Is there an L_x for the Love Equation? PhL 1.10.11

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Overview (1/2 page, written 1.12.11)

I show that, for the Love integral equation's kernel,

$$r(x) = (d/\pi) / [\, d^2 + x^2 \,]$$

there exists no differential operator L_x with constant coefficients (even if we allow infinite order) such that

$$L_x\, r(x) = \delta(x).$$

I show this two different ways and arrive at the same conclusion twice. It follows that there is no L_x such that $L_x\, r(x-y) = \delta(x-y)$ and thus it follows that the Love integral equation cannot be represented as a differential equation of the form

$$L_x[f(x) - 1] = \mp \int_{-1}^{1} dt\, L_x\, r(x-t)\, f(t) \;=\; \mp \int_{-1}^{1} dt\, L_x\, \delta(x-t)\, f(t) \;=\; \mp\, f(x)\, \theta(x>-1)\theta(x<+1)$$

where L_x has the form just stated.

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1. Where this kernel appears.

Sneddon gives the following form for the Love integral equation,

$$f(x) - 1 = \mp \int_{-1}^{1} dt\, r(x-t)\, f(t) \qquad\qquad r(x) = (d/\pi) / [\, d^2 + x^2],$$

where the two parallel disks of r = 1 are located at z = 0 (with $V = V_0$) and z = d (with $V = \pm V_0$), so it is the lower sign which corresponds to the capacitor problem.

2. The related Fourier Transform and Fourier Cosine Transform.

We can look up this Fourier Transform

$$(d^2+x^2)^{-1} = \int_{-\infty}^{\infty} dk\, e^{-ikx} F(k) \qquad F(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} (d^2+x^2)^{-1} = e^{-d|k|}/2d$$

where in the second line I put |k| to allow that k might be negative (it is in half the k integral!) Let's consider this transform a little more carefully. We first write

$$F(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} (d^2+x^2)^{-1} = (1/2\pi) \int_{-\infty}^{\infty} dx[\cos(kx)+i\sin(kx)]\ (d^2+x^2)^{-1}$$

The sin(kx) term is odd in x, so contributes nothing to the integral, so we have

$$F(k) = (1/2\pi) \int_{-\infty}^{\infty} dx \cos(kx)\, (d^2+x^2)^{-1}$$

There are no poles or issues with the integral, so F(k) is real, and it is even in k. Furthermore, the integrand is even in x, so we can fold it to get

$$F(k) = (1/\pi) \int_{0}^{\infty} dx \cos(kx)\, (d^2+x^2)^{-1}$$

and we can regard this if we like as a Fourier Cosine projection. Now we turn to the other direction:

$$(d^2+x^2)^{-1} = \int_{-\infty}^{\infty} dk\, e^{-ikx} F(k) = \int_{-\infty}^{\infty} dk\, [\cos(kx)-i\sin(kx)]\, F(k)$$

Since F(k) is even, the sine term gives nothing and we get

$$(d^2+x^2)^{-1} = \int_{-\infty}^{\infty} dk \cos(kx) F(k) = 2 \int_{0}^{\infty} dk \cos(kx) F(k)$$

Since the integrand is even in k, we can write this as shown on the right above. So we can summarize our transform if we like in this manner, which is the famous Fourier Cosine Integral Transform valid for "even" functions f(x), [even means it would have only even powers if expanded in a power series in x]

$$(d^2+x^2)^{-1} = 2 \int_{0}^{\infty} dk \cos(kx) F(k)$$

$$F(k) = (1/\pi) \int_{0}^{\infty} dx \cos(kx)\, (d^2+x^2)^{-1} = e^{-d|k|}/2d = e^{-dk}/2d$$

Since only k > 0 is involved in this version of the transform, we can remove the || in the last step 2nd line.

