How to find L in 1D $L_x s(x,y)=\delta(x-y)$ PhL 1.9.11

(See Summary in separate summary doc.)

This entire doc considers only 1D situations of $L_x\, s(x,y) = \delta(x-y)$ and special case $L_x r(x) = \delta(x)$.

When I wrote this doc, I did not realize that $L_x g(x)=\delta(x) \not\Rightarrow L_x g(x-x')=\delta(x-x')$ (except for constant coefficient L_x), but I have reread all and have written an Overview, and this does not invalidate much that I wrote, and I comment in the appropriate places.

All this thrashing around boils down to this question stated in section 5 below:

"Given a function r(x), is it, or is it not, a Green's Function of some self-adjoint differential operator? "

I am unable to answer this question, but my gut feeling is that if you select some r(x) at random, such as $r(x) = 1/x^4$, you will find that there is no L_x that will work. I could be wrong. Assuming I am right, one could extend the conclusion to higher dimensions. The gut feeling then is that Green's Functions are a very narrow class of functions in the universe of all functions.

In Section 6 at the very end I think I have proven this theorem:

"If the kernel is of the r(x) form and is even in x and has a Fourier Integral Transform, then $L_x r(x) = \delta(x)$ has no solution L_x which is a differential operator having constant coefficients. "

But in this case I cannot rule out L_x having some other form. The Love integral equation has a kernel that fits this theorem.

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Overview (2 pages, written 1.16.11)

In **Section 1** (8 pages) I consider the problem of finding L_x for $L_x\, s(x,y) = \delta(x-y)$. I Fourier-Integral expand s(x,y) into S(k,y) and expand the δ as well. I first assume in parts (a)→(d) a general form $L_x = \Sigma_{n=0}^{\infty} a_n(x)\, \partial_x^n$. I Fourier expand the $a_n(x)$ as well (which might be a bad idea for a function like x^2!) and I end up with this very ugly and **intractable** equation which has to somehow be valid for all real y in our interval, and the q_n are known,

$$\Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\, q_n(k'-k;\, y) A_n(k) = -(1/2\pi) e^{ik'y}$$

Then in part (e) I start over assuming $L_x = \Sigma_{n=0}^{\infty} a_n \partial_x^n$ with constant coefficients. Our only chance here is if s(x,y) = r(x-y), but then our hopes are dashed by Theorem 2 and 2A which show that, if r(x) is even, which is always the case for me, then R(k) = F(|k|) and is not anlytic in k at k=0 and therefore cannot be matched with a power series in k, and therefore $L_x = \Sigma_{n=0}^{\infty} a_n \partial_x^n$ fails, and this is true on (a,b) or $(-\infty,\infty)$. In the latter case, I can at least diagonalize our integral equation $g(x) = (1/2\pi) \int_{-\infty}^{\infty} dy\, 2\pi r(x-y) f(y)$ into G(k) = 2πR(k)F(k) and then R(k) = F(|k|) is not a problem and I can solve for f using F(k). This assumes of course that g(x) has a FI transform G(k).

In **Section 2** (2 pages) I redo Section 1 but I do a *double* Fourier Integral transform of s(x,y) into S(k,k') and assume L_x has constant coefficients. This leads to this strange requirement

$$[\Sigma_{n=0}^{\infty} a_n (-ik)^n] S(k,k') = (1/2\pi)\delta(k+k')$$
$$[\Sigma_{n=0}^{\infty} a_n \partial_x^n] s(x,y) = \delta(x-y)$$

where I put the starting equation below the result. I don't know how to make a series add up to a delta function. Probably if I assume the r(x-y) situation, S(k,k') will cancel that delta, but I have really conclusively treated this situation in Section 1, so Section 2 brings nothing useful to the party. I was just blindly "trying things" at the time.

In **Section 3** (3 pages) I assume that s(x,y) = r(x-y). I assume general function coefficients $a_n(x)$ for L_x and FI transform them into $A_n(k)$ as I did in Section 1. This leads to

$$\Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\, u_n(k'-k)\, A_n(k) = -(1/2\pi) \qquad \text{where} \qquad u_n(k) = \Sigma_{n=0}^{\infty} (-ik^n \} R(k)$$

which is similar to the first equation in Section 1, where here it has to somehow be true for all k' As in Section 1, I regard this as **intractable**.

In **Section 4** (2 pages) I try the example r(x-y) = |x-y| which appears in Stakgold. But I quickly realize that r(x) has no R(k) FI transform. The lesson here is this: when considering L_x r(x-y) = δ(x-y), if you want to transform both sides of this equation, you better use a convergent transform for r(x). Find some complete set of functions other than e^{ikx} on which to expand both sides!

