Summary of docs in this folder PhL 1.15.11

As of Sunday 1.16.11 Jim has not responded to either of the last two docs. I think I wore him out.

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0. My 1st response to Jims 1st proposed L .doc (1 page, 1.18.11)

On 1/4/11 I presented my Lg=δ problem to Jim on the phone and he said he would think about it. A few days later he called back with what I call "Jim's first proposal". I then sent this doc as an email in which I noted that the dimensions of his proposal were wrong, AND that I did not know how he showed that it was in fact a solution. Jim had mentioned on the phone some sort of square root regulation method which turned out to be exactly the right method. In retrospect, he probably came up with his proposal starting with the 3D known fact, just as I did in my 1.18 wrap-up doc to him. I wish he would have explained this.

1. Jim delta 1_11_11.doc (1.5 pages, 1.11.11)

"Let's start all over".

Question #2: What Cartesian 2D differential operator L_{xy} makes this equation be true:

$$L_{xy}\left(1/\sqrt{(x-x')^2+(y-y')^2}\ \right) = \delta(x-x')\delta(y-y')$$

Answer:

I don't know the answer. It might be that no operator L_{xy} exists which solves Problem #2. Or maybe it is some horrible thing like this, with horrible coefficient functions,

$$L_{xy} = a(x,y)\partial_x + a(y,x)\partial_y + b(x,y)\,\partial_x^2 + b(y,x)\,\partial_y^2 + c(x,y)\partial_x\partial_y + \text{lots of higher terms.}$$

I am pretty sure this is NOT the answer

$L_{xy} = \partial_x^2 + \partial_y^2$

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2. Answer to question 2.doc (11 pages, 1.12.11: this review is a 7 page rewrite!)

(dated 1.12.11, but added to since email was sent)

I found many confusions and various errors in this doc, so basically what follows is a rewrite of it:

Part 1: The Answer to Question #2 (this is a summary of things shown in later parts)

I agree finally that the following is true (I derive it in Part 2 below, perhaps just as you did)

$$- \lim_{\varepsilon \to 0} \partial_r^2 (r[1/\sqrt{r^2+\varepsilon^2}]) = \delta(r)/r = 2\pi\, \delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y) \qquad (1.1)$$

and this is then our regulated definition of the meaning of $- \partial_r^2 (r [1/r])$.

Let's then suppose that my Question #2 was this:

" Find L_{xy} such that $L_{xy}(1/\sqrt{x^2+y^2}) = 2\pi\delta(x)\delta(y)$ "

Then the answer seems to be:

L_{xy} is a differential operator which, when expressed in polar coordinates, has the form

$Lf = - \partial_r^2 (r f) + (1/r)$ (some angular part that involves ∂_θ derivatives) f

I then convert this to Cartesian coordinates adding in no angular part and I get

$$L_{xy}^{(1)} \equiv - \partial_r^2 (r \ldots) = - (1/\sqrt{x^2+y^2}) [\, x^2 \partial_x^2 + y^2 \partial_y^2 + 2xy\partial_x\partial_y + 2x\, \partial_x + 2y\partial_y \,] \qquad (1.2)$$

Part 2: Proof of Equation (1.1)

I prove equation (1.1) above using the Stakgold theorem for n = 2. That is, $- \partial_r^2 (r [1/r]) = \delta(r)/r$.

$$- \lim_{\varepsilon \to 0} \partial_r^2 (r[1/\sqrt{r^2+\varepsilon^2}]) = \delta(r)/r = 2\pi\, \delta^{(2)}(\mathbf{r}) = 2\pi\delta(x)\delta(y) \qquad (2.1)$$

Part 3: Jim's second conjecture

Based on dimensions, I assumed this new "second conjecture" was (here f is any function f)

$$- L_{xy}^{(2)} f \equiv (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\, f) = \nabla^2_{2D}(rf) \quad \Rightarrow \; - L_{xy}^{(2)} = \nabla^2_{2D}(r \ldots) \qquad (3.1)$$

We know that

$$\nabla_{2D}^2 = \partial_x^2+\partial_y^2 = r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2 = \nabla_{2D,radial}^2 + r^{-2}\partial_\varphi^2 \tag{3.2}$$

where

$$\nabla_{2D,radial}^2 = r^{-1}\partial_r(r\,\partial_r) = r^{-1}[r\,\partial_r^2 + \partial_r] = [\partial_r^2 + (1/r)\,\partial_r] \tag{3.3}$$

