the real problem I need solved which is $L_{\mathbf{r}}(1/R) = \delta(\mathbf{r}-\mathbf{r}')$, so I focus on $L_{\mathbf{r}}(1/r) = \delta(\mathbf{r})$. This is all before Shankar and Jim Ball involvement. Early on I identify $L = \nabla^2_{3D,radial} = r^{-2}\partial_{\mathbf{r}}(r^2\partial_{\mathbf{r}}..)$ as a possible candidate solution. I show that $\nabla^2_{2D,radial}$ does not work since it gives $1/r^3$ when acting on (1/r). I don't know about ε regulation, I don't even know that δ(x)δ(y) = δ(r)/2πr, and I am not thinking about the Stakgold Theorem -- these ideas are all in the future. I understand perhaps for the first time in a long time the 2D divergence theorem, but I cannot make φ = 1/r satisfy it (later ε regulation fixes this problem). Late on this day 1/4 I call Jim Ball, and we eventually come up with two solutions to $L_{\mathbf{r}}(1/r) = \delta(\mathbf{r})$, after much labor and learning on my part. It turns out that one of these solutions is exactly $-r\nabla^2_{3D,radial}$ so my initial candidate was pretty close to a solution. But I have no solutions to $L_{\mathbf{r}}(1/R) = \delta(\mathbf{r}-\mathbf{r}')$ which is what I need to convert my 2D integral equation Ku=μu into a PDE EV problem.

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Overview (1 page, written 1.18.11)

In **Section** 1 I state "my problem" $L(1/\sqrt{x^2+y^2}) = 4\pi\delta(x)\delta(y)$.

In **Section 2** I consider the 3D problem $L(1/r) = 4\pi\delta^{(3)}(\mathbf{r})$ and show how the 3D divergence theorem confirms that $L = -\nabla^2$. Since this is similar to my problem, maybe it will help.

In **Section 3** I propose $Lf = r^{-2}\partial_r(r^2\partial_r f)$ as a candidate solution for my problem, on the grounds that I can see that $L(1/r) = 0$ for $r > 0$, and because this is $\nabla^2_{3D,radial}$ which did well in Section 2. But I have no idea whether or not it is a solution (see Section 10 below) .

In **Section 4** I derive the fact that $\nabla^2_{2D} = r^{-1}\partial_r(r\,\partial_r f) + r^{-2}\partial_\theta^2 f$ in polar coordinates. I then note that the $L = \nabla^2_{2D,radial} = r^{-1}\partial_r(r\,\partial_r ...)$ is not a viable candidate solution to my problem because for this L we get $L(1/r) = 1/r^3$ which fails to vanish for $r > 0$.

In **Section 5** I note that $L = \nabla^2_{3D,radial} = r^{-2}\partial_r(r^2\partial_r ..)$ is still a candidate, and I decide I should examine the 2D divergence theorem.

In **Section 6** I write the 2D divergence theorem as $\int dA\ \nabla^2\varphi = \oint_C d\boldsymbol{\ell}\ \hat{\mathbf{n}} \bullet \nabla\varphi = \oint_C d\boldsymbol{\ell}\ \partial_n\varphi$.

In **Section 7** I apply this theorem naively to (1/r) and find that both sides give $-2\pi/a$ except the LHS also has an extra term $+2\pi/0$ which of course diverges, so this is a paradox. I test the 2D divergence theorem on the simple function $\varphi = r^2$ and it works. Then in a noted added 1.18.11 I am able to resolve the paradox by using the regulated $[1/r] = 1/\sqrt{r^2+\varepsilon^2}$, and then for the 2D divergence theorem I get LHS = RHS both before and after the $\varepsilon\to 0$ limit (which gives $-2\pi/a$ on both sides). But when I wrote this section, I never though of doing such regulation !

In **Section 8** I use the 2D divergence theorem to verify that $-[r^{-1}\partial_r(r\,\partial_r\ )]\ [\ -(1/2\pi)\ln(r)] = \delta^2(\mathbf{r})$ which is the same as $-\nabla^2_{2D}\ [\ -(1/2\pi)\ln(r)] = \delta^2(\mathbf{r})$. Thus us Stak's 2D fundy solution of course.

