Jim, PhL 1.14.11 1 AM

Here is part of Stakgold's book where he discusses the δ function theorem. He uses α where we were using symbol ε. I first quote one of his exercises quoted below to give you some motivation. This is exactly the case you concocted on the phone today around 11AM :

$$\lim_{\alpha \to 0} \frac{\alpha}{2\pi(r^2 + \alpha^2)^{3/2}} = \delta(x_1, x_2), \qquad r = (x_1^2 + x_2^2)^{1/2}.$$

So then, here is the Theorem in n dimensions for an arbitrary function f(x_1....x_n)

Example 3. As in Section 1.3, we construct a sequence of functions approaching the delta function (but now in *n* dimensions).

Theorem. *Let* $f(x) = f(x_1, \ldots, x_n)$ *be a* nonnegative *locally integrable function for which* $\int_{R_n} f(x)dx = 1$. *With* $\alpha > 0$, *define*

A

$$f_\alpha(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right) = \frac{1}{\alpha^n} f\left(\frac{x_1}{\alpha}, \ldots, \frac{x_n}{\alpha}\right);$$

then

$$\lim_{\alpha \to 0} f_\alpha(x) = \delta(x).$$

where δ(x) at the end means what I call $\delta^{(n)}(\mathbf{x})$. As he will bring up below, in the radial-only case the above becomes

$g_\alpha(r) = \alpha^{-n}\, g(r/\alpha)$

$\lim_{\alpha \to 0} [\alpha^{-n}\, g(r/\alpha)] = \delta^{(n)}(\mathbf{x})$

and the normalization condition becomes

$\int_{\text{sphere}} d^n V \; g(r) = 1 \; = \;$ (angle integral) * $\int_0^\infty dr\, r^{n-1}\, g(r)$

where the angle integral is 2π in 2D, 4π in 3D, and he calls it $S_n(1)$ in nD.

Here is the essence of the proof:

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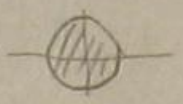
Proof. The substitution $u = x/\alpha$ yields the three properties:

(a) $\int_{R_n} f_\alpha(x)dx = 1;$

(b) $\lim_{\alpha\to 0} \int_{|x|>A} f_\alpha(x)dx = 0$ for each $A > 0;$



(c) $\lim_{\alpha\to 0} \int_{|x|<A} f_\alpha(x)dx = 1$ for each $A > 0.$



You just do what he says, u = x/α, and it is easy to show that each of the three items is true. This must be true for any finite A no matter how small you make A be, so $\lim_{\alpha\to 0} f_\alpha(x) = \delta^{(n)}(\mathbf{x})$. Integral over R_n means over all $dx_1dx_2 \ldots dx_n$ where each variable is -∞ to ∞. He then adds a "technical section" having to do with the convergence of the upper tail end of his integral where he formally uses a "test" function. I don't think this part of the proof is necessary for you and me, but I include it anyway

Our theorem will be proved if it can be shown that, for each test function $\varphi(x)$,

$$\lim_{\alpha \to 0} \int_{R_n} f_\alpha(x)\eta(x)dx = 0, \qquad \text{where } \eta(x) = \varphi(x) - \varphi(0).$$

A

We divide the region of integration R_n into the two parts $|x| < A$ and $|x| > A$, so that

$$\left| \int_{R_n} f_\alpha \eta \, dx \right| \leq \left| \int_{|x|<A} f_\alpha \eta \, dx \right| + \left| \int_{|x|>A} f_\alpha \eta \, dx \right|.$$

Since $\varphi(x)$ is bounded, so is $\eta(x)$; thus $|\eta| \leq M$ for all x. Setting $p(A) = \max_{|x| \leq A} \eta(x)$, and using the nonnegativity of f and property **(a)**, we have

$$\left| \int_{R_n} f_\alpha \eta \, dx \right| \leq p(A) + M\left[\int_{|x|>A} f_\alpha \, dx \right].$$

We want to show that, for each $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$\left| \int_{R_n} f_\alpha \eta \, dx \right| < \varepsilon, \qquad \text{whenever } \alpha < \gamma.$$

ie, thinking $\alpha \to 0$

Since $\eta(x)$ is continuous and $\eta(0) = 0$, $\lim_{A \to 0} p(A) = 0$. We can therefore choose A independently of α such that $p(A) < \varepsilon/2$. With A so chosen, we use property **(b)** to select γ such that

$$\int_{|x|>A} f_\alpha \, dx < \frac{\varepsilon}{2M}, \qquad \text{whenever } \alpha < \gamma.$$

This completes the proof of the theorem.