Since $r(x) = (d/\pi) / [\, d^2 + x^2]$ we could express the above transform as

$$r(x) = \quad 2\,(d/\pi) \int_0^\infty dk\, \cos(kx) F(k) \;=\; (2d/\pi) \int_0^\infty dk\, \cos(kx) F(k)$$

$$F(k) = (1/\pi)(\pi/d) \int_0^\infty dx\, \cos(kx)\, r(x) \;=\; (1/d) \int_0^\infty dx\, \cos(kx)\, r(x) \;=\; e^{-dk}/2d$$

On the other hand, we can still use the full Fourier Integral Transform, which above said

$$(d/\pi)\,(d^2+x^2)^{-1} = (d/\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} F(k)$$

$$F(k) = (1/2\pi)(\pi/d) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, (d/\pi)\,(d^2+x^2)^{-1} = e^{-d|k|}/2d$$

which again we write as

$$r(x) = (d/\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} F(k)$$

$$F(k) = (1/2\pi)(\pi/d) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, r(x) \;= e^{-d|k|}/2d$$

3. Proof using the Fourier Transform.

Now let us consider this problem which we want to solve for L_x

$$L_x\; r(x) = \delta(x)$$

We shall now expand both sides using the full Fourier Integral Transform:

$$L_x\; (d/\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} F(k) \;=\; (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

We then move L_x into the integral with the usual caveat to get

$$(d/\pi) \int_{-\infty}^{\infty} dk\, [\, L_x\, e^{-ikx}\,]\, F(k) \;=\; (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

which we rewrite inserting our known result for F(k)

$$(d/\pi) \int_{-\infty}^{\infty} dk\, [\, L_x\, e^{-ikx}\,]\, e^{-d|k|}/2d \;=\; (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

Now let's make the ansatz that we can represent L_x in this manner

$$L_x = \Sigma_{n=0}^{\infty}\, a_n\, \partial_x^n \qquad\qquad a_n = \text{constants!}$$

We then get

$$(d/\pi) \int_{-\infty}^{\infty} dk \, [\Sigma_{n=0}^{\infty} a_n \, \partial_x^n \, e^{-ikx}] \, e^{-d|k|} /2d \; = \; (1/2\pi) \int_{-\infty}^{\infty} dk \, e^{-ikx}$$

$$(d/\pi) \int_{-\infty}^{\infty} dk \, [\Sigma_{n=0}^{\infty} a_n (-ik)^n \, e^{-ikx}] \, e^{-d|k|} /2d \; = \; (1/2\pi) \int_{-\infty}^{\infty} dk \, e^{-ikx}$$

$$(2d) \int_{-\infty}^{\infty} dk \, e^{-ikx} \, \{ [\Sigma_{n=0}^{\infty} a_n (-ik)^n] \, e^{-d|k|} /2d\} \quad = \quad \int_{-\infty}^{\infty} dk \, e^{-ikx}$$

$$\int_{-\infty}^{\infty} dk \, e^{-ikx} \, \{ [\Sigma_{n=0}^{\infty} a_n (-ik)^n] \, e^{-d|k|}\} \quad = \quad \int_{-\infty}^{\infty} dk \, e^{-ikx}$$

We now appeal to the completeness of the basis set e^{-ikx} and conclude that

$$[\Sigma_{n=0}^{\infty} a_n (-ik)^n] \, e^{-d|k|} \; = \; 1$$

$$[\Sigma_{n=0}^{\infty} a_n (-ik)^n] \; = e^{+d|k|}$$

where we are careful to keep the |k| since k can be either sign in the Fourier Integral Transform. We now want to solve this equation for the a_n coefficients. We can do this, expanding $e^{+d|k|}$,

$$\Sigma_{n=0}^{\infty} a_n (-ik)^n \; = \Sigma_{n=0}^{\infty} d^n |k|^n / n!$$

Now we can write |k| = sign(k) k to get

$$\Sigma_{n=0}^{\infty} a_n (-i)^n k^n \; = \Sigma_{n=0}^{\infty} d^n (\text{sign}(k))^n \; k^n / n!$$