In **Section 9** I am stalled on "my problem" and I call Jim on 1.4.11 and he will think about it. My gut feeling is that my problem in fact has no solution. I mourn that I cannot think of a good web handle and can therefore not do a successful web search for someone who has dealt with "my problem". As it turns out later, Jim will come up with both a regulation method and two proposed and successful solutions! Those solutions will be these, where I now have 2π on the RHS

$$L_{xy}^{(1)}\ [1/r] = -\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = +2\pi\,\delta^{(2)}(\mathbf{r})$$
$$L_{xy}^{(3)}[1/r] = \lim_{\varepsilon\to 0}\,(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = +\ 2\pi\,\delta^{(2)}(\mathbf{r})$$

In **Section 10**, which was added today 1.18.11, I show that *my original candidate* of Section 3 above, when multiplied by -r, also solves my problem:

$$L_{xy}^{(4)}[1/r] = -\,r\,\nabla^2_{3D,radial}[1/r] = -\,r^{-1}\partial_r(r^2\partial_r[1/r]) = +\ 2\pi\,\delta^{(2)}(\mathbf{r})$$

and due to Jim's Theorem $r^{-1}\partial_r(r^2\partial_r f) = \partial_r^2(rf)$ it turns out that $L_{xy}^{(4)}$ is the same as $L_{xy}^{(1)}$. And when written just in terms of r, it is totally obvious that cases (4) and (1) are the same equation. This same logic then shows there are two solutions to the 3D problem:

$$-\nabla^2_{3D,radial}\,([1/r]) = -\,r^{-2}\partial_r(r^2\partial_r[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r})$$ // the well known 3D solution

$r^{-2}\partial_{\mathbf{r}}(r\,[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r})$ // the newly found 3D solution

In **Section 11** I paste an email I am sent to Jim today 1.18 in which I derive both his solutions without using the Stakgold theorem, and thank him for his contributions.

I am still taking my Stakgold mid-term exam. Here is a simple exam problem:

1. Statement of the Problem: What 2D differential operator L causes this to be true:

$$L(1/\sqrt{x^2+y^2}\,) = 4\pi\delta(x)\delta(y)$$

Certainly this is a simply-*stated* problem.

We might first wonder whether such an operator even exists.

In attempting to solve a problem, it is always helpful to study any similar problem that you already know how to solve.

2. A similar problem. We know that this is true

$$-\nabla^2(1/\sqrt{x^2+y^2+z^2}\,) = 4\pi\,\delta(x)\delta(y)\delta(z) = 4\pi\,\delta(\mathbf{r})$$

So how does one prove this is true? I quote first from my integral theorems page

$$\int_V dV\,\nabla\bullet\mathbf{A} = \int_S d\mathbf{S}\bullet\mathbf{A} \qquad \text{// divergence theorem = Gauss's theorem} \qquad (1)\ \text{Schaum 22.59}$$

$$\int_V dV\,\nabla^2\varphi = \int_S d\mathbf{S}\bullet\nabla\varphi = \int_S dS\,\partial_{\mathbf{n}}\varphi \qquad \text{// application of the above with } \mathbf{A} = \nabla\varphi$$

Let's set $\varphi = 1/r$ and set the volume to be a sphere of radius a. We have then

$$\int_V dV\,\nabla^2(1/r) = \int_S dS\,\partial_{\mathbf{n}}\varphi = \int a^2 d\Omega\ \partial_{\mathbf{r}}(1/r)|^{\mathbf{a}} = 4\pi\,a^2\,(-a^{-2}) = -4\pi$$

Using the spherical form of ∇^2, it is trivial to show that $\nabla^2(1/r) = 0$ for $r > 0$. Therefore, in the volume integral on the left, all the contribution must come from the region at $r = 0$. In other words, this equation that $\int_V dV\,\nabla^2(1/r) = -4\pi$ is true no matter how small we make our spherical radius a. Since it is true in the limit that $a\rightarrow 0$, we conclude that $\nabla^2(1/r)$ can only be $-4\pi\delta(\mathbf{r})$. This is a viable definition of a delta function as we learned in Stak.

