Jim's First Proposed L PhL 1.8.11

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Overview (1 page, written 1.17.11)

To his credit, Jim was the first to propose any solution to my "Question #2" equation $L_{\mathbf{x}}[1/r] = 2\pi\delta^{(2)}(\mathbf{r})$. I had no solutions in mind at all when I asked him my Question #2. I was even confused about how to write $\delta^{(n)}(\mathbf{r})$ in terms of $\delta(r)$ and he showed the correct way as noted below (and later confirmed in Stak). Jim was also the first to propose the right method of regulating [1/r] as $\lim_{\varepsilon\to 0} 1/\sqrt{r^2+\varepsilon^2}$.

Note that Section 1 was written AFTER the other sections of this doc!!

In **Section 1**, I give an n=2 Stakgold Theorem proof of this equation, which form was Jim's "first proposal"

$$-\lim_{\varepsilon\to 0} \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}]) = \delta(r)/r = 2\pi\,\delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y) \qquad (1)$$

I think this was my first use of Stak's theorem, my first proof of many to follow later on.

In **Section 2** I wonder just how one would prove the 3D equation $\nabla^2[1/r] = -4\pi\,\delta^{(3)}(\mathbf{r})$ without using the divergence theorem. I wanted to know, because I don't have the divergence theorem available to prove equation (1) above. [The answer I later found is: use the Stakgold theorem, OR, start with the 3D known fact and multiply by r.]

In **Section 3** I peruse Stak for the first time on this subject, and note that he talks about sequences of functions that become delta functions in a limit. I decide to search for a sequence that in some limit reproduces the result shown just above but for the 3D case, where RHS = $-\delta(r)/r^2$.

My first attempt is $s_k(r) = \partial_r(r^2\partial_r(1/[r+k]))$ as $k\to 0$. This is the 3D radial part of ∇^2 with the leading $1/r^2$ removed. Maple tells me that this function $s_k(r)$ happens to have 0 area on $(0,\infty)$ (it has a sign change), so that was discouraging no fly attempt. [I later found out that I misquoted the Maple results here, and still later found out that this method of regulation in fact fails.]

I then note "Stak's Theorem" and ponder it a bit.

My second attempt is to use $-\partial_r(r^2\partial_r(1/r^{1+\alpha}))$ where you regulate the exponent. Same operator, different regulation proposal. This fixes the "thing = 0 for r>0" issue, but in retrospect is not very good since $1/r^{1+\alpha}$ still diverges for small $\alpha > 0$. I cannot make this form "match" the Stak theorem which requires $\lim_{\alpha\to 0}[\,\alpha^{-3}\, g(r/\alpha)\,]$.

At this point I give up. As noted above, the correct regulation method is $1/\sqrt{r^2+\varepsilon^2}$ which Jim came up with. My first attempt was $1/(r+\varepsilon)$ which I later showed fails to give a delta function.

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1. Jim's First Proposal cuts the mustard.

The problem was to find L such that $L(1/r) = 4\pi\delta^{(2)}(\mathbf{r})$.

Jim proposes $Lf = r\,\partial_r^2(rf)$ which certainly meets the requirement that $L(1/r) = 0$ for $r \neq 0$.

Note Added 1.12.10: But as I note elsewhere, $r\,\partial_r^2(r[1/r]) = 4\pi\delta^{(2)}(\mathbf{r})$ has the wrong dimensions, so perhaps we could amend Jim's proposal to say $Lf = \partial_r^2(rf)$ so then $\partial_r^2(r[1/r]) = 4\pi\delta^{(2)}(\mathbf{r})$. Meanwhile, from Stak II page 22 (I just had to search for 8 minutes in Vol II to locate this thing!!!) we do see Jim's claim that in 3D one gets

$\delta^{(3)}(\mathbf{r}) = \delta(r)/[4\pi r^2]$ page 22 A

though this result is concealed in other obscure notation. Just integrate both sides on a small sphere to see that this is true. It just then also follow that