We write this as

$$\Sigma_{n=0}^{\infty} b_n k^n \; = 0 \qquad\qquad b_n = [a_n (-i)^n \; - d^n (\text{sign}(k))^n / n! \,]$$

This must be valid for all k in $(-\infty,\infty)$. But even if it were valid only on some finite interval, we would still conclude that b_n must be 0, see Power Series doc on this subject. Therefore we must have

$$a_n (-i)^n \; = \; (\text{sign}(k))^n \, d^n / n!$$

But this says that a_n is a function k, but we assumed that a_n were just constants like 2.5, and cannot be functions of anything other than n. We cannot just consider k > 0 because our Fourier integrals shown above all involved both signs of k.

The conclusion is that no a_n exists and that therefore our ansatz $L_x = \Sigma_{n=0}^{\infty} a_n \, \partial_x^n$ has failed!

Note: When I did this analysis earlier, I omitted the |k| absolute values and got $a_n = (id)^n/n!$ and this in turn implied that $L_x = \Sigma_{n=0}^{\infty} a_n \, \partial_x^n = \Sigma_{n=0}^{\infty} (id)^n/n! \, \partial_x^n \; = \exp(id\partial_x)$. But that was incorrect!

4. Proof using the Cosine Transform.

Let's repeat the above section using Cosine Transforms instead and see if we can be more successful. Maybe if we only deal with k > 0 things will be somehow better:

Now let us consider this problem which we want to solve for L_x

$$L_x\ r(x) = \delta(x)$$

We shall now expand both sides using the full Cosine Integral Transform. Recall from Section x that we had

$$r(x) = \quad 2\,(d/\pi) \int_0^\infty dk\, \cos(kx)F(k) \;=\; (2d/\pi) \int_0^\infty dk\, \cos(kx)F(k)$$

$$F(k) = (1/\pi)(\pi/d) \int_0^\infty dx\, \cos(kx)\, r(x) \;=\; (1/d) \int_0^\infty dx\, \cos(kx)\, r(x) \;=\; e^{-d|k|}/2d$$

so we then get

$$\text{LHS} = L_x\ [(2d/\pi) \int_0^\infty dk\, \cos(kx)F(k)]$$

Now what is δ(x) expressed in Fourier Cosine language? Well start with

$$\delta(x) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} = (1/2\pi) \int_{-\infty}^{\infty} dk\ [\cos(kx)-i\sin(kx)]$$

I argue now that δ(x) is even in x, so the sin(kx) contribution must vanish. Another argument for this same conclusion is that we think of δ(x) as being real. We then have

$$\delta(x) = (1/2\pi) \int_{-\infty}^{\infty} dk\, \cos(kx) \;=\; (1/\pi) \int_0^\infty dk\, \cos(kx) \qquad (*)$$

Now just out of curiosity, suppose we start with our generic Fourier Cosine Transform

$$f(x) \;=\; 2 \int_0^\infty dk\, \cos(kx)F(k)$$

$$F(k) = (1/\pi) \int_0^\infty dx\, \cos(kx)\, f(x)$$

which I have just carefully shown is consistent with my Summary Case 2 in transforms.doc. If we were to insert f(x) = δ(x) into the second integral here, what do we get?

$$F(k) = (1/\pi) \int_0^\infty dx\, \cos(kx)\, \delta(x) \;= ?$$

Since our Cosine Transform deals only with $x \geq 0$, we pick up "half the delta function" here to get

$$F(k) = (1/\pi) \int_0^\infty dx \cos(kx)\, \delta(x) = (1/2\pi)$$

Then we get

$$\delta(x) = 2 \int_0^\infty dk \cos(kx) F(k) = 2 \int_0^\infty dk \cos(kx)\, (1/2\pi) = (1/\pi) \int_0^\infty dk \cos(kx)$$

in agreement with (*) above. See "disk green attempt2" section 4a for more detail on this idea of picking up half the delta function.