Now he goes on to specialize his delta theorem to the case that $f(x_1, x_2....) = g(r)$:

Example 4. In the special case where $f(x)$ is a function only of $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, that is, $f(x) = g(r)$, the requirement $\int_{R_n} f(x)dx = 1$ of Example 3 reduces to

$$\int_0^\infty r^{n-1} g(r)dr = \frac{1}{S_n(1)},$$

$$f_\alpha(r) = \frac{1}{\alpha^n} g\left(\frac{r}{\alpha}\right)$$

He then goes on with this g(r/α) situation on the next page of his book,

where $S_n(1)$ is the surface area of the unit n-dimensional sphere [see (5.19), of Exercise 5.1 below].

Since $S_3(1) = 4\pi$, it follows that for any nonnegative function $g(r)$ satisfying

A $$\int_0^\infty r^2 g(r)dr = \frac{1}{4\pi},$$

B $$\lim_{\alpha \to 0} \frac{1}{\alpha^3} g\left(\frac{r}{\alpha}\right) = \delta(x_1, x_2, x_3) = \delta(x),$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

As a particular case we observe that

$$\int_0^\infty r^2 e^{-r^2} dr = \frac{1}{4}\pi^{1/2},$$

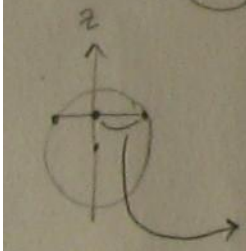
so that

C $$\lim_{\alpha \to 0} \frac{1}{\alpha^3} (\pi)^{-3/2} \exp\left(-\frac{r^2}{\alpha^2}\right) = \delta(x_1, x_2, x_3) = \delta(x).$$

For further examples, see Exercises 5.2 and 5.3.

Now we are going to look at three of his "exercises". He puts lots of useful info into his exercises. The first one is this where he talks about the volume and surface area of a hypersphere in n dimensions. It is best to just accept the results I think. The volume integral is (5.17) and surface integral is (5.18).

EXERCISES



5.1 Let $V_n(r)$ and $S_n(r)$ denote, respectively, the volume and surface area of the n-dimensional sphere of radius r. To obtain a formula for $V_n(r)$ we first relate it to the volume of an $(n-1)$-dimensional sphere as follows. Let z denote a coordinate on an axis through the origin. If A is a constant smaller that r, the intersection of the hyperplane $z = A$ and the n-dimensional sphere is an $(n-1)$-dimensional sphere of radius $(r^2 - z^2)^{1/2}$. (The reader should convince himself of this by drawing appropriate figures for both the three-dimensional and two-dimensional cases.) Thus

$$V_n(r) = \int_{-r}^{r} V_{n-1}[(r^2 - z^2)^{1/2}]dz.$$

fun with hyperspheres

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(3/2) = \sqrt{\pi}/2 = (\frac{1}{2})!$

$\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} = (\frac{3}{2})!$

By induction, show that $V_n(r) = C_n r^n$, where

$$C_{n+1} = 2C_n \int_0^1 (1-u^2)^{n/2}\, du.$$

$(\frac{n}{2})! = \sqrt{\pi}\,\frac{n!!}{2^{(n+1)/2}}$ n odd

Show that

$$\int_0^1 (1-u^2)^{n/2}\, du = \frac{\sqrt{\pi}}{2}\,\frac{(n/2)!}{[(n+1)/2]!},$$

and hence

$\frac{dV_n}{dr} = S_n = \int d\Omega_n$ $(r=1)$ $\qquad V_n(r) = \dfrac{\pi^{n/2} r^n}{(n/2)!}.$ (5.17)

By considering the volume of a thin spherical shell, show that

$$S_n(r) = \frac{n\pi^{n/2} r^{n-1}}{(n/2)!} = \frac{2\pi^{n/2} r^{n-1}}{[(n/2)-1]!}.$$ (5.18)

Using $(1/2)! = \sqrt{\pi}/2$, check that the formulas (5.17) and (5.18) are correct for $n = 2$ and $n = 3$.