$\delta^{(2)}(\mathbf{r}) = \delta(r)/[2\pi r]$

and here we would integrate this over a small disk. The powers of r cancel against the volume element, and the angle integral is done. Using this last result, Jim's proposal becomes

$k\partial_r^2(r[1/r]) = 4\pi\delta^{(2)}(\mathbf{r}) = 4\pi\,\delta(r)/[2\pi r] = 2\,\delta(r)/r$

Again, where k is a constant TBD,

$\partial_r^2(r[1/r]) = (2/k)\,\delta(r)/r$

So now the problem is entirely in one dimension, r that runs $(0,\infty)$. Jim proposes regulating the LHS this way

$\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon}\,]) = (2/k)\,\delta(r)/r$

According to Maple, the LHS is this

```
f := Diff(Diff(r/sqrt(r^2+epsilon),r),r);
```

$$f := \frac{\partial^2}{\partial r^2}\frac{r}{\sqrt{r^2+\varepsilon}}$$

```
f := diff(diff(r/sqrt(r^2+epsilon),r),r);
```

$$f := -3\frac{r}{(r^2+\varepsilon)^{\left(\frac{3}{2}\right)}} + 3\frac{r^3}{(r^2+\varepsilon)^{\left(\frac{5}{2}\right)}}$$

```
simplify(%);
```

$$-3\frac{r\varepsilon}{(r^2+\varepsilon)^{\left(\frac{5}{2}\right)}}$$

which is to say, we then have

LHS = $\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon}\,]) = -3r\varepsilon / (r^2+\varepsilon)^{5/2}$

so then the question is: can we show this:

$\lim_{\varepsilon\to 0}[-3r\varepsilon / (r^2+\varepsilon)^{5/2}] = (2/k)\,\delta(r)/r$

Let's rewrite the RHS as it used to be, so since $\delta^{(2)}(\mathbf{r}) = \delta(r)/[2\pi r]$

$(2/k)\,\delta(r)/r = (2/k)\,2\pi\,\delta^{(2)}(\mathbf{r}) = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$

Then we want to show that

$\lim_{\varepsilon\to 0}[-3r\varepsilon / (r^2+\varepsilon)^{5/2}] = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$?

Let's right now replace $\varepsilon \to \varepsilon^2$ so ε and r have the same dimension. Then we have

$\lim_{\varepsilon\to 0}[-3r\varepsilon^2 / (r^2+\varepsilon^2)^{5/2}] = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$?

But

$(r^2+\varepsilon^2)^{5/2} = (\,\varepsilon^2[(r/\varepsilon)^2+1]\,)^{5/2} = \varepsilon^5[(r/\varepsilon)^2+1]^{5/2}$ dim = L^5

$r\varepsilon^2 = (r/\varepsilon)\varepsilon^3$

Then we want to show this

$\lim_{\varepsilon\to 0}[-3(r/\varepsilon)\varepsilon^3 / \{\,\varepsilon^5[(r/\varepsilon)^2+1]^{5/2}] = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$?

$\lim_{\varepsilon\to 0}[-3(r/\varepsilon)\varepsilon^{-2} / \{\,[(r/\varepsilon)^2+1]^{5/2}] = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$?

$$\lim_{\varepsilon\to 0}[\varepsilon^{-2}\{-3(r/\varepsilon) / \{ [(r/\varepsilon)^2+1]^{5/2}\}] = (4\pi/k)\,\delta^{(2)}(\mathbf{r}) \qquad ?$$

Suppose we now define

$$h(r) = -3r / (r^2+1)^{5/2}$$

Then we want to ask it this is true:

$$\lim_{\varepsilon\to 0}[\varepsilon^{-2}\, h(r/\varepsilon)] = (4\pi/k)\,\delta^{(2)}(\mathbf{r})$$

This at least has a limit form as we see used in Stak Vol II Section 5.4 (page 13 bottom) where Stak says

$$\delta^{(n)}(r) = \lim_{\varepsilon\to 0}\ [\,\varepsilon^{-n} g(r/\varepsilon)]$$

where the only condition on g(r) is that

$$\int_0^\infty dr\, r^{n-1}\, g(r) = 1/S_n(1)$$

I think Jim has done it! If we go to n = 2 we get

$$\delta^{(2)}(\mathbf{r}) = \lim_{\varepsilon\to 0}\ [\,\varepsilon^{-2} g(r/\varepsilon)]$$