In **Section 5** (3 pages) I follow the suggestion of Section 4. I assume some complete set $\varphi_n(x)$ and I define $L_{n'n} = \langle \varphi_{n'} | L_x | \varphi_n \rangle$ and end up with this infinite matrix equation: $\Sigma_{n'=0}^{\infty} L_{nn'} r_{n'} = \varphi_n(0)$, which is what has become of L_x r(x) = δ(x) [I did not realize at this time that solving this does not solve the equation L_x r(x-y) = δ(x-y), so this whole section is wrongly aimed.] r_n are of course the coefficients of $r(x) = \Sigma_{n=0}^{\infty} r_n \varphi_n(x)$. The ∞ matrix equation is $L\mathbf{r} = \boldsymbol{\varphi}(0)$ which *maybe* you could solve numerically. If I redo this sectino properly for L_x r(x-y) = δ(x-y), I end up with this

$\Sigma_{n=0}^{\infty}\varphi_n(x-y)\,\Sigma_{n'=0}^{\infty} r_{n'}\, L_{nn'} = \Sigma_{n=0}^{\infty}\varphi_n(x)\,\varphi_n(y)$

and I cannot then appeal to completeness of the $\varphi_n(x)$ to get a simple result, end of the line. The only way out of course is to use $\varphi_n(x) = \exp(ik_n x)$ on the interval (a,b) so $\varphi_n(x-y) = \varphi_n(x)\,\varphi_n(y)$ and then it seems you end up with infinite matrix equation $L\mathbf{r} = \mathbf{1}$ which you might solve for some working $L_{n'n}$. I then make the "radical step" of assuming that $\varphi_n(x)$ are eigenfunctions of our unknown L_x. This leads to being able to write $r(x-y) = \Sigma_{n=0}^{\infty}\,\varphi_n(x)\varphi_n(y)/\lambda_n$ which is the usual Green's Function form, but this is really useless since I don't know L_x, so I don't know λ_n or $\varphi_n(x)$! It's just a circle. At this point I sent off my first Richard Price email.

In **Section 6** (2 pages) I repeat Section 3 r(x-y), but with L_x having constant coefficients. I end up with

$$\Sigma_{n=0}^{\infty} b_n k^n = (1/2\pi)(1/R(k))$$

which is what I found at the end of Section 1, and again I consider $r(x) = (d/\pi) / (d^2 + x^2)$ and $R(k) = e^{-d|k|}/2d$ and how then no series exists. And I conclude again this situation arises for any function r(x) which is even in x.

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1. s(x,y) kernel which has single Fourier Int expansion and general $a_n(x)$ coefficients in L_x.

As Shankar suggested, I think the easiest way is to expand f and the delta function into a Fourier Transform in the appropriate number of dimensions.

(a) First, let's consider the action of a differential operator L_x on an integral equation in 1D. I assume I can move L_x through the integral, which needs to be validated in a rigorous analysis, but not here:

$$g(x) = \int dy\, s(x,y)f(y) \qquad (1.1)$$

$$L_x\, g(x) = \int dy\, L_x\, s(x,y)f(y)$$

(b) In order for this integral equation to be converted into a differential equation, I have to find L_x which has this property

$$L_x\, s(x,y) = \delta(x-y) \qquad (2.1)$$

and if I can find such an L_x then acting on our integral equation results in this differential equation:

$$L_x\, g(x) = f(x)$$

Thus, a differential equation corresponding to our integral equation only exists if L_x exists such that

$L_x\, s(x,y) = \delta(x-y)$. If $L_x\, s(x,y)$ produces just a function h(x,y), then the integral remains and I have then an integro-differential equation which is not what I want.

Notice that y is a dummy integration variable in (1.1). I certainly cannot have L_x being a function of y! That is why I write it as L_x. Apart from that, I are not narrowing our world of possible L_x operators at this time, everything is possible.

(c) So how can I determine whether such an L_x exists for a given kernel s(x,y) ? A first idea is to treat y as a bystander parameter and try to find an expansion of this form:

$$s(x,y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, S(k, y) \qquad \Rightarrow \qquad S(k, y) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, s(x, y)$$

I am using capital letters for momentum-space objects. I can at the same time expand

$$2\pi\delta(x) = \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

Then our equation (2.1) becomes

$$L_x \int_{-\infty}^{\infty} dk\, e^{-ikx}\, S(k, y) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)}$$

Again we assume we can move L_x through the integral (and rewrite things on both sides a bit) to get

$$\int_{-\infty}^{\infty} dk\ [\, L_x\, e^{-ikx}\,]\, S(k, y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, [\, (1/2\pi) e^{iky}\,]$$

(d) Now we shall make an assumption about L_x, namely,

$$L_x = \Sigma_{n=0}^{\infty}\, a_n(x)\, \partial_x^{\,n}$$

Each ∂_x on e^{-ikx} brings down a factor if (-ik), so we then have

$$\int_{-\infty}^{\infty} dk\, S(k, y)\ [\Sigma_{n=0}^{\infty}\, a_n(x)\, (-ik)^n\,]\, e^{-ikx} = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, [\, (1/2\pi) e^{iky}\,]$$

Next, I suppose we have to Fourier expand each of our coefficient functions

$$a_n(x) = \int_{-\infty}^{\infty} dk'\, e^{-ik'x}\, A_n(k') \qquad \Rightarrow \qquad A_n(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, a_n(x)$$

