Answer to Question 2 PhL 1.12.11 → 1.15.11

Note: the version of this I emailed to Jim on 1.12.11 had only Part 1 and Part 2.
I added Part 3 later and sent Jim another email which I quote below at the end of this doc as Part 4.
That email quoted below was sent with subject "The delta saga continues... direction 1.13.11"
Later, in "summary of docs in this folder.doc", when I was reviewing this doc, I found various confusions and bad notation and errors. I tried to rescue this doc by editing in changes in red and for the most part I guess that has worked. But then I basically ended up rewriting this doc in "summary of docs", something that took me an entire day due to constant algebra and other errors. So I think it is best to ignore this and read that summary version. However, there are details that only appear in THIS version, so both must be kept. What a mess! The cause of this mess was that this stuff was all new to me, I was learning it for the first time and could not even believe the results I was getting, and I was in a rush because I Jim pondering things and did not want to hold back on what I was learning.

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**Part 1: The Answer to Question #2**

I agree finally that the following is true (I derive it in Part 2 below, perhaps just as you did)

$$- \lim_{\varepsilon \to 0} \partial_{\mathbf{r}}^2 (r[1/\sqrt{r^2+\varepsilon^2}]) = \delta(r)/r = 2\pi\, \delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y) \qquad (1)$$

and this is then our regulated definition of the meaning of $- \partial_{\mathbf{r}}^2 ( r [1/r])$ .

Let's then suppose that my Question #2 was this:

" Find $L_{\mathbf{xy}}$ such that $L_{\mathbf{xy}}(1/\sqrt{x^2+y^2}) = 2\pi\delta(x)\delta(y)$ "

Then the answer seems to be:

$L_{\mathbf{xy}}$ is a differential operator which, when expressed in polar coordinates, has the form

$- Lf = - \partial_{\mathbf{r}}^2 (r f) + (1/r)$ ( some angular part that involves $\partial_{\boldsymbol{\theta}}$ derivatives) f

Any dimensionless angular part involving $\partial_{\boldsymbol{\theta}}$ derivatives will do, including no angular part (since the angular part has no effect on (1/r) ) . So let's take "no angular part".

So now we want to express $\partial_{\mathbf{r}}^2 (r f)$ in Cartesian coordinates. We have $x = r\cos\theta$ and $y = r\sin\theta$ so that

$$\partial_{\mathbf{r}} = \partial x/\partial r\, \partial_{\mathbf{x}} + \partial y/\partial r\, \partial_{\mathbf{y}} = \cos\theta\partial_{\mathbf{x}} + \sin\theta\partial_{\mathbf{y}} = (x/r)\, \partial_{\mathbf{x}} + (y/r)\, \partial_{\mathbf{y}} = (1/r)(x\partial_{\mathbf{x}} + y\partial_{\mathbf{y}})$$

Then we have

$$-L_{xy}^{(1)} f = \partial_r^2(rf) = [(x/r)\,\partial_x + (y/r)\,\partial_y][(x/r)\,\partial_x + (y/r)\,\partial_y](rf) \qquad // = Lf \text{ in Maple}$$

where $r = \sqrt{x^2+y^2}$ . So it is just a matter of computing this mess. Let's write

$$g = [(x/r)\,\partial_x + (y/r)\,\partial_y](rf)$$
$$L = [(x/r)\,\partial_x + (y/r)\,\partial_y]g$$

Here is what Maple has to say

```
restart;
r := sqrt(x^2+y^2);
```

$$r := \sqrt{x^2+y^2}$$

```
g := (x/r)*diff(r*f(x,y), x) + (y/r)*diff(r*f(x,y), y);
```

$$g := \frac{x\left(\frac{f(x,y)\,x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\left(\frac{\partial}{\partial x} f(x,y)\right)\right)}{\sqrt{x^2+y^2}} + \frac{y\left(\frac{f(x,y)\,y}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\left(\frac{\partial}{\partial y} f(x,y)\right)\right)}{\sqrt{x^2+y^2}}$$

```
L := (x/r)*diff(g, x) + (y/r)*diff(g, y):simplify(%);
```

$$\frac{\left(\frac{\partial^2}{\partial x^2} f(x,y)\right)x^2 + 2\,x\,y\left(\frac{\partial^2}{\partial y\,\partial x} f(x,y)\right) + 2\,x\left(\frac{\partial}{\partial x} f(x,y)\right) + 2\,y\left(\frac{\partial}{\partial y} f(x,y)\right) + y^2\left(\frac{\partial^2}{\partial y^2} f(x,y)\right)}{\sqrt{x^2+y^2}}$$