Therefore we can write

$$-L_{xy}^{(2)} f = \nabla^2_{2D}(rf) = [\nabla_{2D,radial}^2 + r^{-2}\partial_\varphi^2](rf)$$

$$= [\partial_r^2 + (1/r)\,\partial_r]\,(rf) + r^{-1}\partial_\varphi^2 f \tag{3.4}$$

which we can apply to this special radial function case,

$$-L_{xy}^{(2)}\,[1/r] = (\partial_r^2(r\,[1/r]) + (1/r)\partial_r(r\,[1/r]) = -L_{xy}^{(1)}\,[1/r] + L_{xy}^{(3)}\,[1/r] \tag{3.5}$$

where $L_{xy}^{(1)}$ was defined in Part 1 above, and I here define the second term to be

$$L_{xy}^{(3)} \equiv (1/r)\partial_r(r\,...) = (1/r)[r\,\partial_r + 1\,] = \partial_r + (1/r) \tag{3.6}$$

Note carefully that we do *not* have $-L_{xy}^{(2)} = -L_{xy}^{(1)} + L_{xy}^{(3)}$ in general. This is only true when applied to some F(r) ! The full connection between these three operators can be computed as follows:

$$-L_{xy}^{(2)} \equiv \nabla^2_{2D}(r\,...) = \{\, r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2\}\,(r\,...)$$

$$= \partial_r^2(r...) + (1/r)\partial_r(r...) + r^{-1}\partial_\varphi^2\,(...) \tag{3.7}$$

$$-L_{xy}^{(1)} \equiv \partial_r^2(r\,...) \qquad (3.8) = (1.2)$$

$$L_{xy}^{(3)} \equiv (1/r)\partial_r(r\,...) \tag{3.6}$$

Therefore we find that

$$-L_{xy}^{(2)} = -L_{xy}^{(1)} + L_{xy}^{(3)} + r^{-1}\partial_\varphi^2$$

or

$$L_{xy}^{(1)} - L_{xy}^{(2)} = L_{xy}^{(3)} + r^{-1}\partial_\varphi^2 \tag{3.9}$$

I then study this second term $L_{xy}^{(3)}[1/r]$ and apply to it the Stakgold n = 2 analysis to get

$$L_{xy}^{(3)}\,[1/r] = \lim_{\varepsilon\to 0}(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = +\,2\pi\,\delta^{(2)}(\mathbf{r}) \tag{3.10}$$

$$L_{xy}^{(1)}\,[1/r] = -\lim_{\varepsilon\to 0}\,\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) \quad = +2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (3.11) = (2.1)$$

where on the second line I show my Part 2 result, **so now I have two different candidate solutions** to my Question #2! Since above we have

$$-L_{xy}^{(2)}[1/r] = -L_{xy}^{(1)}[1/r] + L_{xy}^{(3)}[1/r], \qquad (3.11a)$$

an immediate implication of this result is this: for the operator which was "Jim's second conjecture", we get

$$-L_{xy}^{(2)}\,[1/r] = \nabla^2_{2D}(r[1/r]) = -L_{xy}^{(1)}[1/r] + L_{xy}^{(3)}[1/r] = -2\pi\,\delta^{(2)}(\mathbf{r}) + 2\pi\,\delta^{(2)}(\mathbf{r}) = 0 \qquad (3.12)$$

Since this is a full ∇^2 operator, I am able to apply the 2D divergence theorem, and this then provides ***independent confirmation*** that in fact $-L_{xy}^{(2)}\,[1/r] = \nabla^2_{2D}(r[1/r]) = 0$!

I then convert Jim's $L_{xy}^{(2)}$ to Cartesian coordinates and I get

$$L_{xy}^{(2)} = -\nabla^2_{2D}(r..) = -(1/\sqrt{x^2+y^2}\,)[\,x^2(\partial_x^2+\partial_y^2) + y^2(\partial_x^2+\partial_y^2) + 2x\partial_x + 2y\partial_y + 1] \qquad (3.13)$$

At this point we then have these Cartesian results (I do $L_{xy}^{(3)}$ below)

$$L_{xy}^{(1)} \equiv -\partial_r^2(r\,...) = -(1/\sqrt{x^2+y^2})\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y\,] \qquad (3.14)$$

$$L_{xy}^{(2)} = -\nabla^2_{2D}(r..) = -(1/\sqrt{x^2+y^2}\,)[\,x^2(\partial_x^2+\partial_y^2) + y^2(\partial_x^2+\partial_y^2) + 2x\partial_x + 2y\partial_y + 1] \qquad (3.13)$$

