Dissing the arguments I presented in my to-jim email of 1.13.11 5 PM **1.14.11**

I agree with Jim that the following is probably not true: (but it is true, see below)

$$\lim_{\varepsilon\to 0} [(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] = +2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (2.11)$$

I need to show three things.
(1) why the Stak theorem is failing in this case (probably not LI)
(2) why my 2D divergence theorem argument is invalid
(3) why the further support from $L_x^{(3)} \sim \partial_\varphi^2$ is also wrong.

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Overview (1/2 page, written 1.16.11)

When I wrote this doc, I was pretty much convinced (and in agreement with Jim Ball) that (2.11) shown above was NOT true. But I had a lot of evidence that it WAS true. So here I am trying to show why each of my three pieces of evidence was wrong.

In **Section 1** I examined the Stakgold Theorem n=1 proof of (2.11). But this proof shows conclusively that 2.11 IS true! For a while I had a factor of 1/2 discrepancy, but I then learned (Section 4) about this factor which arises only in the n=1 case.

In **Section 2** I examine the Stakgold Theorem itself. I review how one proves the three facts he calls (a),(b),(c) on page 13 of his book. I convince myself it is correct and there is nothing special about my application which should make it not be correct. In fact g(r) can blow up at r=0, it is only the normalization condition that matters.

In **Section 3** I convince myself that the correct n=1 normalization condition has 1/2 on the RHS.

In **Section 4** I explain away the factor of 1/2 discrepancy I was getting in Section 1.

By **Section 5** I have finally realized that (2.11) really is true, and I list off the pieces of evidence.

In **Section 6** I do for Jim another Stakgold n=2 case $\lim_{\varepsilon\to 0} [\,\varepsilon / (r^2+\varepsilon^2)^{3/2}\,] = \delta^2(\mathbf{r})$. I prove it here, and then I note that it appears in Stakgold as an exercise. I think Jim felt that you really needed some derivatives ∂_r (and preferably two as in ∂_r^2) in order to construct a limit expression which makes a delta function, and this Section 6 case is a counterexample. I think Jim is a little amazed by these limit forms, and I don't think despite 30 years of teaching Math Methods that he ever ran into them before.

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1. Argument A why (2.11) might be wrong

Here perhaps is a simple argument for why 2.11 is wrong. We know we can write

$$2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

which has a sort of double-infinity blow up in that at r = 0 you get ∞/0. So consider 2.11

$$\lim_{\varepsilon\to 0}\,[(1/r)\partial_{\mathbf{r}}(r/\sqrt{r^2+\varepsilon^2}\,)\,] \;= \delta(r)/r$$

We ought to be able to cancel the 1/r to get

$$\lim_{\varepsilon\to 0}\,[\partial_{\mathbf{r}}(r/\sqrt{r^2+\varepsilon^2}\,)\,] \;= \delta(r)$$

So let's study this case. Maple says

$$\partial_{\mathbf{r}}(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^2/(r^2+\varepsilon^2)^{3/2} \;= \varepsilon^2/[\,\varepsilon^2\,((r/\varepsilon)^2+1)]^{3/2} \;= \varepsilon^2/[\,\varepsilon^3\,((r/\varepsilon)^2+1)^{3/2}]$$

$$= \varepsilon^{-1}/\,((r/\varepsilon)^2+1)^{3/2}$$

Therefore, adding a factor of K to both sides, we have shown that

$$K\,\partial_{\mathbf{r}}(r/\sqrt{r^2+\varepsilon^2}\,) \;= \varepsilon^{-1}\,\{\,K/\,((r/\varepsilon)^2+1)^{3/2}\,)$$

So define

$$g(x) = K/(x^2+1)^{3/2}$$

The Stak theorem for n says

$$\int_0^\infty dr\; r^{n-1}\,g(r) = 1/S_n(1) \qquad S_n(1) \text{ = area of n dimensional sphere}$$

$$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-n}\,g(r/\varepsilon)\,] \;= \delta^{(n)}(\mathbf{r}) \qquad \text{Note: } = \delta^{(n)}(\mathbf{r}) \;= \delta(x_1)\delta(x_2)....\delta(x_n)$$