Then we have

$$\int_{-\infty}^{\infty} dk\, S(k, y)\ [\Sigma_{n=0}^{\infty}\, a_n(x)\, (-ik)^n\,]\, e^{-ikx} = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, [\, (1/2\pi) e^{iky}\,]$$

$$\int_{-\infty}^{\infty} dk\, S(k, y) \left[\Sigma_{n=0}^{\infty} \{ \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k')\} (-ik)^n \right] e^{-ikx} = \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[(1/2\pi) e^{iky} \right]$$

$$\int_{-\infty}^{\infty} dk\, S(k, y) \int_{-\infty}^{\infty} dk'\, e^{-i(k+k')x} \left[\Sigma_{n=0}^{\infty} \{ A_n(k')\} (-ik)^n \right] = \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[(1/2\pi) e^{iky} \right]$$

Now replace k' by k" = k+k' on the LHS to get

$$\int_{-\infty}^{\infty} dk\, S(k, y) \int_{-\infty}^{\infty} dk"\, e^{-ik"x} \left[\Sigma_{n=0}^{\infty} A_n(k"-k) (-ik)^n \right] = \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[(1/2\pi) e^{iky} \right]$$

Now swap variable names k↔k" on the LHS

$$\int_{-\infty}^{\infty} dk"\, S(k", y) \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[\Sigma_{n=0}^{\infty} A_n(k-k") (-ik")^n \right] = \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[(1/2\pi) e^{iky} \right]$$

$$\int_{-\infty}^{\infty} dk\, e^{-ikx} \int_{-\infty}^{\infty} dk"\, S(k", y) \left[\Sigma_{n=0}^{\infty} A_n(k-k") (-ik")^n \right] = \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[(1/2\pi) e^{iky} \right]$$

Now, finally, all the x dependence is in the e^{-ikx} factors and we can claim these are a complete set and we then can equate the integrands. This gives

$$\int_{-\infty}^{\infty} dk"\, S(k", y) \left[\Sigma_{n=0}^{\infty} A_n(k-k") (-ik")^n \right] = (1/2\pi) e^{iky}$$

Now let k - k" = k' on the LHS

$$- \int_{-\infty}^{\infty} dk'\, S(k-k', y) \left[\Sigma_{n=0}^{\infty} A_n(k') (-i)^n (k-k')^n \right] = (1/2\pi) e^{iky}$$

and now swap k ← k'

$$\int_{-\infty}^{\infty} dk\, S(k'-k, y) \left[\Sigma_{n=0}^{\infty} A_n(k) (-i)^n (k'-k)^n \right] = - (1/2\pi) e^{ik'y}$$

$$\Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\, A_n(k) \{ (-i)^n (k'-k)^n S(k'-k, y) \} = - (1/2\pi) e^{ik'y}$$

$$\Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} q_n(k'-k; y) A_n(k)\, dk = - (1/2\pi) e^{ik'y}$$

where $q_n(k'-k; y) = (-i)^n (k'-k)^n S(k'-k, y)$ $\qquad$ $q_n(k; y) = (-i)^n k^n S(k, y)$

What we have here is a single equation with an infinite series on the LHS. Each term in this series is an integral of an L_x Fourier coefficient against a known kernel. But this equation must be true for each value

of y, so it is then a continuous infinite system of equations with an infinite discrete number of unknown $A_n(k)$ functions,

$$\Sigma_{n=0}^{\infty} Q_n(y)\, A_n = -(1/2\pi)e^{ik'y} \qquad y = \text{all values in } (-\infty,\infty)$$

The kernels at least are of the difference form. We could go on here, but it seems fairly intractable. There could in theory be solutions $A_n(k)$, I guess we cannot rule them out. Or solutions might not exist.

(e) Suppose we assumed <u>constant coefficients</u> instead? Back up a bit:

$$\int_{-\infty}^{\infty} dk\ [\, L_x\, e^{-ikx}\,]\, S(k, y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, [\, (1/2\pi)e^{iky}\,]$$

$$L_x = \Sigma_{n=0}^{\infty} a_n\, \partial_x^n$$

$$\int_{-\infty}^{\infty} dk\ [\Sigma_{n=0}^{\infty} a_n(-ik)^n\, e^{-ikx}\,]\, S(k, y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, [\, (1/2\pi)e^{iky}\,]$$

$$[\Sigma_{n=0}^{\infty} a_n(-ik)^n\,]\, S(k, y) = (1/2\pi)e^{iky} \qquad (*)$$

which compare to starting equation

$$[\, \Sigma_{n=0}^{\infty} a_n\, \partial_x^n]\, s(x,y) = \delta(x-y)$$

We could write (*) this way with $b_n = a_n(-i)^n$

$$[\Sigma_{n=0}^{\infty} b_n\, k^n\,] = (1/2\pi)[\, e^{iky} / S(k, y)]$$

The equation has to be true for all y. So the only chance you would have for a solution here was if it happened that

$$e^{iky} / S(k, y)] = 1/R(k) = \text{ independent of y} \qquad \Rightarrow \qquad S(k, y) = R(k)\, e^{iky}$$

Therefore, for some general s(x,y) you come up with (which has a single Fourier Transform on the first variable), it is pretty unlikely you will get a constant coefficient L_x that works. In fact, you would have to have $S(k, y) = e^{iky}\, R(k)$ as the form. What would that imply for the kernel?