If $f(x)$ depends only on r [say, $f(x) = g(r)$], show that

$\int g\,dV = \int g S_n(r)\,dr = \int g\, r^{n-1} S_n(1)\,dr$

$$\int_{R_n} f(x)dx = S_n(1) \int_0^\infty g(r) r^{n-1}\, dr.$$ (5.19)

as in p13 B

The case n = 1 is interesting and has a bit of a trick to it. The sphere of radius a is a line segment (-a,a) of length 2a, which 5.17 confirms. The "surface" area is $S_1(1) = 2$ which 5.18 tells us, you get a "1 from each endpoint" of the segment. An much easier way to see this is as follows:

$\int_{R1} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_{0}^{\infty} dr\, g(r) = 1/2 = 1/S_1(1)$

The trick for n = 1 is the "half a delta function" issue one often runs into, (you only get half a loaf!)

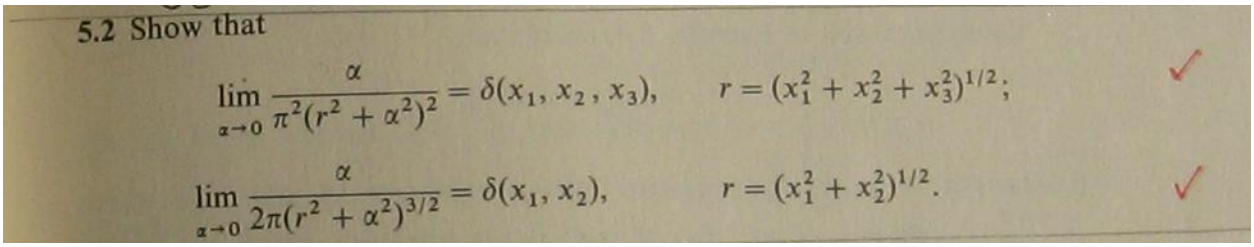
$\lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(x/\alpha)] = \delta(x)$ on the domain x in $(-\infty,\infty)$

but

$\lim_{\alpha \leftrightarrow 0} [\alpha^{-1} g(r/\alpha)] = (1/2)\ "\delta(r)"$ on the domain r in $(0,\infty)$ (*)

if you define the "radial δ(r)" by $\int_{0}^{\infty} dr\ "\delta(r)"\varphi(r) = \varphi(0)$ for test function φ(r) . This is I think what people usually mean when they write δ(r) in a spherical coordinates context, so you get that extra factor of (1/2) shown in (*).

Here then is the next exercise which has two parts:



5.2 Show that

$$\lim_{\alpha \to 0} \frac{\alpha}{\pi^2(r^2+\alpha^2)^2} = \delta(x_1, x_2, x_3), \qquad r = (x_1^2 + x_2^2 + x_3^2)^{1/2};$$

$$\lim_{\alpha \to 0} \frac{\alpha}{2\pi(r^2+\alpha^2)^{3/2}} = \delta(x_1, x_2), \qquad r = (x_1^2 + x_2^2)^{1/2}.$$

The second one above is the one you brought up today on the phone 1.14.11.
And then here is the next exercise:

5.3 From elementary calculus, we have, for $a > 0$,

$$\int_0^\infty e^{-ar^2}\,dr = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}, \qquad \int_0^\infty re^{-ar^2}\,dr = \frac{1}{2a}.$$

Differentiating these expressions repeatedly with respect to a, show that

$$\int_0^\infty r^{2n}e^{-ar^2}\,dr = \left(\frac{\pi}{a}\right)^{1/2} a^{-n}2^{-2n}\frac{(2n-1)!}{(n-1)!},$$

$$\int_0^\infty r^{2n+1}e^{-ar^2}\,dr = \frac{n!}{2a^{n+1}}.$$

Consult Exercise 5.1 to show that, for any positive integer n,

$$\lim_{\alpha\to 0}\frac{1}{\alpha^n}e^{-r^2/\alpha^2} = (\pi)^{n/2}\delta(x_1, \ldots, x_n), \tag{5.20}$$

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$.

$S_1(\alpha) = \frac{1\cdot\sqrt{\pi}\cdot 1}{\sqrt{\pi}/2} = 2$

That's all folks!