Our equation with both sides expanded as above is now

$$L_x\ (2d/\pi) \int_0^\infty dk \cos(kx) F(k) = (1/\pi) \int_0^\infty dk \cos(kx) \qquad x \geq 0$$

We now move L_x inside the integral to get

$$(2d) \int_0^\infty dk\, [\, L_x \cos(kx)]\, F(k) = \int_0^\infty dk \cos(kx) \qquad x \geq 0$$

I keep saying $x \geq 0$ because our Cosine Transform is only defined on $x \geq 0$. Now let's assume the following form for L_x

$$L_x = \Sigma_{n=0,2}^{\infty} a_n\ \partial_x^n$$

Here I assume only even powers because odd powers will generate sin(kx) terms which are odd in x, but the RHS is even in x. (by saying g(x) is "even" for $x \geq 0$ I mean that if expanded in an x^n power series, only even powers would appear.) So we get,

$$[(2d) \int_0^\infty dk\, [\Sigma_{n=0,2}^{\infty} a_n\ \partial_x^n \cos(kx)]\, F(k) = \int_0^\infty dk \cos(kx) \qquad \text{all } x$$

Now for even n we know that

$$\partial_x^2 \cos(kx) = -k^2 \cos(kx)$$
$$\partial_x^4 \cos(kx) = k^4 \cos(kx)$$
$$\ldots.$$
$$\partial_x^n \cos(kx) = (-1)^{n/2}\ k^n \cos(kx)$$

So we then have

$$(2d)\ \int_0^\infty dk\ [\Sigma_{n=0,2}^{\infty} a_n\ \ (-1)^{n/2}\ k^n \cos(kx)]\ F(k)\ =\ \int_0^\infty dk \cos(kx) \qquad \text{all } x$$

Now let's install our F(k). Since we are only thinking $k \geq 0$ now we use

$$F(k) = e^{-dk}/2d$$

to get

$$\int_0^\infty dk\ [\Sigma_{n=0,2}^{\infty} a_n\ \ (-1)^{n/2}\ k^n \cos(kx)]e^{-dk}\ =\ \int_0^\infty dk \cos(kx) \qquad \text{all } x$$

$$\int_0^\infty dk \cos(kx)\ [\Sigma_{n=0,2}^{\infty} a_n\ \ (-1)^{n/2}\ k^n]e^{-dk}\ =\ \int_0^\infty dk \cos(kx) \qquad \text{all } x$$

We now appeal to cos(kx) being complete for even functions of x to conclude that

$$\Sigma_{n=0,2}^{\infty} a_n\ \ (-1)^{n/2}\ k^n\ =\ e^{dk}\ =\ \cosh(dk) + \sinh(dk)$$

or

$$\Sigma_{n=0,2}^{\infty} a_n\ \ (-1)^{n/2}\ k^n - \cosh(dk)\ = \sinh(dk)$$

No matter what a_n we choose, the LHS has only even powers of k and the RHS has only odd powers of k, so these cannot be equal! Therefore, no a_n exists for the assumed form of our L_x.

This is the same conclusion we reached in the previous section!

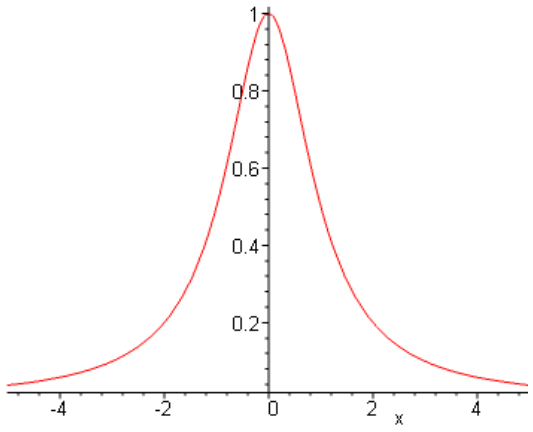
5. Solution to $L_x g(x) = \delta(x)$ for C^∞ functions g(x)