$$s(x,y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, S(k, y) = \int_{-\infty}^{\infty} dk\, e^{-ikx}\, e^{iky}\, R(k)$$

$$= \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)}\, R(k) = g(x-y)$$

Interesting. If the kernel is of "convolution form" then we stand a chance. So let's assume that

$s(x,y) = r(x-y)$

Then from this kernel we compute

$$R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, r(x)$$

Then we need to find the b_n coefficients such that

$$[\Sigma_{n=0}^{\infty} b_n\, k^n] = (1/2\pi)[\, e^{iky} / S(k, y)] = (1/2\pi)\,(\, 1/R(k)\,)$$

Assuming we can find the b_n, our solution is then

$$L_x = [\, \Sigma_{n=0}^{\infty} a_n\, \partial_x^n] \qquad \text{where } a_n = b_n (i)^n$$

Wow, we have a solution! Let's state this as a Theorem:

Theorem 1. Our integral equation of interest is as follows, and we seek L_x to do as shown

$$g(x) = \int_{-\infty}^{\infty} dy\, s(x,y) f(y).$$

$$L_x\, g(x) = \int_{-\infty}^{\infty} dy\, L_x\, s(x,y) f(y) = \int_{-\infty}^{\infty} dy\, \delta(x-y) f(y) = f(x) \qquad L_x\, s(x,y) = \delta(x-y)$$

We consider only the special case where $s(x,y) = r(x-y)$. We assume that $r(x)$ has a Fourier Transform

$$R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx}\, r(x)$$

We assume we can find coefficients b_n such that (ie, we can expand $1/R(k)$ in a power series in k)

$$\Sigma_{n=0}^{\infty} b_n\, k^n = 1/[2\pi R(k)]$$

Then the solution L_x is this: (that is, we have constant coefficients)

$$L_x = \Sigma_{n=0}^{\infty} a_n\, \partial_x^n \qquad \text{where } a_n = (i)^n b_n$$

Is there an easier way to understand what is happening here? We have

$$g(x) = (1/2\pi) \int_{-\infty}^{\infty} dy\, 2\pi h(x-y) f(y).$$

$$L_x\, g(x) = \int_{-\infty}^{\infty} dy\, L_x\, r(x-y) f(y) = \int_{-\infty}^{\infty} dy\, \delta(x-y) f(y) = f(x) \qquad L_x\, h(x-y) = \delta(x-y)$$

The first equation is of convolution form with the correct group range. So it will diagonalize into

$G(k) = 2\pi R(k)F(k)$ // see "Diag. of Convolution...doc"

Then we have

$F(k) = G(k) / [2\pi R(k)]$

So we know how to solve the first equation without even using the $L_x\, h(x-y) = \delta(x-y)$ idea. But OK, let's now start with just $L_x\, h(x-y) = \delta(x-y)$. We Fourier Expand the h and the δ

$$L_x \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)}\, R(k) = \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)} (1/2\pi)$$

Assume $L_x = \Sigma_{n=0}^{\infty} a_n \partial_x^n$ and move it in to get

$$\int_{-\infty}^{\infty} dk\, [\, \Sigma_{n=0}^{\infty} a_n (-ik)^n\,]\, e^{-ik(x-y)}\, R(k) = \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)} (1/2\pi)$$

Now claim completenss to say

$$[\, \Sigma_{n=0}^{\infty} a_n (-ik)^n\,]\, R(k) = (1/2\pi)$$

Therefore we have

$$\Sigma_{n=0}^{\infty} a_n (-ik)^n = 1/ [2\pi R(k)]$$

which replicates the claim of Theorem 1.

Now how is this different with finite endpoints?

$g(x) = (1/2\pi) \int_a^b dy\, 2\pi r(x-y) f(y)$.

$L_x\, g(x) = \int_a^b dy\, L_x\, r(x-y) f(y) = \int_a^b dy\, \delta(x-y) f(y) = f(x)\theta(x>a)\theta(x<b)$ $\quad L_x\, r(x-y) = \delta(x-y)$

The result now has theta functions. But if we can still say this:

$L_x\, g(x) = f(x)$ valid for $a \le x \le b$

so we still seem to have a solution if we can find L_x such that $L_x\, r(x-y) = \delta(x-y)$. In the original integral equation, r(x-y) has an argument that runs from a-b up to b-a which is a range of 2(b-a) = 2L. In other words, the function h(x) need only be defined on the range (-L,L) centered on the origin. In order to do our Fourier Integral expansion idea, we would need to somehow "extend" the function h(x) out beyond this range to both endpoints. For the moment, perhaps $r(x) = 1/x^2$ and this naturally extends itself to the two limits. In fact, let's do exactly this case. Then

$$R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} [\, 1/x^2] = (1/2\pi) \int_{-\infty}^{\infty} dx \cos(kx) [\, 1/x^2] = (1/\pi) \int_{0}^{\infty} dx \cos(kx)/x^2$$

But this integral diverges at the origin. So instead let's do our Love case of $r(x) = 1/(d^2 + x^2)$ and then we know

$$R(k) = (1/\pi) \int_{0}^{\infty} dx \cos(kx)/ (d^2 + x^2) = e^{-d|k|}/2d$$

Then we are stuck with this series problem:

$$\Sigma_{n=0}^{\infty} a_n (-ik)^n = 1/ [2\pi R(k)] = (1/2\pi)\, 2d\ e^{+d|k|} = (1/\pi d)\, e^{+d|k|}$$

and we have the famous problem that our function $f(k) = e^{+d|k|}$ is not analytic at k = 0. In fact we know that there is a slope discontinuity there since f '(0+) = d and f '(0-) = -d . Thus, you cannot have a power series around k = 0. Thus, there is no solution L_x of this form.