This says then that

$$(4\pi/k)\,\delta^{(2)}(\mathbf{r}) = \lim_{\varepsilon\to 0}\ [\,\varepsilon^{-2}(4\pi/k) g(r/\varepsilon)]$$

Comparing with the above, I have

$$(4\pi/k)g(r/\varepsilon) = h(r/\varepsilon) \qquad \Rightarrow \qquad (4\pi/k)g(r) = h(r)$$

$$\Rightarrow \qquad g(r) = (k/4\pi)\, h(r) = (k/4\pi)[\, -3r / (r^2+1)^{5/2}]$$

Now to get the normalization we need to get

$$\int_0^\infty dr\, r^1\, (k/4\pi)[\, -3r / (r^2+1)^{5/2}] = 1/2\pi$$

$$-(3k/4\pi) \int_0^\infty dr\, r^1\, [\, r / (r^2+1)^{5/2}] = 1/2\pi$$

$$-(3k/2) \int_0^\infty dr\, r^1\, [\, r / (r^2+1)^{5/2}] = 1$$

$$-(3k/2)\int_0^\infty dr\, r^2 / (r^2+1)^{5/2} = 1$$

Maple says the integral is 1/3

```
Int(r^2/(1+r^2)^(5/2),r=0..infinity);
```

$$\int_0^\infty \frac{r^2}{(1+r^2)^{\left(\frac{5}{2}\right)}}\,dr$$

```
int(r^2/(1+r^2)^(5/2),r=0..infinity);
```

$$\frac{1}{3}$$

So we end up with

$$-(3k/2)(1/3) = 1$$

$$-(k/2) = 1 \qquad \Rightarrow k = -2$$

Therefore I have shown that

$$\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = (2/k)\,\delta(r)/r = -\,\delta(r)/r$$

Again

$$-\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = \delta(r)/r = 2\pi\,\delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y)$$

So this is indeed a major step forward. So we then know that, in spherical coordinates,

$$L = \partial_r^2(r\,..) + \text{angular part}$$

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(these are older sections)

2. How do you prove the usual 3D case without using the divergence theorem?

[Note Added: In a later doc, I do prove the 3D case without using the divergence theorem! I use an n=3 application of Stakgold's Theorem. I sent this to Jim in an email, it was part of a large doc.]

So now one needs to show this:

$$\int_{\text{small disk}} dV \; L(1/r) = 4\pi$$

By analogy, we know in 3D that the corresponding fact is this

$$\int_{\text{small sphere}} dV \; [-\nabla^2(1/r)] = 4\pi$$

How would we prove this if we did not have the divergence theorem? I guess you install a test function:

$$\int_{\text{small sphere}} dV \; \varphi(r) \; [-\nabla^2(1/r)] = 4\pi\varphi(0) \qquad ?$$

$$\int_{\text{small sphere}} dV \; \varphi(r) \; r^{-2}\partial_r(r^2\partial_r\,(1/r)) = -\,4\pi\varphi(0) \qquad ?$$

$$\int_0^a r^2 dr \; \varphi(r) \; r^{-2}\partial_r(r^2\partial_r\,(1/r)) = -\,\varphi(0) \qquad ?$$

$$\int_0^a dr \; \varphi(r) \; \partial_r(r^2\partial_r\,(1/r)) = -\,\varphi(0) \qquad ?$$

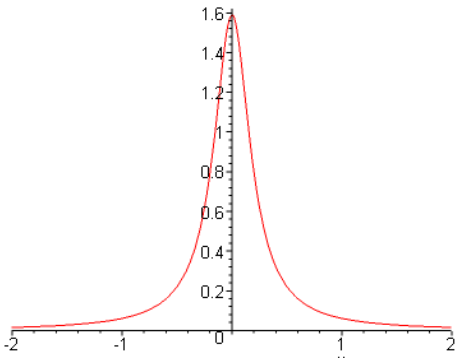
Now what do you do, Mr. Stak? Cannot compute the derivative because it gives 0. Parts is pretty unclear. So I guess I have to go back and review "delta functions" in Stak as part of this mid term exam.