3. Back to our 2D problem. If we work by analogy, it seems that the first thing we need is to find an operator L such that L(1/r) = 0 for r > 0. This must be true for L, otherwise we don't stand a chance. Suppose we try

$$Lf = r^{-2}\partial_r(r^2\partial_r f)$$

where we are now in 2D polar coordinates. We know that L(1/r) = 0 for r>0, very obvious, same idea as in the 3D case, but this is not a 2D r. So this is a candidate at least.

Lots of questions are arising now:

1. Is this L somehow ∇^2 in 2D?
2. Is there a 2D version of our divergence theorem?

4. Question 1: how to you write ∇^2 in polar coordinates?

Answer: we go to our tensors doc. In Section 9b we see this section:

(d) Example C: 2D Cartesian to Polar coordinates

There we compute the metric tensor and find it to be

$$g'_{ab} = \begin{pmatrix} g'_{rr} & g'_{r\theta} \\ g'_{\theta r} & g'_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

and we then identify $Q_1 = Q_r = 1$ and $Q_2 = Q_\theta = r$. We then fast-forward to the Laplacian section where we have

$$\nabla^2 f = (1/[Q_1Q_2])\{\partial_1[(Q_2/Q_1)(\partial_1 f)] + \partial_2[(Q_1/Q_2)(\partial_2 f)]\}$$

so this says

$$\nabla^2 f = (1/[r])\{\partial_r[(r)(\partial_r f)] + \partial_\theta[(1/r)(\partial_\theta f)]\}$$

$$= r^{-1}\partial_r(r\,\partial_r f) + r^{-2}\partial_\theta^2 f$$

and to no great surprise we see this is the non-z part of the cylindrical coordinates ∇^2.

Notice that $\nabla^2(1/r) = r^{-1}\partial_r(r\,\partial_r r^{-1}) = -r^{-1}\partial_r(r\,r^{-2}) = -r^{-1}\partial_r(r^{-1}) = +r^{-1}r^{-2} = r^{-3}$

and that therefore this ∇^2 is NOT a candidate for our L operator above! This being the case, I guess we won't bother with question 2 above.

Note: I compute this ∇^2 "the long way" in doc "polar coordinates.doc" in math misc.

5. Status at this point: We still have a candidate $Lf = r^{-2}\partial_r(r^2\partial_r f)$ for L, but this is NOT ∇^2 in polar coordinates. Thus even if there is a 2D divergence theorem, it won't help us out. So our analogy with the known 3D case seems not too helpful! But let's do Question 2 anyway!

6. Question 2: what is the 2D divergence theorem?

I answer this question in doc "Proof of the Divergence Theorem.doc" in math misc, and I also prove it. The result is this (after I repaired my notation in that doc!)

$$\int dA\ \ \nabla\bullet\mathbf{F} = \oint_C d\boldsymbol{\ell}\ \hat{\mathbf{n}} \bullet \mathbf{F} \qquad\qquad 2D$$

where on the left we integrate over some 2D area, and on the right we go around its boundary. If we wanted, we could set $\mathbf{F} = \nabla\varphi$ and this would then say

$$\int dA\ \ \nabla^2\varphi = \oint_C d\boldsymbol{\ell}\ \hat{\mathbf{n}} \bullet \nabla\varphi \ \ = \ \oint_C d\boldsymbol{\ell}\ \partial_n\varphi \qquad\qquad 2D$$

7. Digression on a Paradox

Let's apply our 2D divergence theorem above to $\varphi = 1/r$. We claimed earlier that $\nabla^2(1/r) = r^{-3}$ where we use our 2D Laplacian. Then our theorem applied to a disk of radius a says

$$\int rdr \int d\theta\ \ r^{-3} \ \ = \ \int ad\theta\ \partial_r(1/r)|^{r=a}$$

We easily find that

$$RHS = 2\pi\ a\ (-1/a^2) \ = -2\pi/a$$

$$LHS = 2\pi\ \int_0^a dr\ r^{-2} = 2\pi\ [\ -r^{-1}]|^a_0 \ = -2\pi/a\ + 2\pi/0$$

So what is going on here? The divergence theorem fails for $\varphi = 1/r$, it would seem.