So if this has all been done correctly, the answer is this:

Question:

" Find $L_{xy}$ such that $\qquad L_{xy}(1/\sqrt{x^2+y^2}) ) = 2\pi\delta(x)\delta(y)$

Answer: Here is one $L_{xy}$ that works (it is not unique because we could have added an angular part)

$$L_{xy}^{(1)} = -(1/\sqrt{x^2+y^2})\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y\,]$$

The angular part mentioned above is just a solution to the homogeneous equation which we always can add in

$$L_{xy,\,\text{angular part}}[1/r] = 0$$

so that this would also be a viable solution to our problem

$$L_{xy}^{(1)} + L_{xy,\,\text{angular part}}$$

Perhaps by assuming angular parts of different forms, one could obtain an alternative $L_{xy}$ which has a simpler Cartesian form. In any event, knowing the result, we can rewrite it as

$$L_{xy}^{(1)} = -(1/\sqrt{x^2+y^2})\,[(x\partial_x+y\partial_y)^2 + (x\partial_x+y\partial_y)\;]$$

$$= -(1/\sqrt{x^2+y^2})\,(x\partial_x+y\partial_y)\,[1 + (x\partial_x+y\partial_y)\;]$$

and this then makes the result more obvious, since $r\partial_r = (x\partial_x+y\partial_y)$,

$$-L_{xy}^{(1)} = (1/r)\,(r\partial_r)[\,1 + (r\partial_r)] = (1/r)\,[\,(r\partial_r) + (r\partial_r)\,(r\partial_r)]$$

$$= [\,(\partial_r) + (\partial_r)\,(r\partial_r)] = [\,(\partial_r) + (\partial_r^2 + \partial_r)] = \partial_r^2 + 2\partial_r = \partial_r^2(r\;)$$

which agrees with our result above.

**Part 2: Proof of Equation (1)**

The claim is this

$$-\lim_{\varepsilon\to 0}\,\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = \delta(r)/r = 2\pi\,\delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y) \qquad (1)$$

where I chose $\varepsilon^2$ so $\varepsilon$ and r have the same dimensions.

My starting point is this general theorem described in my Stakgold book page 13 (Section 5.4) [ the proof of the theorem is pretty simple.]

**Theorem n**: Any non-negative locally integrable function g(r) for which this is true

$$\int_0^\infty dr\, r^{n-1}\, g(r) = 1/S_n(1) \qquad S_n(1) = \text{area of n dimensional sphere}$$

has the property that

$$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-n}\,g(r/\varepsilon)\,] = \delta^{(n)}(\mathbf{r})$$

So we can restate this for n = 2:

**Theorem n=2**: Any non-negative locally integrable function g(r) for which this is true

$$\int_0^\infty dr\, r\, g(r) = 1/2\pi$$

has the property that

$$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-2}\,g(r/\varepsilon)\,] = \delta^{(2)}(\mathbf{r})$$

Now consider this candidate g(r)  (motivation is given below)

$$g(r) = (1/2\pi)[\, 3r / (r^2+1)^{5/2}]$$

We find that

$$\int_0^\infty dr\, r\, g(r) = (3/2\pi) \int_0^\infty dr\, r^{\,2}/ (r^2+1)^{5/2} = (3/2\pi)\ (1/3) = 1/2\pi$$

which is the desired result for normalization of g. Here is Maple on this integral

```
Int(r^2/(1+r^2)^(5/2),r=0..infinity);
```

$$\int_0^\infty \frac{r^2}{(1+r^2)^{\left(\frac{5}{2}\right)}}\, dr$$

```
int(r^2/(1+r^2)^(5/2),r=0..infinity);
```

$$\frac{1}{3}$$

Therefore, we know that the following is true

$$\lim_{\varepsilon\to 0} [\, \varepsilon^{-2}(1/2\pi)[\, 3(r/\varepsilon) / [(r/\varepsilon)^2+1]^{5/2}] = \delta^{(2)}(\mathbf{r})$$

which we write as

$$\lim_{\varepsilon\to 0} [\, \varepsilon^{-2}[\, 3(r/\varepsilon) / [(r/\varepsilon)^2+1]^{5/2}] = 2\pi\, \delta^{(2)}(\mathbf{r}) \qquad (2)$$