Is this valid for n = 1 ? On page 12 he does not rule this out. Then the claim would be

$$\int_0^\infty dr\; g(r) = 1/S_1(1)$$

But what is the area of a n=1 dimensional sphere of unit radius? He has a formula on page 15 but he never talks about n = 1. For n=1 the formula gives

$$S_1(r) = 2\,\pi^{1/2}r^0/\,(\,1/2\text{-}1)! \;= 2\,\sqrt{\pi}\,/(-1/2)! \;=\; 2\,\sqrt{\pi}\,/\Gamma(1/2) \;= 2\sqrt{\pi}\,/\,\sqrt{\pi} \;=\; 2$$

Perhaps the n = 1 sphere is a line segment going from x = -a to x = +a. The boundary of this sphere is just the two interval endpoints. Perhaps each endpoint somehow counts as a 1. [I computed this same result in pencil on the bottom of page 15.]

Well, let's just try the theorem and see what happens.

We now want to find the value of K which provides our desired normalization $\int_0^\infty dx\, g(x) = 1/2$. So

$$1/2 = \int_0^\infty dx \{ K/(x^2+1)^{3/2} \} = K \int_0^\infty dx / (x^2+1)^{3/2} = K * 1 = K$$

This says K = 1/2 and we then get

$$(1/2)\partial_r(r/\sqrt{r^2+\varepsilon^2}) = \varepsilon^{-1} \{ (1/2)/ ((r/\varepsilon)^2 + 1)^{3/2} \} = \varepsilon^{-1} g(r/\varepsilon)$$

Then if the theorem applies, we get

$$\lim_{\varepsilon\to 0}[(1/2)\partial_r(r/\sqrt{r^2+\varepsilon^2})] = \lim_{\varepsilon\to 0}[\varepsilon^{-1} g(r/\varepsilon)] = \delta(r) \qquad (1/2)\,\delta(r)$$

which says

$$\lim_{\varepsilon\to 0}[\partial_r(r/\sqrt{r^2+\varepsilon^2})] = 2\delta(r) \qquad = \delta(r)$$

But my claim above was this (obtained from multiplying both sides by r of the 5 PM result)

$$\lim_{\varepsilon\to 0}[\partial_r(r/\sqrt{r^2+\varepsilon^2})] = \delta(r)$$

so we have a **factor of 2 discrepancy [fixed]**. For the n = 1 case, the good news is that g(x) is cleanly locally integrable. The bad news is that we don't really know if the theorem is true or not for n = 1. On this last, in equation (5.20), Stak has an example where he seems to allow n = 1 to be included.

Now go back to this earlier ingredient

$$\lim_{\varepsilon\to 0}[\partial_r(r/\sqrt{r^2+\varepsilon^2})] = \lim_{\varepsilon\to 0}[\varepsilon^2 / (r^2+\varepsilon^2)^{3/2}]$$

For r > 0, there no question that the limit is 0. For r = 0, we get $\lim_{\varepsilon\to 0}[1/\varepsilon]$ which is infinite, so it is sort of reasonable.

So this argument does not get rid of the δ, but it changes the factor.

2. Let us review Stak's theorem, in particular, the three items top of page 13.

Property (a): $\int_{R^n} f_\alpha(x)dx = 1$? $\qquad x = x_1 \ldots . x_n$

Here R_n means over the entire space R_n , not some region within R_n.

Let $u = x/\alpha$ and use $f_\alpha(x) = \alpha^{-n} f(x/\alpha)$. Then $dx = \alpha^n du$. So

$$\int_{Rn} f_\alpha(x)dx = \int_{Rn} \alpha^{-n} f(x/\alpha)\, dx = \int_{Rn/\alpha} \alpha^{-n} f(u)\, \alpha^n\, du = \int_{Rn/\alpha} f(u)du$$

But R_n/α for finite α is still "the entire space R_n ", so we get

$$= \int_{Rn} f(u)du$$

This is then equal to 1 because that was an assumed property of f(u) in this theorem.