Let's first make sure our divergence theorem works for a more reasonable φ. Try

$$\varphi = r^2$$
$$\nabla^2(r^2) = r^{-1}\partial_r(r\ \partial_r r^2) = r^{-1}\partial_r(r\ 2r) = 2\ r^{-1}\partial_r(r^2) = 4\ r^{-1}r^1 = 4$$
$$\int dA\ \ \nabla^2\varphi = 4A \ = 4\pi a^2$$
$$\oint_C d\boldsymbol{\ell}\ \partial_n\varphi \ = \ 2\pi\ a\ \partial_r(r^2)|^{r=a} \ = 2\pi\ a\ 2a = 4\pi a^2$$

I guess my conclusion is that a divergence theorem is not valid if one side or the other involves a divergent integral. I think the fact that $\nabla^2(1/r) = r^{-3}$ is correct for all r, and there is no delta function here.

Note added 1.18.11. What happens to our paradox if we regulate 1/r using $1/\sqrt{r^2+\varepsilon^2}$? We then have

$$\nabla^2[1/r] \equiv \nabla^2(1/\sqrt{r^2+\varepsilon^2}) = r^{-1}\partial_r(r\,\partial_r\{1/\sqrt{r^2+\varepsilon^2}\}) = (r^2-2\varepsilon^2)/(r^2+\varepsilon^2)^{5/2} \qquad \text{// Maple}$$

Then for LHS we then get

$$\int_0^a rdr \int d\theta\ \nabla^2[1/r] = 2\pi \int_0^a rdr(r^2-2\varepsilon^2)/(r^2+\varepsilon^2)^{5/2} = -2\pi a^2/(a^2+\varepsilon^2)^{3/2} \quad \rightarrow -2\pi/a \qquad \text{// Maple}$$

For the RHS we first compute

$$\partial_r[1/r]|^{r=a} = \partial_r(1/\sqrt{r^2+\varepsilon^2})^{r=a} = -\,r/(r^2+\varepsilon^2)^{3/2} = -\,a/(a^2+\varepsilon^2)^{3/2}$$

Then

$$\text{RHS} = \int ad\theta\ \partial_r(1/r)|^{r=a} = 2\pi a\,[-\,a/(a^2+\varepsilon^2)^{3/2}] = -2\pi a^2/(a^2+\varepsilon^2)^{3/2} \rightarrow -2\pi/a$$

Now we get LHS = RHS both before and after the limit, so the divergence theorem is happy again!

8. A reminder of the 2D Fundy Solution of Stak Vol II p 50

The claim is this:

$$-[r^{-1}\partial_r(r\,\partial_r\ )]\ [\ -(1/2\pi)\ln(r)] = \delta^2(\mathbf{r})$$

First, it is clear that the LHS is 0 for r>0. So then we have to apply our 2D divergence theorem:

$$\int dA\ \nabla^2\varphi = \oint_C d\boldsymbol{\ell}\ \hat{\mathbf{n}} \bullet \nabla\varphi$$

We want to apply this to the case $\varphi = -(1/2\pi)\ln(r)$ and we apply it to a disk of radius a.

For the RHS, we have $\hat{\mathbf{n}}$ **is** $\hat{r}$ so $\hat{\mathbf{n}} \bullet \nabla\varphi = \partial_r\varphi = \partial_r\,[\ -(1/2\pi)\ln(r)] = -(1/2\pi)(1/r)$. Then

$$\oint_C d\boldsymbol{\ell}\ \hat{\mathbf{n}} \bullet \nabla\varphi = \int a\ d\theta\ [-(1/2\pi)(1/r)]|^a = 2\pi\ a\ [-(1/2\pi)(1/a)] = -1$$

We are claiming that $-\nabla^2\varphi = \delta^2(\mathbf{r})$ so inserting this on the LHS gives -1, QED.