Part 4: Email to Jim (a section of the doc)

I quote the results shown earlier in this manner

$$(1/r)\partial_r(r\,[1/r]) = \lim_{\varepsilon\to 0}\,\{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,])\,\} = +2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (4.1) = (3.10)$$

$$\partial_r^2(r[1/r]) = \lim_{\varepsilon\to 0}\,\{\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,])\} = -2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (4.2) = (2.1)$$

and I show that these add up to $\nabla^2_{2D}(r\,[1/r])$ and the fact that they are equal and opposite is *verified* by the 2D divergence theorem.

Using these facts,

$$x = r\cos\varphi \qquad \partial_\varphi x = -r\sin\varphi = -y \qquad \partial_r x = \cos\varphi = (x/r)$$
$$y = r\sin\varphi \qquad \partial_\varphi y = r\cos\varphi = x \qquad \partial_r y = \sin\varphi = (y/r)$$

$$\partial_\varphi = (\partial_\varphi y)\partial_y + (\partial_\varphi x)\partial_x = (x\partial_y - y\partial_x)$$
$$\partial_r = (\partial_r y)\partial_y + (\partial_r x)\partial_x = (1/r)(y\partial_y + x\partial_x)$$

we get

$$\partial_{\varphi} = (x\partial_y - y\partial_x) \tag{4.3}$$
$$r\partial_r = (y\partial_y + x\partial_x) \tag{4.4}$$

I then compute from the expressions shown above,

$$L_{xy}^{(1)} - L_{xy}^{(2)} = (1/\sqrt{x^2+y^2})[\, x^2\partial_y^2 + y^2\partial_x^2 - 2xy\partial_x\partial_y + 1] \tag{4.5}$$

$$= (1/\sqrt{x^2+y^2})\,[\,(x\partial_y - y\partial_x)^2 + (y\partial_y + x\partial_x) + 1] \tag{4.6}$$

$$= (1/r)\,\partial_{\varphi}^2 + \partial_r + (1/r) \tag{4.7}$$

According to (3.9) above we should have

$$L_{xy}^{(1)} - L_{xy}^{(2)} = L_{xy}^{(3)} + r^{-1}\partial_{\varphi}^2 \tag{4.8 = 3.9}$$

Comparing these two results requires that

$$L_{xy}^{(3)} = [\,\partial_r + (1/r)] \tag{4.9}$$

and this agrees with (3.6).

Comment: We have shown that both these guys solve our little equation $L_{xy}[1/r] = 2\pi\,\delta^{(2)}(\mathbf{r})$.

$$L_{xy}^{(3)}[1/r] = \lim_{\varepsilon\to 0}(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,]) = +\,2\pi\,\delta^{(2)}(\mathbf{r}) \tag{3.10}$$
$$L_{xy}^{(1)}[1/r] = -\lim_{\varepsilon\to 0}\partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = +\,2\pi\,\delta^{(2)}(\mathbf{r}) \tag{3.11 = 2.1}$$

The difference of these two solutions is given by

$$L_{xy}^{(3)} - L_{xy}^{(1)} = -L_{xy}^{(2)} - r^{-1}\partial_{\varphi}^2 = \nabla^2_{2D}(r..) - r^{-1}\partial_{\varphi}^2\,.$$

The difference must satisfy the homogeneous equation $L_{xy}[1/r] = 0$. We have show already that the first term solves this equation (as we prove using the divergence theorem for example), and it is obvious that the second term also solves the homo equation.

As a further check, let's express $L_{xy}^{(3)}$ in Cartesians:

$$L_{xy}^{(3)} \equiv (1/r)\partial_r(r\,...) = (1/r)^2(x\partial_x + y\partial_y)\,(r\,...) \qquad \text{since} \qquad \partial_r = (1/r)(x\partial_x + y\partial_y)$$

Here is what Maple says:

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r := sqrt(x^2+y^2);
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$$r := \sqrt{x^2+y^2}$$

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(1/r^2)* ( x*Diff(r*f(x,y),x) + y*Diff(r*f(x,y),y));
```

$$\frac{x\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}\,f(x,y)\right)+y\left(\frac{\partial}{\partial y}\sqrt{x^2+y^2}\,f(x,y)\right)}{x^2+y^2}$$