Now why is this an interesting g(r)?  We take "the Jim proposal" and do this computation in Maple:

$$\partial_{\mathbf{r}}^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = ?$$

```
restart;
Diff(Diff(r/sqrt(r^2+epsilon^2),r),r);
```

$$\frac{\partial^2}{\partial r^2}\frac{r}{\sqrt{r^2+\varepsilon^2}}$$

```
diff(diff(r/sqrt(r^2+epsilon^2),r),r): simplify(%);
```

$$-3\frac{r\varepsilon^2}{(r^2+\varepsilon^2)^{\left(\frac{5}{2}\right)}}$$

so this shows that,

$$\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}]) = -3\, r\varepsilon^2/(r^2+\varepsilon^2)^{5/2}$$

but then we write

$$(r^2+\varepsilon^2)^{5/2} = (\varepsilon^2[(r/\varepsilon)^2+1])^{5/2} = \varepsilon^5[(r/\varepsilon)^2+1]^{5/2} \qquad \dim = L^5$$
$$r\varepsilon^2 = (r/\varepsilon)\varepsilon^3$$

to get

$$-\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}]) = 3(r/\varepsilon)\varepsilon^3/[\varepsilon^5[(r/\varepsilon)^2+1]^{5/2}] = \varepsilon^{-2}\, 3\,(r/\varepsilon)/[(r/\varepsilon)^2+1]^{5/2}$$

But according to equation (2) above we have

$$\lim_{\varepsilon\to 0}[\varepsilon^{-2}[3(r/\varepsilon)/[(r/\varepsilon)^2+1]^{5/2}] = 2\pi\,\delta^{(2)}(\mathbf{r})$$

Therefore we conclude that

$$-\lim_{\varepsilon\to 0}\{\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}])\} = 2\pi\,\delta^{(2)}(\mathbf{r}) \qquad \text{QED}$$

**Part 3: Jim's second conjecture**

He called to say he thought my work was unnecessary and that the answer is this: (using $2\pi$ for the moment).

$$-L_{xy}f = (1/\sqrt{x^2+y^2})(\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\, f) \qquad \text{and } L_{xy}(1/r) = 2\pi\,\delta^2(\mathbf{r}) = 2\pi\delta(r)/(2\pi r) = \delta(r)/r$$

Right off the bat, the dimensions are wrong. $\dim(L_{xy}(1/r)) = L^{-3}$ while $\dim \delta^2(\mathbf{r}) = L^{-2}$. So I will fix him up by removing the 1/r out front. So Jim must be claiming this (I give it a superscript label)

$$-L_{xy}^{(2)}f = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\, f) \qquad \text{and } L_{xy}(1/r) = 2\pi\,\delta^2(\mathbf{r}) = 2\pi\delta(r)/(2\pi r) = \delta(r)/r$$

Jim did not provide any explanation of this conjecture as to why it should be true. So I will test it. It can be written this way

$$-L_{xy}^{(2)}f = \nabla^2_{2D}(rf) \qquad \text{where } r = \sqrt{x^2+y^2}$$

and we assume $f = f(r)$. We know then that (from cylindricals info)

$$\nabla^2_{2D}F(r) = (1/r)\partial_r(r\partial_r F(r)) = (1/r)[r\partial_r^2 F(r) + \partial_r F(r)]$$

$$= \partial_r^2 F(r) + (1/r)\partial_r F(r) = (\partial_r^2 + (1/r)\partial_r)F(r)$$

or

$$\nabla^2_{2D,radial} = [\partial_r^2 + (1/r)\partial_r]$$

So it would seem that Jim is claiming that (when acting on f = f(r) )

$$- L_{xy}^{(2)} f = [\partial_r^2 + (1/r)\partial_r] (rf) = \partial_r^2(rf) + (1/r)\partial_r(rf)$$