9. Resume our problem.

We seek L such that

$$L(1/\sqrt{x^2+y^2}\,) = 4\pi\delta(x)\delta(y)$$

If we had this, then we could say

$$L(1/\sqrt{(x-x')^2+(y-y')^2}\,) = 4\pi\delta(x-x')\delta(y-y')$$

which is what I am really after. In 2D polar coordinates we have then

$$L(1/r) = 4\pi\,\delta^{(2)}(\mathbf{r})$$

I just called Jim (1.4.11) and I think he understood the question. He babbled about strings and monopoles, but they I asked him just to tell me what L is. He said he would have to think about it. So maybe I will go do something else for a while. If there is no solution, I would like to know why? Later (1.5.11) I emailed Shankar with this same question. Maybe I will let it ride for a while and see if either of these people tells me the answer.

I don't know how to look up this kind of problem on the web. Why?
(1) the symbols in the above equation don't work well in web searches
(2) there are many different ways the equation could be written
(3) this is not a "standard problem". One usually has L and asks for g.
(4) there is not much 2D work in the world
(5) there is no phrase I can use as a handle, like "two dimensional delta function".

I will let some days go by, and if I get no response, maybe I will continue some more on this problem.

Shankar responded and I think convinced me that if there is some L, it is not something with constant coefficients, so it is not a "simple" differential operator. I could not get him to say more. Jim is still pending.

10. What happens when we try $\mathbf{L = \nabla^2_{3D,radial}}$ using Jim regulation?

This section was added 1.18.11. If this were a solution, we would have to have

$$\nabla^2_{3D,radial}[1/r] \;=\; r^{-2}\partial_r(r^2\partial_r[1/r]) \;=\; \text{const} * \delta^{(2)}(\mathbf{r})$$

First off we have to knock out a power of r to get correct dimensions, so we would maybe expect

$$\text{(r)}\;\nabla^2_{3D,radial}[1/r] \;=\; r^{-1}\partial_r(r^2\partial_r[1/r]) \;=\; \text{const} * \delta^{(2)}(\mathbf{r}) \;= \text{const}\,\delta(r)/(2\pi r)$$

As usual, we start with Maple's help and compute

$$r^{-1}\partial_r(r^2\partial_r[1/r]) = r^{-1}\partial_r(r^2\partial_r(1/\sqrt{r^2+\varepsilon^2}\,)) \;=\; -3\,r\varepsilon^2 / (r^2+\varepsilon^2)^{5/2} \;= \varepsilon^{-2}\;\{\,-3\,(r/\varepsilon) / [(r/\varepsilon)^2+1]^{5/2}\,\}$$

so we have shown that

$$-K\, r^{-1}\partial_r(r^2\partial_r(1/\sqrt{r^2+\varepsilon^2})) = \varepsilon^{-2}\ g(r/\varepsilon) \qquad \text{where } g(x) = 3Kx/(x^2+1)^{5/2}$$

Our normalization in 2D then says

$$1/2\pi = \int_0^\infty dx\ x\ g(x) = \int_0^\infty dx\ x\{3Kx/(x^2+1)^{5/2}\} = 3K \int_0^\infty dx\ x^2/(x^2+1)^{5/2} = 3K\ \{\ 1/3\ \}$$

so we find $K = 1/2\pi$ and therefore we have shown that

$$\lim_{\varepsilon\to 0}\{ - r^{-1}\partial_r(r^2\partial_r(1/\sqrt{r^2+\varepsilon^2}))\} = 2\pi\ \lim_{\varepsilon\to 0}\{\ \varepsilon^{-2}\ g(r/\varepsilon)\} = 2\pi\ \delta^{(2)}(\mathbf{r}) \qquad (*)$$

and therefore

$$L = \lim_{\varepsilon\to 0}\{ - r^{-1}\partial_r(r^2\partial_r(1/\sqrt{r^2+\varepsilon^2}))\}$$

is a third solution to my problem! But probably we can see this another way. Recall Jim's Theorem

$$r^{-2}\partial_r(r^2\partial_r f) = r^{-1}\partial_r{}^2(rf) \qquad // \ = 2r^{-1}\partial_r f + \partial_r{}^2 f \ \text{, did this in Maple just now}$$