```
(1/r^2)* ( x*diff(r*f(x,y),x) + y*diff(r*f(x,y),y)): simplify(%);
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$$\frac{\left(\frac{\partial}{\partial x}f(x,y)\right)x+f(x,y)+y\left(\frac{\partial}{\partial y}f(x,y)\right)}{\sqrt{x^2+y^2}}$$

so we conclude that

$$L_{xy}^{(3)} = (1/\sqrt{x^2+y^2}\,)\,[\,x\partial_x + y\partial_y + 1\,] = (1/r)\,[\,r\partial_r + 1\,] = \partial_r + (1/r) \qquad (4.10)$$

which also agrees with (3.6).

At this point I want to make this comment. We have found two solutions to

$$L_{xy}([1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r})$$

Those solutions are $L_{xy}^{(1)}$ and $L_{xy}^{(3)}$ as shown in (3.10) and (3.11) above. If we take the *difference* of these solutions, we should find that

$$L_{xy}^{(1)}\,[1/r] - L_{xy}^{(3)}\,[1/r] = 0$$

From (3.11a) above we see that this implies that

$$L_{xy}^{(2)}\,[1/r] = 0$$

From (3.1) we can express this as

$$\nabla^2_{2D}(r\,[1/r]) = 0$$

and this agrees with (3.12).

Once again, here are the two solutions I have found to the equation $L_{xy}([1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r})$:

$$L_{xy}^{(3)} = (1/\sqrt{x^2+y^2}\,)\,[\,x\partial_x + y\partial_y + 1\,] = [\,\partial_r + (1/r)\,] = (1/r)\partial_r(r\ldots) \qquad (4.11)$$

$$L_{xy}^{(1)} \equiv -(1/\sqrt{x^2+y^2}\,)\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y\,] = -\partial_r^2(r\ldots)$$

Certainly the first operator has the simpler form when written out in Cartesians. The conclusion is that

$$(1/r)\partial_r(r\,[1/r]\,) = 2\pi\,\delta^{(2)}(\mathbf{r}) = L_{xy}^{(3)}([1/r]\,) \qquad (4.12)$$

or

$$(1/\sqrt{x^2+y^2}\,)\;\;[\;x\partial_{\mathbf{x}} + y\partial_{\mathbf{y}} + 1]\;(\sqrt{x^2+y^2}\,)\;[1/\sqrt{x^2+y^2}\,]\;\; = 2\pi\,\delta(x)\delta(y)\;(*) \qquad (4.13)$$

where we think of

$$[1/\sqrt{x^2+y^2}\,] = \lim_{\varepsilon\to 0}\,(1/\sqrt{x^2+y^2+\varepsilon^2}\,)$$

Notice that the work done here does not seem to solve this problem

$$L_{\mathbf{xy}}\,(1/\sqrt{(x-x')^2+(y-y')^2}\,)\;\; = \;2\pi\,\delta(x-x')\delta(y-y') \qquad (4.14)$$

and THIS is the thing I was *really* trying to solve. Suppose we define

$$x = X-X' \qquad \partial_{\mathbf{x}} = \partial_{\mathbf{X}}$$
$$y = Y-Y' \qquad \partial_{\mathbf{y}} = \partial_{\mathbf{Y}}$$

Then (*) above says

$$(1/\sqrt{(X-X')^2+(Y-Y')^2}\,)[(X-X')\partial_{\mathbf{X}}+(Y-Y')\partial_{\mathbf{Y}}+1]\;\{\sqrt{(X-X')^2+(Y-Y')^2}\;\;[1/\sqrt{(X-X')^2+(Y-Y')^2}\,]\,\}$$

$$=\;\;2\pi\,\delta(X-X')\delta(Y-Y')\;\;=2\pi\,\delta^{(2)}(\mathbf{R}-\mathbf{R}')$$

Having derived this, let's now change everything to lower case

$$(1/\sqrt{(x-x')^2+(y-y')^2}\,)[(x-x')\partial_{\mathbf{x}}+(y-y')\partial_{\mathbf{y}}+1]\;\{\sqrt{(x-x')^2+(y-y')^2}\;\;[1/\sqrt{(x-x')^2+(y-y')^2}\,]\,\}$$

$$=\;\;2\pi\,\delta(x-x')\delta(y-y')\;\;=2\pi\,\delta^{(2)}(\mathbf{x}-\mathbf{x}') \qquad (4.15)$$