Then applied to [1/r] this says

$$- L_{xy}^{(2)} [1/r] = (\partial_r^2(r [1/r]) + (1/r)\partial_r(r [1/r]) \qquad \text{// both terms are } L^{-2}$$

which disagrees with his original proposal which was that

$$- L_{xy}^{(1)} [1/r] = \partial_r^2( r [1/r])$$

So basically this is a new proposed form for L which would have to be tested We know that the first term produces the delta function, so the second term would have to also be proportional to a delta function.

Let's see:

$$(1/r)\partial_r(r [1/r]) = \lim_{\varepsilon\to 0} (1/r)\partial_r(r [1/\sqrt{r^2+\varepsilon^2}\,])$$

and we have our Maple assist,

```
restart;
(1/r)*Diff(r/sqrt(r^2+epsilon^2),r);
```

$$\frac{\frac{\partial}{\partial r}\frac{r}{\sqrt{r^2+\varepsilon^2}}}{r}$$

```
(1/r)*diff(r/sqrt(r^2+epsilon^2),r):simplify(%);
```

$$\frac{\varepsilon^2}{r\left(r^2+\varepsilon^2\right)^{\left(\frac{3}{2}\right)}}$$

which says that

$$(1/r)\partial_r(r [1/\sqrt{r^2+\varepsilon^2}\,]) = \varepsilon^2/ [\, r\,(r^2+\varepsilon^2)^{3/2}] = (\varepsilon^2/r) / (r^2+\varepsilon^2)^{3/2} \qquad = L/L^3 = L^{-2} \text{ so ok}$$

and we now use

$$(r^2+\varepsilon^2)^{3/2} = (\, \varepsilon^2[(r/\varepsilon)^2+1]\,)^{3/2} = \varepsilon^3[(r/\varepsilon)^2+1]^{3/2} \qquad \dim = L^3$$
$$\varepsilon^2/r = \varepsilon\,(\varepsilon/r) \qquad \dim = L$$

to get

$$(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) \;=\; (\varepsilon^2/r)\,/\,(r^2+\varepsilon^2)^{3/2} \;=\; \varepsilon\,(\varepsilon/r)/\,\{\,\varepsilon^3[(r/\varepsilon)^2+1]^{3/2}\qquad\}\qquad L^{-2}$$

$$= \varepsilon^{-2}\,(\varepsilon/r)\,[(r/\varepsilon)^2+1]^{3/2} = \;\varepsilon^{-2}\;\; 1\;/\;\{\,(r/\varepsilon)\,[(r/\varepsilon)^2+1]^{3/2}\,\}$$

$$= \varepsilon^{-2}\,h(r/\varepsilon) \qquad\qquad h(r) = r^{-1}/\;[r^2+1]^{3/2}$$

And indeed, this is again the right form! So let's get the normalization. We need

$$\int_0^\infty dr\; r\; g(r) = 1/2\pi$$

so let's assume

$$g(r) = K\,h(r) = K\,r^{-1}/\;[r^2+1]^{3/2}$$

$$\int_0^\infty dr\; r\; g(r) \;=\; K\int_0^\infty dr\,/\;[r^2+1]^{3/2} \;=\; K$$

says Maple,

```
Int(1/(1+r^2)^(3/2),r=0..infinity);
```

$$\int_0^\infty \frac{1}{\left(\frac{3}{2}\right)}\,dr$$

```
int(1/(1+r^2)^(3/2),r=0..infinity);
```

Therefore we have K = 1/2π. Thus our normalized g(r) is this:

$$g(r) = (1/2\pi)\,r^{-1}/\;[r^2+1]^{3/2} \qquad\qquad g(r) = (1/2\pi)h(r)$$

We showed above that

$$(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) \;=\; \varepsilon^{-2}\,h(r/\varepsilon) \qquad\qquad \text{where } h(r) = \;r^{-1}/\;[r^2+1]^{3/2}$$