or

$$r^{-1}\partial_r(r^2\partial_r f) = \partial_r{}^2(rf)$$

Thus, we can take our conclusion (*) above

$$\lim_{\varepsilon\to 0}\{ - r^{-1}\partial_r(r^2\partial_r(1/\sqrt{r^2+\varepsilon^2}))\} = 2\pi\ \delta^{(2)}(\mathbf{r}) \qquad (*)$$

and restate it as

$$\lim_{\varepsilon\to 0}\{ - \partial_r{}^2(r(1/\sqrt{r^2+\varepsilon^2}))\} = 2\pi\ \delta^{(2)}(\mathbf{r})$$

and this is the same as the first of our known two solutions

$$L_{xy}^{(1)}\ [1/r] = - \lim_{\varepsilon\to 0}\ \partial_r{}^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = + 2\pi\ \delta^{(2)}(\mathbf{r})$$
$$L_{xy}^{(3)}[1/r] = \lim_{\varepsilon\to 0}\ (1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = + 2\pi\ \delta^{(2)}(\mathbf{r})$$

The solution we just showed is the same as the first is this

$$L_{xy}^{(4)}[1/r] = - r\ \nabla^2_{3D,\,radial}[1/r] = - r^{-1}\partial_r(r^2\partial_r[1/r])$$

In retrospect, my very first candidate with this extra factor of r is a solution to my problem.
So here then are three operators which work:

$$L_{xy}^{(4)} = - r^{-1}\partial_r(r^2\partial_r..) = - r\ \nabla^2_{3D,\,radial} \qquad L_{xy}^{(1)} = L_{xy}^{(4)}$$
$$L_{xy}^{(1)} = - \partial_r{}^2(r...)$$
$$L_{xy}^{(3)} = (1/r)\partial_r(r..)$$

and due to Jim's Theorem $r^{-1}\partial_r(r^2\partial_r\,..) = \partial_r{}^2(r..)$, the first two are the same.

So in the end we can compare these two results:

$-\nabla^2{}_{3D,radial}([1/r]) = 4\pi\delta^3(\mathbf{r}) = \delta(r)/r^2$ // the famous 3D result

$-r\,\nabla^2{}_{3D,radial}([1/r]) = 2\pi\delta^2(\mathbf{r}) = \delta(r)/r$ // one of our 2D results

and, when expressed just in terms of r, these two equations are really the same equation.

What this then tells us is that we also have a second solution to the 3D problem as follows:

Our second solution to the 2D problem is this:

$$L_{xy}{}^{(3)}[1/r] = \lim_{\varepsilon\to 0}(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = +\,2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

Divide both sides by 1/r to get

$$\lim_{\varepsilon\to 0}(1/r^2)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = \delta(r)/r^2 = 4\pi\,\delta^{(3)}(\mathbf{r})$$

Therefore we have these two 3D solutions

$r^{-2}\partial_r(r\,[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r})$ // the newly found 3D solution

$-\nabla^2{}_{3D,radial}([1/r]) = -\,r^{-2}\partial_r(r^2\partial_r[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r})$ // the well known 3D solution

What happens if we subtract these two solutions? The difference should say

$r^{-2}\partial_r(r\,[1/r]) + r^{-2}\partial_r(r^2\partial_r[1/r]) = 0$ (*)

or

$\partial_r(r\,[1/r]) + \partial_r(r^2\partial_r[1/r]) = 0$

or

$[\,\partial_r(r..) + \partial_r(r^2\partial_r..)]\,[1/r] = 0$

or

$[\,r^2\partial_r{}^2 + 3r\partial_r + 1]\,[1/r] = 0$ // says Maple

and in fact, if we just write $[1/r]$ as r^{-1}, this last equation is true for all r (Maple). This then verifies our second 3D solution.

Line (*) by Jim's Theorem $r^{-2}\partial_r(r^2\partial_r f) = r^{-1}\partial_r{}^2(rf)$ can be written this way

$r^{-2}\partial_r(r\,[1/r]) + r^{-1}\partial_r{}^2(r[1/r])) = 0$

or

$r^{-1}\partial_r(r\,[1/r]) + \partial_r{}^2(r[1/r])) = 0$

which says that when you subtract our known 2D solutions, you properly get 0.