In fact, for any even r(x), we know that if it exists we will get R(k) = F(|k|). Since |k| is not analytic, even if F is continuous, F(|k|) won't be analytic and there will be no possible series. Most kernels I am familiar with are in fact even.

Theorem 2: Consider this situation

$$g(x) = (1/2\pi) \int_{a}^{b} dy\, 2\pi r(x-y) f(y).$$

$$L_x\, g(x) = \int_{a}^{b} dy\, L_x\, r(x-y) f(y) = \int_{a}^{b} dy\, \delta(x-y) f(y) = f(x)\theta(x>a)\theta(x<b) \qquad L_x\, r(x-y) = \delta(x-y)$$

and assume that r(x) is even and can be naturally extended to $(-\infty,\infty)$. And assume that the Fourier Transform of r(x) exists, which we call R(k). Then our theorem states that R(k) has the form F(|k|), and therefore Theorem 1 fails because we cannot find coefficients b_n , and therefore we cannot find L_x of the form $L_x = \Sigma_{n=0}^{\infty} a_n \partial_x^n$ with constant coefficients. This theorem applies for any fixed endpoints a,b includiong $-\infty$ and $+\infty$. Let's restate this result more compactly:

Theorem 2A. Consider the problem $L_x\, r(x-y) = \delta(x-y)$ on an interval (a,b) where we seek L_x given r. Then if r(x) is even, and if when extended to $(-\infty,\infty)$ it has a Fourier transform R(k), then there is no solution L_x which has constant coefficients.

2. s(x,y) kernel which has double Fourier Int expansion and constant a_n coefficients in L_x.

(a) Let's back up and take a different path just to see where it leads. Go all the way back to here:

$$L_x\, s(x,y) = \delta(x-y) \qquad (2.1)$$

Now let's assume we can do a double Fourier of s(x,y) and see what happens:

$$s(x,y) = \int_{-\infty}^{\infty} dk\, e^{-ikx} \int_{-\infty}^{\infty} dk'\, e^{-ik'y}\, S(k,k')$$

$$S(k,k') = (1/2\pi)^2 \int_{-\infty}^{\infty} dx\, e^{+ikx} \int_{-\infty}^{\infty} dy\, e^{+ik'y}\, s(x,y)$$

Then we have

$$L_x\, s(x,y) = \delta(x-y)$$

$$L_x \int_{-\infty}^{\infty} dk\, e^{-ikx} \int_{-\infty}^{\infty} dk'\, e^{-ik'y}\, S(k,k') = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)}$$

Move L_x inside,

$$\int_{-\infty}^{\infty} dk\, [L_x\, e^{-ikx}] \int_{-\infty}^{\infty} dk'\, e^{-ik'y}\, S(k,k') = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ik(x-y)}$$

Negate k on the RHS only

$$\int_{-\infty}^{\infty} dk\, [L_x\, e^{-ikx}] \int_{-\infty}^{\infty} dk'\, e^{-ik'y}\, S(k,k') = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{+ik(x-y)}$$

Swap dummy variable names $k \leftrightarrow k'$ on LHS only

$$\int_{-\infty}^{\infty} dk'\, [L_x\, e^{-ik'x}] \int_{-\infty}^{\infty} dk\, e^{-iky}\, S(k',k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{+ikx}\, e^{-iky}$$

Rewrite

$$\int_{-\infty}^{\infty} dk\, e^{-iky} \int_{-\infty}^{\infty} dk'\, [L_x\, e^{-ik'x}]\, S(k',k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-iky}\, e^{+ikx}$$

Now use completeness of e^{-iky} to claim that

$$\int_{-\infty}^{\infty} dk'\, [L_x\, e^{-ik'x}]\, S(k',k) = (1/2\pi)\, e^{+ikx}$$

Now swap $k \leftrightarrow k'$ again

$$\int_{-\infty}^{\infty} dk\, [L_x\, e^{-ikx}]\, S(k,k') = (1/2\pi)\, e^{+ik'x}$$

Now assume for the moment the simple case that L_x has constant coefficients

$$L_x = \Sigma_{n=0}^{\infty} a_n \, \partial_x^n$$

Then we have

$$\int_{-\infty}^{\infty} dk \, [\Sigma_{n=0}^{\infty} a_n \, (-ik)^n] \, e^{-ikx} \, S(k,k') = (1/2\pi) \, e^{+ik'x}$$

This seems to require that

$$[\Sigma_{n=0}^{\infty} a_n \, (-ik)^n] \, S(k,k') = (1/2\pi)\delta(k+k') \qquad (*)$$

which compare to starting point

$$[\Sigma_{n=0}^{\infty} a_n \, \partial_x^n \,] \; s(x,y) \quad = \delta(x-y)$$

I have no idea how to deal with an equation like (*), so I now let this path come to rest. I think the single Fourier analysis of the last section is better, the double is unnecessary for this application.