3. Stakgold on Delta Functions

Vol I page 22. Here Stak considers the sequence

$$s_k(x) = k/[\pi(1+k^2x^2)]$$

The claim is that this becomes a delta function as $k \to \infty$. Plotting for $k = 5$ is promising:



So on page 22-23 he throws in a test function $\varphi(x)$ and he proves what he needs to prove, there is nothing too fancy in this proof. The area of $s_k(x) = 0$.

So can I come up with some kind of sequence for my 3D case above?

$$\int_0^a dr\ \varphi(r)\ \partial_r(r^2\partial_r\ (1/r)) \ = -\ \varphi(0) \qquad\qquad ?$$

Perhaps I replace r by r+k and then later take the limit $k \to 0$. When $k > 0$, I can compute stuff. Let

$$s_k(r) = \partial_r(r^2\partial_r\ (1/[r+k]))$$
$$= -2(r+k)^{-2} + 4r(r+k)^{-3} \qquad = 2(r-k)(r+k)^{-3} \qquad \text{according to Maple}$$

There are **two problems**.

First, I am not sure how to deal with the fact that we are on $(0,\infty)$ so I somehow don't have access to the left half of the delta function! [this is resolved elsewhere]

Second, Maple says that for any k, the area under my proposed sequence curve is 0. So I guess this is not a good "candidate" for the second reason. [wrong, it turns out in fact to be infinite!]

Note added 1.17.11: Somehow I entered things wrong, the Maple result above is wrong! This is one place where I failed to *quote* the Maple result, and naturally I got it wrong.

```
Diff(r^2*Diff(1/(r+k),r),r);
```

$$\frac{\partial}{\partial r} r^2 \left(\frac{\partial}{\partial r} \frac{1}{r+k} \right)$$

```
diff(r^2*diff(1/(r+k),r),r);
```

$$-2\frac{r}{(r+k)^2} + 2\frac{r^2}{(r+k)^3}$$

```
simplify(%);
```

$$-2\frac{rk}{(r+k)^3}$$

I show in Section 6 of "jim delta 1.13.11 5PM.doc" that this method of regulation fails for our usual 3D problem being considered here. That is to say, sequence $s_\varepsilon(x) = \partial_r(r^2\partial_r\ (1/(r+\varepsilon))$ does not approach a delta function $\delta(r)$ in the limit $\varepsilon \to 0$. It approaches something bigger than a delta function!

Regrettably, after studying Stak for a year, I have no idea what to do here. Where do I start reviewing? This is part of the game. You once read about a subject, and now you have a problem in that subject, so you have to review it because the attic is not large enough to retain it.

I think Vol II Section 5.4 (page 10) might have what I need. I just reviewed Example 3 page 12 which is rather amazing and this is a deal in general R^n . Consider f(x) which has a unit volume integral in R^n. Then consider $f_\alpha(x) = \alpha^{-n} f(x/\alpha)$ where you scale each coordinate as in x_i/α. The amazing claim is that in the limit that $\alpha \to 0$, this thing = $\delta(x)$ in n dimensions! The proof is pretty straightforward.

Example 4 is even closer to our problem at hand! Suppose f(x) has the simple form g(r). Then of course we have $f_\alpha(x) = \alpha^{-n}\, g(r/\alpha)$. The integral requirement is that $S_n(1)\int dr\, r^{n-1}\, g(r) = 1$ where the S thing is the angular integral in n dimensions. A candidate for n=3 is this

$$g(r) = \pi^{-3/2} \exp(-r^2) \qquad \text{since } S_n(1)\int dr\, r^2\, g(r) = 1$$

Then we find that

$$\lim_{\alpha\to 0}\ [\ \alpha^{-3}\, g(r/\alpha)\ ] = \delta^3(\mathbf{r})$$

So this seems to me a pretty powerful way to create delta functions.