Property (c): $\lim_{\alpha\to 0} \int_{sphere=A} f_\alpha(x)dx = 1$

For n = 1, the sphere is the line running from -A to A. We know n = 2 and 3. The volume element dx in n dimensions is $dV_n = S_n(r)dr$ with $S_n(r)$ as in 5.18. For n = 1, $S_1(r) = 2$ so $dV_1 = 2\, dr$. I think I might explain that by saying we are dealing only with r = (0,A) and if we want both parts of our line segment running from (-A,A) which is our "sphere", then we have this factor of 2. But in general we have

$$\int_{sphere=A} f_\alpha(x)dx = \int dV_n\, f_\alpha(x) = \int S_n(r)\, dr\, f_\alpha(x) = S_n(1) \int_0^A r^{n-1}dr\, f_\alpha(x)$$

where the last step is only justified if we have $f_\alpha(x) = g_\alpha(r)$, which I now will assume. Then we continue

$$= S_n(1) \int_0^A r^{n-1}dr\, g_\alpha(r) = S_n(1) \int_0^A r^{n-1}dr\, \alpha^{-n}\, g(r/\alpha)$$

Now we define variable $u = r/\alpha$ so that

$$r^{n-1}dr\, \alpha^{-n} = (\alpha u)^{n-1}\, \alpha du\, \alpha^{-n} = u^{n-1}\, du$$

and then we have

$$= S_n(1) \int_0^{A/\alpha} du\, u^{n-1}\, g(u)$$

Now we take $\alpha\to 0$ and this becomes

$$= S_n(1) \int_0^\infty du\, u^{n-1}\, g(u) = S_n(1) \int_0^\infty dx\, x^{n-1}\, g(x)$$

But our normalization condition is that this thing is 1! Thus we have shown that

$\lim_{\alpha\to 0} [\int_{\text{sphere=A}} g_\alpha(r)dx] = 1$ for any A> 0

and this then is the claim of item (c), thus it is proved.

Property (b): $\lim_{\alpha\to 0} \int_{|x|>A} g_\alpha(r)dx = 1$

To show (b), you repeat the above with $\int_0^a dr$ replaced by $\int_a^\infty dr$ which then becomes $\int_{a/\varepsilon}^\infty dx$ which becomes $\int_\infty^\infty dx$ which gives 0. This part concerns only the convergence of the tail of the integral and does not involve any issues near the origin.

Comment: How bad can $g_\alpha(r)$ be near r = 0? Looking above, we deal only with this integral

$$\int_0^A r^{n-1}dr\, g_\alpha(r) = \int_0^A r^{n-1}dr\, \alpha^{-n}\, g(r/\alpha)$$

So for finite A and finite α, our concern is this

$$\int_0^\varepsilon r^{n-1}dr\, g(r/\alpha) < \infty$$

This means that

power $(r^{n-1}\, g(r/\alpha)) > -1$

For n = 2 which we did in the 5PM doc, this says

power $(r\, g(r/\alpha)) > -1$

Our g in question was this

$g(r) = -3Kr/[(r^2+1)^{5/2}] \sim 1/r$

But we then have

power $(r\ 1/r) = 0 > -1$

so this should be OK!

3. Stakgold normalization condition for n = 1

The normalization condition for n = 1 is written

$$\int_{R1} f(x)\, dx = 1$$

where R_1 is the entire x axis. Now assume that $f(x) = g(|x|) = g(r)$. Then the normalization condition is

$$\int_{-\infty}^{\infty} dx\, g(|x|) = 1$$

which we can write as

$$\int_{0}^{\infty} dr\, g(r) = 1/2$$

This agrees with the general formula since $S_1(1) = 2$!

4. Going after our factor of 2 discrepancy.

Let's do Stak in the case n = 1 and just see what happens. The normalization condition is this

$$\int_{R1} f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Let's now verify page 13 (a) at the top,

$$\int_{R1} f_\alpha(x)\, dx = \int_{-\infty}^{\infty} f_\alpha(x)\, dx = ?$$

We then write

$$f_\alpha(x) = \alpha^{-1} f(x/\alpha)$$

Then we have

$$\int_{-\infty}^{\infty} dx\ f_\alpha(x) = \int_{-\infty}^{\infty} dx\ \alpha^{-1} f(x/\alpha)$$

Then define $u = x/\alpha$ and we get (for finite α) $dx\, \alpha^{-1} = du$

$$= \int_{-\infty}^{\infty} du\, f(u) = 1$$

according to the normalization condition. Therefore we have shown that

$$\int_{R1} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

and (a) is then verified for the case n = 1.