3. r(x-y) kernel which has Fourier Int expansion and general $a_n(x)$ coefficients in L_x.

Now, finally, let's make the **Big Assumption** that our original kernel has the form

$$s(x,y) = r(x-y)$$

so our integral equation is then

$$g(x) = \int dy \; s(x,y) f(y) = \int dy \; r(x-y) f(y)$$

and we then go back to

$$L_x \, s(x,y) \; = \delta(x-y) \qquad (2.1)$$

$$L_x \, r(x-y) \; = \delta(x-y) \qquad (2.1)$$

Notice that this is the case for the Love integral equation, and for my 2D 1/r problem.

Since this must be true for all y, it is true for y = 0 and this must be true

$$L_x \, r(x) \; = \delta(x)$$

Now we construct Fourier Integral expansions for r(x) and δ(x)

$$r(x) = \int_{-\infty}^{\infty} dk \, e^{-ikx} \, R(k) \qquad \Rightarrow \qquad R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx \, e^{+ikx} \, r(x)$$

Then we have

$$L_x \int_{-\infty}^{\infty} dk\, e^{-ikx} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

$$\int_{-\infty}^{\infty} dk\, L_x\, e^{-ikx} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} \qquad (6.1)$$

Now let's first try the most general case that

$$a_n(x) = \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k') \qquad \Rightarrow \qquad A_n(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} a_n(x)$$

$$L_x = \Sigma_{n=0}^{\infty} a_n(x)\, \partial_x^n = \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k')\, \partial_x^n$$

$$\Rightarrow L_x\, e^{-ikx} = \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k')\, \partial_x^n\, e^{-ikx}$$

$$L_x\, e^{-ikx} = \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k') (-ik)^n\, e^{-ikx}$$

Then (6.1) becomes

$$\int_{-\infty}^{\infty} dk \{ \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk'\, e^{-ik'x} A_n(k') (-ik)^n\, e^{-ikx} \} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

$$\int_{-\infty}^{\infty} dk \{ \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk'\, e^{-i(k+k')x} A_n(k') (-ik)^n \} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

Now we shall follow the same steps followed earlier in the more general case. Let k" = k+k' on LHS

$$\int_{-\infty}^{\infty} dk \{ \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk''\, e^{-ik''x} A_n(k''-k) (-ik)^n \} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

And now swap k and k" on the LHS only

$$\int_{-\infty}^{\infty} dk'' \{ \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\, e^{-ikx} A_n(k-k'') (-ik'')^n \} R(k'') = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

$$\int_{-\infty}^{\infty} dk\, e^{-ikx} \int_{-\infty}^{\infty} dk'' \{ \Sigma_{n=0}^{\infty} A_n(k-k'') (-ik'')^n \} R(k'') = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

Now use completeness of the e^{-ikx} to claim

$$\int_{-\infty}^{\infty} dk'' \{ \Sigma_{n=0}^{\infty} A_n(k-k'') (-ik'')^n \} R(k'') = (1/2\pi)$$

Now replace k'' so that k-k'' = k' and keep k where it was

$$\int_{-\infty}^{\infty} dk' \{ \Sigma_{n=0}^{\infty} A_n(k') (-i[k-k']^n \} R(k-k') = (1/2\pi) \qquad \text{for all k}$$

Now swap k and k'

$$\int_{-\infty}^{\infty} dk \{ \Sigma_{n=0}^{\infty} A_n(k) (-i[k'-k]^n \} R(k'-k) = (1/2\pi) \qquad \text{for all k'}$$

Now we identify a kernel here

$$u_n(k'-k) = \Sigma_{n=0}^{\infty} (-i[k'-k]^n \} R(k'-k) \qquad u_n(k) = \Sigma_{n=0}^{\infty} (-ik^n \} R(k)$$

and we then have

$$\Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\, u_n(k'-k)\, A_n(k) = -(1/2\pi)$$

And so we arrive at a conclusion very similar to what we got before. There are an infinite discrete number of unknown functions $A_n(k)$, and the equation must be true for an infinite continuum of values of k', so it is an infinite system of equations with an infinite number of unknowns. It happens that the RHS is independent of k' in this case. Again, this is pretty hopeless. There may or may not be solutions.

4. Known r(x-y) example does not have a Fourier Transform!

6A. I happen to know that this is true (Stak II p51 top)

$$\partial_x^2[\ (1/2)|x-y|\ ] = \delta(x-y) \qquad s(x,y) = |x-y|/2 \qquad r(x) = |x|/2$$

Since this is a case that we know has a solution, we want to just see "how it works out". We had

$$r(x) = \int_{-\infty}^{\infty} dk\, e^{-ikx} R(k) \qquad \Rightarrow \qquad R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} r(x)$$

So we then have

$$R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} |x|/2 = (1/4\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} |x|$$

Since |x| is even in x, only the cosine term survives and we have

$= (1/4\pi) \int_{-\infty}^{\infty} dx \cos(kx)\,|x| = (1/2\pi) \int_{0}^{\infty} dx \cos(kx)\,|x| = (1/2\pi) \int_{0}^{\infty} dx \cos(kx)\, x$