11. Email to Jim with this attachment called "jim delta 1_18_11.doc".

Jim,

I'm sure you are pretty bored by now with the problem I proposed, but here is simple way to verify the two solutions you came up with, where there is no need for Stakgold's Theorem and all that stuff.

If you don't want to look at equations, at least jump down to item 3.

1. Use this shorthand notation:

$$[1/r] \equiv \lim_{\varepsilon \to 0} 1/\sqrt{r^2 + \varepsilon^2} \quad . \tag{0}$$

We have this 3D known fact that

$$-\nabla^2_{3D,radial}([1/r]) = -r^{-2}\partial_r(r^2\partial_r[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r}) \tag{1}$$

and we know that $\delta^{(3)}(\mathbf{r}) = \delta(r)/[4\pi r^2]$ so the above says

$$-r^{-2}\partial_r(r^2\partial_r[1/r]) = \delta(r)/r^2 \tag{2}$$

Multiply both sides by r

$$-r^{-1}\partial_r(r^2\partial_r[1/r]) = \delta(r)/r \tag{3}$$

Use the fact that $\delta^{(2)}(\mathbf{r}) = \delta(r)/[2\pi r]$ so the above says

$$-r^{-1}\partial_r(r^2\partial_r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{4}$$

Now use "Jim's Theorem" which says this

$$r^{-2}\partial_r(r^2\partial_r f) = r^{-1}\partial_r{}^2(rf) \qquad // \quad = 2r^{-1}\partial_r f + \partial_r{}^2 f \tag{5}$$

and multiply it by r to get

$$r^{-1}\partial_r(r^2\partial_r f) = \partial_r{}^2(rf) \tag{6}$$

and install this into (4) to get

$$-\partial_r{}^2(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{7}$$

and this then gives **your first solution** to my posed problem $L([1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r})$

2. **Your second solution** is this:

$$(1/r)\partial_r(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{8}$$

If this is correct, then if we subtract (7) from (8) we should get zero,

$$(1/r)\partial_r(r[1/r]) + \partial_r^2(r[1/r]) = 0 \tag{9}$$

We shall now show that this is in fact zero and in this way we verify your second solution.
For the first step we have Maple show this fact,

$$(1/r)\partial_r(rf(r)) + \partial_r^2(rf(r)) = (1/r)\,[\,r^2\partial_r^2 + 3r\partial_r + 1]f(r) \tag{10}$$

If we take f(r) = (1/r), this thing is trivially true:

$$[\,r^2\partial_r^2 + 3r\partial_r + 1](1/r) = 2/r - 3/r + 1/r = 0 \tag{11}$$

Not surprisingly, if we take f(r) = [1/r], it is still true. Maple gives

$$[\,r^2\partial_r^2 + 3r\partial_r + 1][1/r] = -\varepsilon^2\ (2r^2-\varepsilon^2)/(r^2+\varepsilon^2)^{5/2} \to \varepsilon^2\ 2/r^3 \to 0 \tag{12}$$

Setting f(r) = [1/r] in (10) and using (12) we conclude that

$$(1/r)\partial_r(r[1/r]) + \partial_r^2(r[1/r]) = 0$$

which is the fact (9) we sought to show, and your second solution is then verified.

3. I wish to give **proper acknowledgement to Jim** for solving this problem and, in doing so, pointing out these useful facts:

(a) The fact that $\delta^{(n)}(\mathbf{r}) = \delta(r)/[S_n(1)r^{n-1}]$ where $S_n(1)$ is the surface area of a sphere in n dimensions.

(b) That the proper way to regulate 1/r is to say $[1/r] \equiv \lim_{\varepsilon\to 0} 1/\sqrt{r^2+\varepsilon^2}$. Other methods I tried did not work, such as $[1/r] \equiv \lim_{\varepsilon\to 0} 1/(r+\varepsilon)$ or $[1/r] \equiv \lim_{\varepsilon\to 0} 1/r^{1+\varepsilon}$.

(c) The fact that $\partial_r(r^2\partial_r f) = r^{-1}\partial_r^2(rf)$ which I used above.

(d) for coming up with both solutions noted above. When I asked Jim about this problem, I had no solutions at all.