This is of the form

$$L_{\mathbf{xy,x'y'}}\;(\;[1/\sqrt{(x-x')^2+(y-y')^2}\,]\;) = \;2\pi\,\delta^{(2)}(\mathbf{x}-\mathbf{x}') \qquad (4.16)$$

but the form I am always looking for is this

$$L_{\mathbf{xy}}(\;[1/\sqrt{(x-x')^2+(y-y')^2}\,]\;) = \;2\pi\,\delta^{(2)}(\mathbf{x}-\mathbf{x}') \qquad (4.17)$$

Therefore, my solving this problem

$$L_{\mathbf{xy}}([1/r]) = \;2\pi\,\delta(x)\delta(y)$$

has not helped one bit ! Many hours down the drain, but interesting results found. This probably means my Richard Price email has problems. // This evening 1.15.11 I updated my Richard email so at least the above error is removed from it.

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3. Jim delta 1_13_11.doc (2.5 pages, 1.13.11)

Here I respond to Jim's suggestion that maybe this is true

$$L_{xy}([1/r]) = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,([1/r]) = \nabla^2(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \qquad \text{// not true!}$$

I give two arguments why I think in fact $\nabla^2(r[1/r]) = 0$ rather than a delta function. On argument is the divergence theorem which I present in some but not total detail. The other is that I *claim* I have shown

$$\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = -2\pi\,\delta^{(2)}(\mathbf{r})$$
$$\lim_{\varepsilon\to 0}\ \{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,])\} = +2\pi\,\delta^{(2)}(\mathbf{r})$$

and if you add these up, you get $\nabla^2(r[1/r]) = 0$. But I don't back up these two claims.

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4. Jim delta 1.13.11 5PM.doc (8 pages, 1.17.11)

This is a longer doc where I back up my various claims. In **Section** 1 I quote Stak's theorem. In **Sections 2 and 3** I apply it for two different sequences. In **Section 3** I show that these results are consistent with the 2D divergence theorem which says the results of Sections 1 and 2 must add up to 0. In **Section 4** I show why the 2D and 3D divergence theorems give results of a different nature (0 and non 0) for our problems of interest. In **Section 5** I show how to apply the Stak n=3 theorem to the usual 3D case we know and love. In **Section 6**, added just today, I show that, while the regulation method $[1/r] \equiv \lim_{\varepsilon\to 0}(1/\sqrt{r^2+\varepsilon^2}\,)$ works for the 3D case treated in Section 5, the regulation method $[1/r] \equiv \lim_{\varepsilon\to 0}(1/(r+\varepsilon))$ fails!

Here is the TOC for this doc:

0. The Stakgold book theorem says this: 1

1. Showing that $\partial_r^2\,(r\,[1/r]) = -2\pi\,\delta^{(2)}(r)$ 2

2. Showing that $(1/r)\,\partial_r(r\,[1/r]) = +2\pi\,\delta^{(2)}(r)$ 3

3. Confirmation from the 2D Divergence Theorem.....4

4. 3D Divergence Theorem. Why does this give a different kind of result? 6

5. How do we apply the Stakgold theorem to the 3D case?.....7

6. Attempt the same 3D problem using 1/(r+ε) regulation (this section added 1.17.11).....8

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5. Jim delta 1_14_11 11AM.doc (4 pages, 1.14.11))

In the previous doc in item 3 I did the n=2 Stakgold proof that $\lim_{\varepsilon\to 0}\,[(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] = \delta(r)/r$. I realized this ought to work if you multiply both sides by r to get an n=1 Stakgold theorem situation. This is my first n = 1 derivation and two special issues come up which I explain. First is the factor of 1/2 that you must add to the theorem due to "half a delta function", and second is the way to understand why boundary integral is 2. I put this stuff into a little Appendix.

After Jim read this, he said he didn't believe any of this delta stuff, so I sent the next item.

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6. Stakgold on the delta theorem.doc (7 pages, 1.14.11)

I took photos of two Stakgold pages, then pasted pieces of these pages into a doc interleaved with my comments. Luckily Stak included an exercise which was exactly a case Jim was challenging. I want to put Jim into battle with Stakgold rather than me.

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7. Jim delta 1_18_11.doc (2 pages, 1.18.11)

This is a little wrap-up and thank-you doc sent to Jim on this day. In it I derive both his solutions without any need for the Stakgold theorem math, and I acknowledge his contributions.