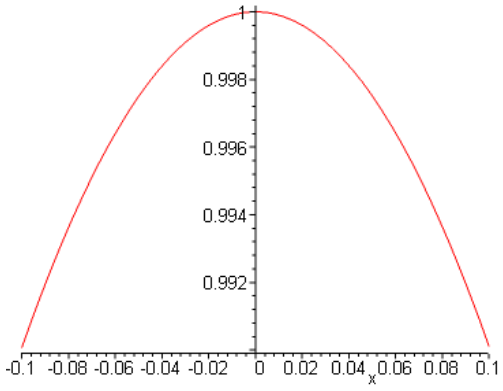
In order to have $L_x g(x) = \delta(x)$, it just seems to me that something dramatic needs to happen with f(x) at the point x = 0. For the string problem we know that the derivative takes a jump at x = 0. The Green's Function f(x) is a sawtooth that peaks at x = 0 (in this case), and so it has zero curvature away from x = 0 and something dramatic happens at x = 0. Similarly for the case of 1/r in 2D or 3D, this function does something dramatic at r = 0.

Now suppose g(x) is a boring function at x = 0, such as our Love kernel

$$g(x) = 1/(x^2+a^2) \qquad a = \text{fixed}$$

If you plot this close to x = 0, you get a nice smooth parabola as shown on the right below,





All derivatives exist and are finite at x = 0, so I suppose this thing is C^∞ . So now can some L_x find something dramatic at x = 0 so $L_x g(x) = \delta(x)$? If you try L_x with some coefficient functions $a_n(x)$, you get this

```
restart;
R := 1/(x^2+a^2);
```

$$R := \frac{1}{x^2 + a^2}$$

```
diff(R,x$14):simplify(%);
```

$$87178291200 \frac{15 x^{14} - 455 x^{12} a^2 + 3003 x^{10} a^4 - 6435 x^8 a^6 + 5005 x^6 a^8 - 1365 x^4 a^{10} + 105 x^2 a^{12} - a^{14}}{(x^2 + a^2)^{15}}$$

$\partial_n g(x) = C_n \; p_n(x) / (x^2 + a^2)^{n+1}$ $p_n(x)$ = poly of degree n C_n = a number

So then

$$L_x g(x) = \Sigma_{n=0}^{\infty} a_n(x) \, \partial_n g(x) = \Sigma_{n=0}^{\infty} a_n(x) \, C_n \; p_n(x) / (x^2 + a^2)^{n+1}$$

If the $a_n(x)$ are polynomials of some degree, then each term is some poly over $(x^2 + a^2)^{n+1}$ which is analytic at x = 0. How would you make this sum vanish near x = 0 but not at x = 0 ? In order to have any kind of kink at x = 0, it seems to me you need to use some complete basis functions. We have

$$\delta(x-x') = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \, \varphi_n(x') \qquad \delta(x) = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \, \varphi_n(0)$$

So we need to arrange for this to happen

$$\Sigma_{n=0}^{\infty} a_n(x) \, C_n \; p_n(x) / (x^2 + a^2)^{n+1} = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \, \varphi_n(0)$$

for some set of basis functions. So that says

$$a_n(x) \, C_n \; p_n(x) / (x^2 + a^2)^{n+1} = \varphi_n(x)^* \, \varphi_n(0)$$

$$a_n(x) = \varphi_n(x)^* \varphi_n(0) \ (x^2 + a^2)^{n+1} / [C_n p_n(x)]$$

For example, we could select normalized Legendre Polynomials,

$$\Sigma_{n=|m|}^{\infty} (1/K_n^m) P_n^m(z') P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1} f(n,m)$$

$$\Sigma_{n=0}^{\infty} (1/K_n) P_n(z') P_n(z) = \delta(z'-z) \qquad K_n = (n+1/2)^{-1}$$

$$\varphi_n(x) = P_n(z)/\sqrt{K_n} = \sqrt{n+1/2} \ P_n(z)$$

Then a solution would be

$$a_n(x) = (n+1/2) P_n(x) P_n(0) \varphi_n(0) \ (x^2 + a^2)^{n+1} / [C_n p_n(x)]$$