Therefore we now have

$$(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) \;=\; \varepsilon^{-2}\,h(r/\varepsilon) \;=\; \varepsilon^{-2}\,2\pi\,g(r/\varepsilon) = 2\pi\,[\varepsilon^{-2}\,g(r/\varepsilon)]$$

and therefore according to Stak's theorem, we have

$$\lim_{\varepsilon\to 0}\{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,])\} = 2\pi \lim_{\varepsilon\to 0}\{\varepsilon^{-2}\, g(r/\varepsilon)\} = 2\pi\,\delta^{(2)}(\mathbf{r})$$

In Part 2 above I showed that

$$-\lim_{\varepsilon\to 0}\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = \delta(r)/r = 2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (1)$$

So I now summarize these two results:

$$\lim_{\varepsilon\to 0}\{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,])\} = 2\pi\,\delta^{(2)}(\mathbf{r})$$
$$\lim_{\varepsilon\to 0}\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = \delta(r)/r = -2\pi\,\delta^{(2)}(\mathbf{r})$$

Amazing. So Jim's result is then this

$$-L_{xy}^{(2)}(1/r) = (\partial_r^2(r\,[1/r]) + (1/r)\partial_r(r\,[1/r])) = -2\pi\,\delta^{(2)}(\mathbf{r}) + 2\pi\,\delta^{(2)}(\mathbf{r}) = 0\,!$$

There must be an easier way to show this. Consider the divergence theorem:

$$\int_V dV\,\nabla^2\varphi = \int_S d\mathbf{S}\bullet\nabla\varphi = \int_S dS\,\partial_n\varphi \qquad \text{// application of the above with } \mathbf{A} = \nabla\varphi$$

In 2D with a disk of radius a this says

$$\int_V dV\,\nabla^2\varphi = \int_S dS\,\partial_n\varphi$$

$$\int_0^\infty rdr\int_0^{2\pi} d\theta\,\nabla^2\varphi = \int_0^\infty rd\theta\,\partial_r\varphi = a\int_0^\infty d\theta\,\partial_r\varphi$$

If we apply this to $\varphi = rF(r)$ it says

$$\int_0^\infty rdr\int_0^{2\pi} d\theta\,\nabla^2[rF(r)] = a\int_0^\infty d\theta\,\partial_r[rF(r)]$$

Now suppose $F(r) = [1/r]$. Then we get

$$\int_0^\infty rdr\int_0^{2\pi} d\theta\,\nabla^2[r[1/r]] = a\int_0^\infty d\theta\,\partial_r[r[1/r]] = a\int_0^\infty d\theta\,0 = 0\,!$$

**Finally**, let's just compute out Jim's proposed form directly in Cartesians:

$$-L_{xy}^{(2)} f = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\, f) = \nabla_{2D}^2(r\, f) \qquad \text{// = Lf for Maple}$$

```
restart;
r := sqrt(x^2+y^2);
```

$$r := \sqrt{x^2+y^2}$$

```
Lf := Diff(Diff(r*f(x,y),x),x) + Diff(Diff(r*f(x,y),y),y);
```

$$Lf := \left(\frac{\partial^2}{\partial x^2}\sqrt{x^2+y^2}\,f(x,y)\right) + \left(\frac{\partial^2}{\partial y^2}\sqrt{x^2+y^2}\,f(x,y)\right)$$

```
Lf := diff(diff(r*f(x,y),x),x) + diff(diff(r*f(x,y),y),y): simplify(%);
```

$$\frac{x^2\left(\frac{\partial^2}{\partial x^2}f(x,y)\right) + x^2\left(\frac{\partial^2}{\partial y^2}f(x,y)\right) + 2\left(\frac{\partial}{\partial x}f(x,y)\right)x + \left(\frac{\partial^2}{\partial x^2}f(x,y)\right)y^2 + f(x,y) + \left(\frac{\partial^2}{\partial y^2}f(x,y)\right)y^2 + 2\left(\frac{\partial}{\partial y}f(x,y)\right)y}{\sqrt{x^2+y^2}}$$

'