But we are suddenly dead in the water. The Fourier Transform only exists in general for functions which are square or abs integrable,

Existence of the Fourier Transform

Conditions for the *existence* of the Fourier transform are complicated to state in general [12], but it is *sufficient* for $x(t)$ to be *absolutely integrable*, *i.e.*,

$$\|x\|_1 \triangleq \int_{-\infty}^{\infty} |x(t)|\,dt < \infty.$$

This requirement can be stated as $x \in L1$, meaning that x belongs to the set of all signals having a finite $L1$ norm ($\|x\|_1 < \infty$). It is similarly sufficient for $x(t)$ to be *square integrable*, *i.e.*,

$$\|x\|_2^2 \triangleq \int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty,$$

or, $x \in L2$. More generally, it suffices to show $x \in Lp$ for $1 \le p \le 2$ [12, p. 47].

We can see that $\int_{0}^{\infty} dx \cos(kx)\, x$ diverges and is meaningless. So our whole method here breaks down completely. One could add that only integrable $a_n(x)$ are allowed in what we did above.

Can this example be worked using some other transform? Let's go back to :

$\partial_x^2 [\ (1/2)|x|\ ] = \delta(x)$

Well, I know we can prove this doing a finite integral

$\int_{-\varepsilon}^{\varepsilon} \partial_x^2 [\ (1/2)|x|\ ] = 1$

LHS $= \partial_x [\ (1/2)|x|\ ]\,|_{-\varepsilon}^{\varepsilon} = (1/2) - (-1/2) = 1$

which is just the usual jump business in 1D. I suppose we could try expanding $|x|$ and $\delta(x)$ on some orthogonal set on some interval like -1 to 1, perhaps the $P_n(x)$.

The lesson here is this: when you consider a priori finding L_x such that

$L_x\, r(x) = \delta(x)$

and if you are going to try to solve this by doing some transform expansion, you better use transform such that r(x) has an expansion in that transform! In all my stuff above, I was always using the Fourier Transform.

5. Expand r(x) in a complete set of basis functions

Here we generalize from Fourier Transform with basis functions e^{-ikx} to an arbitrary $\varphi_n(x)$ basis.

Let's assume we have some orthonormal basis functions $\varphi_n(x)$ on some interval surrounding x = 0. And assume that r(x) is expandable in this basis. Then we have

$$r(x) = \Sigma_{n=0}^{\infty} r_n \varphi_n(x)$$

$$\delta(x-x') = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(x')$$

So then we have for $L_x\, r(x) = \delta(x)$,

$$L_x[\Sigma_{n=0}^{\infty} r_n \varphi_n(x)] = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

$$[\Sigma_{n=0}^{\infty} r_n\, L_x\, \varphi_n(x)] = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

Now since $\varphi_n(x)$ is a complete set, we can write

$$L_x\, \varphi_n(x) = \Sigma_{n'=0}^{\infty} L_{n'n}\, \varphi_{n'}(x) \qquad\qquad L_{n'n} = \langle \varphi_{n'} |\, L_x\, | \varphi_n \rangle$$

Then we have

$$\Sigma_{n=0}^{\infty} r_n \{ \Sigma_{n'=0}^{\infty} L_{n'n}\, \varphi_{n'}(x)\} = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

$$\Sigma_{n=0}^{\infty} \Sigma_{n'=0}^{\infty} \varphi_{n'}(x)\, r_n\, L_{n'n} = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

Now swap $n \leftrightarrow n'$ on the LHS

$$\Sigma_{n=0}^{\infty} \Sigma_{n'=0}^{\infty} \varphi_n(x)\, r_{n'}\, L_{nn'} = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

$$\Sigma_{n=0}^{\infty} \varphi_n(x) \Sigma_{n'=0}^{\infty} r_{n'}\, L_{nn'} = \Sigma_{n=0}^{\infty} \varphi_n(x)\, \varphi_n(0)$$

and then appeal to completeness to claim

$$\Sigma_{n'=0}^{\infty} L_{nn'}\, r_{n'} = \varphi_n(0)$$

$$L\, R = \boldsymbol{\varphi}(0)$$

where we have an infinite matrix equation.

Now let's take a radical next step. Let's assume our $\varphi_n(x)$ are not just ANY orthonormal functions, but they are that particular set of orthonormal functions that diagonalize L_x. We know this set is the set of eigenfunctions of L_x (assuming it has eigenfunctions as an operator). So suppose

$$L_x\ \varphi_n(x) = \lambda_n\ \varphi_n(x)$$

$$L_{n'n} = \langle\varphi_{n'}|\ L_x\ |\ \varphi_n\rangle\ = \lambda_n \langle\varphi_{n'}|\ \varphi_n\rangle\ = \lambda_n\ \delta_{n,n'}$$

Then we go back to our last equation above

$$\Sigma_{n'=0}^{\infty}\ L_{nn'}\ r_{n'}\ = \varphi_n(0)$$

$$\Sigma_{n'=0}^{\infty}\ \lambda_n\ \delta_{n,n'}\ r_{n'}\ = \varphi_n(0)$$

$$\lambda_n\ r_n = \varphi_n(0)$$

We know r_n , but we don't know anything about L_x so we don't know the φ_n or the λ_n.