So here is my candidate L_x

$$L_x = \Sigma_{n=0}^{\infty} (n+1/2) P_n(x) P_n(0) \varphi_n(0) \ (x^2 + a^2)^{n+1} / [C_n p_n(x)] \ \partial_x^n$$

So maybe I can generalize this as follows:

$$\delta(x) = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \varphi_n(0)$$

$$L_x = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \varphi_n(0) [\partial_x^n g(x)]^{-1} \partial_x^n$$

Then

$$L_x \, g(x) = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \varphi_n(0) [\partial_x^n g(x)]^{-1} \partial_x^n \, g(x) = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \varphi_n(0) = \delta(x)$$

so this seems to give an answer for any g(x) you want as long as it is C^{∞} so all those derivatives exist. They have to all exist so you can write L_x as shown with no divide by zeros.

Theorem(?): If g(x) is C^{∞} in an interval of interest surrounding the origin x = 0, then a solution to the posed question:

$$L_x \, g(x) = \delta(x)$$

is this:

$$L_x = \Sigma_{n=0}^{\infty} \varphi_n(x)^* \varphi_n(0) [\partial_x^n g(x)]^{-1} \partial_x^n$$

where $\varphi_n(x)$ is any orthonormalized complete set of basis functions on that interval.

This theorem is applicable to the Love kernel, but of course it is a pretty ugly L_x and of infinite order.

6. Is the Love kernel Hilbert-Schmidt?

Here is our Love integral equation

$$f(x) - 1 = \mp \int_{-1}^{1} dt\, r(x-t)\, f(t) \qquad r(x) = (d/\pi) / [\, d^2 + x^2],$$

So the question is whether or not the following integral is finite:

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \;|\; (d/\pi) / [\, d^2 + (x-y)^2] \,|^2 < \infty \qquad ?$$

or

$$I \equiv \int_{-1}^{1} dy \int_{-1}^{1} dx \;\; 1 / [\, d^2 + (x-y)^2]^{\,2} < \infty \qquad ?$$

Define new variable z = x-y to replace x and dx = dz so have

$$I = \int_{-1}^{1} dy \int_{-1-y}^{1-y} dz \;\; 1 / (d^2 + z^2)^2$$

Things are pretty messy from this point on, so here is Maple on the subject:

```
> restart;
> assume(d>0);
> f := collect(int(1/(d^2+z^2)^2,z=-1-y..1-y),arctan);
```

$$f := \frac{1}{2}\frac{(d^4+2d^2+2d^2y^2+1-2y^2+y^4)\arctan\left(\frac{1+y}{d}\right)}{d^3(d^2+1-2y+y^2)(d^2+1+2y+y^2)} + \frac{1}{2}\frac{(-d^4-2d^2-2d^2y^2-1+2y^2-y^4)\arctan\left(\frac{-1+y}{d}\right)}{d^3(d^2+1-2y+y^2)(d^2+1+2y+y^2)} + \frac{1}{2}\frac{2d^3+2d-2dy^2}{d^3(d^2+1-2y+y^2)(d^2+1+2y+y^2)}$$

```
> int(f,y=-1..1);
```

$$2\frac{\arctan\left(2\frac{1}{d}\right)}{d^3}$$

Therefore we find that

$$\int_{-1}^{1} dx \int_{-1}^{1} dy \;|\; (d/\pi) / [\, d^2 + (x-y)^2] \,|^2$$

$$= (d/\pi)\, I = (d/\pi) * (2/d^3) \tan^{-1}(2/d) = (2/\pi d^2)\tan^{-1}(2/d)$$

and as long as $d > 0$, this is finite. Therefore the Love kernel is indeed a Hilbert-Schmidt kernel.