So this says ( I omitted the +1 at the end in my emailed version of this doc)

$$-L_{xy}^{(2)} = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,f) = (1/\sqrt{x^2+y^2})[\,x^2(\partial_x^2+\partial_y^2) + y^2(\partial_x^2+\partial_y^2) + 2x\partial_x + 2y\partial_y + 1]\,f$$

and this is different from my earlier result which was

$$-L_{xy}^{(1)} = (1/\sqrt{x^2+y^2})\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y\,]$$

**Part 4: Email to Jim:**

1. I think you proposed by phone that a solution to $L_{xy}(1/r) = 2\pi\,\delta^{(2)}(\mathbf{r})$ might be this

$$L_{xy}([1/r]) = (1/\sqrt{x^2+y^2})\;(\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

But dimensionally this says $L^{-3} = L^{-2}$, so I think what you intended was:

$$-L_{xy}^{(2)}([1/r]) = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,([1/r])) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$ // not true

2. Assuming this is what you meant, you are then claiming this (perhaps apart from a constant)

$$\nabla^2_{2D}(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$ // it will turn out that $\nabla^2_{2D}(r[1/r]) = 0$

where $\nabla^2_{2D} = \partial_x^2+\partial_y^2$ is the 2D Laplacian. In *cylindrical* coordinates we know that

$$\nabla_{3D}^2 = r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2 + \partial_z^2$$

So for *polar* coordinates r and φ it must be true that

$$\nabla_{2D}^2 = \partial_x^2+\partial_y^2 = r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2$$

Therefore

$$\nabla^2_{2D}F(r) = r^{-1}\partial_r(r\,\partial_r F(r)) = r^{-1}(r\,\partial_r^2 F(r) + \partial_r F(r)) = \partial_r^2 F(r) + r^{-1}\,\partial_r F(r)$$

Let's take F(r) = r[1/r] so this then says

$$\nabla^2_{2D}(r[1/r]) = \partial_r^2(r[1/r]) + r^{-1}\,\partial_r(r[1/r])$$

Thus, a new term has been added to your original proposal which was just $\partial_r^2(r[1/r])$.

3. Using my Part 2 methods of the last doc, I was able to show these interesting facts:

$$\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}]) = -2\pi\,\delta^{(2)}(\mathbf{r})$$
$$\lim_{\varepsilon\to 0}\ \{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}])\} = +2\pi\,\delta^{(2)}(\mathbf{r})$$

After doing this, I realized that it was obvious from the divergence theorem applied to $\nabla\varphi$,

$$\int_V dV\,\nabla^2\varphi = \int_S dS\,\partial_n\varphi$$

which applied to a 2D disk of radius a says

$$\int_0^a rdr\ \int_0^{2\pi} d\theta\,\nabla^2\varphi = a\int_0^{\infty} d\theta\,\partial_r\varphi$$

Setting $\varphi = r\,[1/r]$ we get

$$\int_0^a rdr\ \int_0^{2\pi} d\theta\,\nabla^2(r\,[1/r]) = a\int_0^{\infty} d\theta\,\partial_r(r\,[1/r])|^{r=a} = a\int_0^{\infty} d\theta\,\partial_r(1) = 0$$

So that

$$\nabla^2(r\,[1/r]) = 0$$

4. So the bad news is that your two terms exactly cancel so that

$$-L_{xy}^{(2)}\,[1/r] = \nabla^2_{2D}\,(r[1/r]) = -2\pi\,\delta^{(2)}(\mathbf{r}) + 2\pi\,\delta^{(2)}(\mathbf{r}) = 0$$

The good news is that we now have *two* different solutions to the problem which are these

$$-\partial_r^2(r\,[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$
$$+(1/r)\partial_r(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

which correspond to (Maple) [ added the missing 1 at the end ]

$$L_{xy}^{(1)} = -(1/\sqrt{x^2+y^2})\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\partial_x + 2y\partial_y\,]$$

$$L_{xy}^{(2)} = -(1/\sqrt{x^2+y^2})[\, x^2(\partial_x^2+\partial_y^2) + y^2(\partial_x^2+\partial_y^2) + 2x\partial_x + 2y\partial_y + 1]$$

I have deleted the rest of this Part 4 Email since it was in error.

See the "rewrite" of this doc in "summary of docs" in this folder which at the end talks about the difference between two successful candidate $L_{xy}$ operators.