Is there some way I could use this idea to show that

$$[-\nabla^2(1/r)] = 4\pi\delta(\mathbf{r}) \qquad n = 3$$

$$-\partial_r(r^2\partial_r\,(1/r)) = 4\pi\delta(\mathbf{r})$$

This is a little different because there are derivatives on the LHS. I would have to write

$$-\partial_r(r^2\partial_r\,(1/r)) = (1/4\pi)\lim_{\alpha\to 0}\ [\ \alpha^{-3}\, g(r/\alpha)\ ]$$

so I would have to find some g(r) that makes this true. How do you do that? All you can do is try different things. For example

$$-\partial_r(r^2\partial_r\,(1/r^{1+\alpha})) = -\alpha(\alpha+1)/r^{1+\alpha} \quad =?= \quad (1/4\pi)\,\alpha^{-3}\, g(r/\alpha)$$

$$\Rightarrow\ g(r/\alpha) = 4\pi\alpha^3\,[-\alpha(\alpha+1)/r^{1+\alpha}\ ] = -4\pi\alpha^4(\alpha+1)\,(r/\alpha)^{-1-\alpha}\ \alpha^{-1-\alpha} = -4\pi\alpha^{3-\alpha}(\alpha+1)\,(r/\alpha)^{-1-\alpha}$$

$$\Rightarrow\ g(r) = -4\pi\alpha^{3-\alpha}(\alpha+1)\,(r)^{-1-\alpha}$$

But g(r) is not allowed to depend on α, so this trial does not fly. Here is another trial

$$-\partial_r(r^2\partial_r\,(1/(r+\alpha))) = 2\,r\alpha\,(r+\alpha)^{-3} = 2\,r\alpha\,(r/\alpha+1)^{-3}\,\alpha^{-3} = 2\,r\alpha^{-2}\,(r/\alpha+1)^{-3} = 2\alpha^{-1}(r/\alpha)\,(r/\alpha+1)^{-3}$$

$$=?= \quad (1/4\pi)\,\alpha^{-3}\, g(r/\alpha)$$

$$\Rightarrow g(r/\alpha) = 4\pi\alpha^3\, 2\alpha^{-1}(r/\alpha)\,(r/\alpha+1)^{-3} = 8\pi\alpha^2(r/\alpha)\,(r/\alpha+1)^{-3}$$

$$\Rightarrow\ g(r) = 8\pi\alpha^2\, r(r+1)^{-3}$$

This seems to be at least a better trial, but we still have α dependence, though in a simple way. Here is a third trial

$- \partial_r(r^{2+\beta}\partial_r\,(1/r^{1+\alpha})) = -2(r+\beta)(\beta-\alpha)/(r+\alpha)^3 = -2(r/\alpha + \beta/\alpha)\alpha(\beta-\alpha)/[\,(r/\alpha + 1)^3\,\alpha^3]$

$=?= \quad (1/4\pi)\,\alpha^{-3}\,g(r/\alpha)$

$\Rightarrow \; g(r/\alpha) = 4\pi\alpha^3 * -2(r/\alpha + \beta/\alpha)\alpha(\beta-\alpha)/[\,(r/\alpha + 1)^3\,\alpha^3]$

$= -8\pi\,\alpha(\beta-\alpha)\,(r/\alpha + \beta/\alpha)/\,(r/\alpha + 1)^3$

But you are now stuck with α(β-α) which is α dependent. I think I could spend 100 hours here searching for a g(r).

In his fundy section, Stak always calls upon the divergence theorem to get what he needs. What "handle" can one attach to Stak's discussion of g(r) ? I could scan some of my "famous math books" like Courant and Hilbert.

Conclusion: I don't know how to prove the 3D case without the divergence theorem! Thus, I have no precedent on which to try to prove or disprove the 2D case. I don't really know how to do a web search on this subject.

So Jim's suggestion is of no use because there is no way. [But later a "way" was found!]