So far then we have shown that

$$\int_{R1} f_\alpha(x)\,dx = \int_{-\infty}^{\infty} du\, f(u) = \int_{R1} du\, f(u) = 1$$

But the RHS is our normalization integral and is therefore 1, so (a) is verified.

Now what is the claim of the theorem anyway for n = 1 ?

$$\lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(x/\alpha)] = \delta(x) \qquad \text{on the domain } (-\infty,\infty)$$

which really says, where φ(x) is a test function

$$\int_{-a}^{a} dx\, \varphi(x) \lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(x/\alpha)] = \int_{-a}^{a} dx\, \varphi(x)\, \delta(x) = \varphi(0)$$

But suppose we want our domain to be r in the range (0,∞). It seems that you would then get

$$\int_{0}^{a} dx\, \varphi(x) \lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(x/\alpha)] = \int_{0}^{a} dx\, \varphi(x)\, \delta(x) = (1/2)\, \varphi(0)$$

with the idea that you only pick up half the delta function in this case. This would argue that

$$\lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(x/\alpha)] = (1/2)\, \delta(x) \qquad \text{on the domain } (0,\infty)$$

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5. Review of reasons why I think these two claims are valid

$$\lim_{\varepsilon \to 0} [\partial_r (r/\sqrt{r^2+\varepsilon^2}\,)\,] = \delta(r) \qquad (1)$$

$$\lim_{\varepsilon \to 0} [(1/r)\, \partial_r (r/\sqrt{r^2+\varepsilon^2}\,)\,] = \delta(r)/r = 2\pi\delta(x)\delta(y) = 2\pi\delta^{(2)}(\mathbf{r}) \qquad (2)$$

(1) I reviewed the Stakgold theorem and it seems correct as long as the normalization integral converges. That was my only concern about the proof of (2).

(2) I did a separate n = 1 proof of (1), so now both (1) and (2) are separately proved as n = 1 and n = 2 cases of the Stakgold theorem. I did have to use the "half delta" thing for the n = 1 case.

(3) The 2D divergence theorem supports (2) since (2) must be the negative of a result we believe gives

$-2\pi\delta^{(2)}(\mathbf{r})$

(4) When I computed the difference between the L_{xy} implied by the two claims, it came out being something proportional to ∂_φ^2 which makes no contribution acting on a function of r, supporting the idea that both L_{xy} results are valid. [This is not quite right: as shown in a certain Comment located here:

"summary of docs..." / review of "Answer to question 2.doc" / Part 4: Email

the two solutions differ by $\nabla^2_{2D}(r..) - r^{-1}\partial_\varphi^2$ and *each* of these two terms is a solution to the homogeneous equation in question $L_{xy}[1/r] = 0$.]

6. Jim suggested checking this no-derivative example:

$\lim_{\varepsilon\to 0} [\ \varepsilon / (r^2+\varepsilon^2)^{3/2}\ ] = \delta^2(\mathbf{r})$

We get

$\varepsilon / (r^2+\varepsilon^2)^{3/2} = \varepsilon\ / [\ \varepsilon^3\ (\ (r/\varepsilon)^2 + 1)^{3/2}] = \varepsilon^{-2}\ \{\ 1/\ (r/\varepsilon)^2 + 1)^{3/2}\ \}$

Then we have

$K\ \varepsilon / (r^2+\varepsilon^2)^{3/2} = \varepsilon^{-2}\ \{\ K/\ (r/\varepsilon)^2 + 1)^{3/2}\ \} = \varepsilon^{-2}\ g(r/\varepsilon)$

and $g(x) = K / (x^2+1)^{3/2}$ and 2D normalization requires that

$\int_0^\infty dx\ x\ K / (x^2+1)^{3/2} = 1/2\pi \qquad K * 1 = 1/2\pi \quad \Rightarrow K = 1/2\pi$

Therefore

$\lim_{\varepsilon\to 0} [(1/2\pi)\ \varepsilon / (r^2+\varepsilon^2)^{3/2}\ ] = \lim_{\varepsilon\to 0} [\varepsilon^{-2}\ g(r/\varepsilon)] = \delta^2(\mathbf{r})$

Interestingly, this actually appears in Stakgold's book.