This seems to me to be a pretty stringent requirement. For example, suppose we were to just try as candidate L_x = the Legendre operator, and we know that $L_x\ \varphi_n(x) = n(n+1)\varphi_n(x) = \lambda_n\ \varphi_n(x)$. We then have our candidate set of eigenfunctions and eigenvalues. The equation $L_x\ r(x) = \delta(x)$ only has a solution if it *happens* that $r_n = \varphi_n(0)/\lambda_n$. This says r(x) has to be the Green's function of L_x. For Legendre, we know that

$$\Sigma_{n=0}^{\infty}(1/K_n^0)\ P_n(x)\ P_n(0)\ =\ \delta(x) \qquad \text{that is} \qquad \Sigma_{n=0}^{\infty}\ \varphi_n(x)\varphi_n(0) = \delta(x)$$

Then we consider $L_x\ r(x) = \delta(x)$

$$L_x\ r(x)\ =\ \Sigma_{n=0}^{\infty}\ \varphi_n(x)\varphi_n(0)$$

$$L_x\ \Sigma_{n=0}^{\infty}\ r_n\varphi_n(x) =\ \Sigma_{n=0}^{\infty}\ \varphi_n(x)\varphi_n(0)$$

$$\Sigma_{n=0}^{\infty}\ r_n\lambda_n\ \varphi_n(x) =\ \Sigma_{n=0}^{\infty}\ \varphi_n(x)\varphi_n(0)$$

$$r_n = \varphi_n(0)/\lambda_n$$

Then we have

$$r(x) = \Sigma_{n=0}^{\infty}\ r_n\varphi_n(x)\ =\ \Sigma_{n=0}^{\infty}\ \varphi_n(0)/\lambda_n\ \varphi_n(x)$$

and this is just the famous result

$$r(x;0) = \Sigma_{n=0}^{\infty}\ \varphi_n(x)\varphi_n(0)/\lambda_n\ =\ g(x;\ 0)$$

I guess I am just unable to answer this question:

Question: Given a function r(x), is it, or is it not, a Green's Function of some self-adjoint differential operator?

I have only scratched the surface of this question. I know how to show that r(x) is or is not the Green's Function of a differential operator with constant coefficients. There are an infinite number of self-adjoint differential operators, so there are an infinite number of Green's functions. But, I think if you pick a typical function like $r(x) = 1/x^3$, you will find that it is not a Green's function of any self-adjoint differential operator. I just don't know how to prove whether it is or is not. I just send off an email to Richard Price, maybe he will have something to add.

6. r(x-y) kernel which has Fourier Int expansion and constant a_n coefficients in L_x.

Now let's maintain our Big Assumption of section 6 that s(x,y) = r(x-y), and this time let's try to search for an L_x with constant coefficients. We assume that r(x) has a Fourier Transform. We then have

$$g(x) = \int dy\, r(x-y) f(y) \qquad (1.1) \qquad \text{// this is just the associated integral equation}$$

$$L_x\, r(x) = \delta(x)$$

$$r(x) = \int_{-\infty}^{\infty} dk\, e^{-ikx} R(k) \qquad \Rightarrow \qquad R(k) = (1/2\pi) \int_{-\infty}^{\infty} dx\, e^{+ikx} r(x)$$

$$\int_{-\infty}^{\infty} dk\, L_x\, e^{-ikx} R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx} \qquad (6.1)$$

$$L_x = \Sigma_{n=0}^{\infty} a_n\, \partial_x^n$$

$$L_x\, e^{-ikx} = \Sigma_{n=0}^{\infty} a_n\, (-ik)^n\, e^{-ikx}$$

Then we have from 6.1,

$$\int_{-\infty}^{\infty} dk\, \{ \Sigma_{n=0}^{\infty} a_n\, (-ik)^n\, e^{-ikx} \}\, R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

$$\int_{-\infty}^{\infty} dk\, e^{-ikx} \{ \Sigma_{n=0}^{\infty} a_n\, (-ik)^n \}\, R(k) = (1/2\pi) \int_{-\infty}^{\infty} dk\, e^{-ikx}$$

and completeness right away tells us that

$$\{ \Sigma_{n=0}^{\infty} a_n\, (-ik)^n \}\, R(k) = (1/2\pi)$$

Let's define

$b_n = (-i)^n a_n$

then we have (this must be valid for all k in the range $(-\infty,\infty)$!)

$$\Sigma_{n=0}^{\infty} b_n k^n = (1/2\pi)(1/R(k))$$

If we can express 1/R(k) as a power series in k, then we have our solution!

In the Love example, treated in a separate doc, we find

$$r(x) = (d/\pi) / (d^2 + x^2)$$

$$R(k) = e^{-d|k|} /2d$$

and the absolute value makes things not work and there is no solution L_x with constant coefficients.

I think if r(x) is any even function, as in the Love example, which has a Fourier Transform, you are going to find that R(k) is even in k, and is therefore R(|k|) and you will be stuck with

$$\Sigma_{n=0}^{\infty} b_n k^n = (1/2\pi)(1/R(|k|)) = f(|k|)$$

and then there will be no solution for L_x which has constant coefficients.