The name of this doc is: " jim delta 1.13.11 5PM.doc " . PhL 1.13.11

Jim, here are some of the details the I omitted from the previous doc. I am doing my best here to be detailed and accurate, but I know that I always make mistakes. At least you can show me where (so I put equation numbers this time in some places).

Contents (you can do shift-click on any of these items to go to that section)

0. The Stakgold book theorem says this: **1**

1. Showing that ∂_r^2 (r [1/r]) = – 2π $\delta^{(2)}$(r) **2**

2. Showing that (1/r) ∂_r(r [1/r]) = + 2π $\delta^{(2)}$(r) **3**

3. Confirmation from the 2D Divergence Theorem. **4**

4. 3D Divergence Theorem. Why does this give a different kind of result? **6**

5. How do we apply the Stakgold theorem to the 3D case? **7**

6. Attempt the same 3D problem using 1/(r+ε) regulation (this section added 1.17.11) **8**

0. The Stakgold book theorem says this:

Theorem n: Any non-negative locally integrable function g(r) for which this is true

$$\int_0^\infty dr\, r^{n-1}\, g(r) = 1/S_n(1) \qquad S_n(1) = \text{area of n dimensional sphere}$$

has the property that

$$\lim_{\varepsilon\to 0}\left[\, \varepsilon^{-n}\, g(r/\varepsilon)\,\right] = \delta^{(n)}(\mathbf{r}) \qquad \text{Note: } = \delta^{(n)}(\mathbf{r}) = \delta(x_1)\delta(x_2)....\delta(x_n)$$

Theorem for n=2: Any non-negative locally integrable function g(r) for which this is true

$$\int_0^\infty dr\, r\, g(r) = 1/2\pi$$

has the property that

$$\lim_{\varepsilon\to 0}\left[\, \varepsilon^{-2}\, g(r/\varepsilon)\,\right] = \delta^{(2)}(\mathbf{r}) \qquad \text{Note: } = \delta^{(2)}(\mathbf{r}) = \delta(x)\delta(y)$$

Detail: Function g(r) is "locally integrable" means that $\int_K dr\,|g(r)| < \infty$ where K is any bounded subset of the *open* interval $(0,\infty)$. So for example $g(r) = 1/[r(r^2+1)^{3/2}]$ meets this requirement even though the integral diverges at the lower endpoint. As shown below, the *normalization* integral $\int_0^\infty dr\, r\, g(r)$ is finite.

1. Showing that $\partial_r{}^2$ (r [1/r]) = – 2π $\delta^{(2)}$(r) .

Consider $\partial_r{}^2(r[1/r])$, where we define as a shorthand notation

$$[1/r] \equiv \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2}) \tag{1.1}$$

so the operator is $\partial_r{}^2(r..)$ and the thing operated on is $[1/r] \equiv \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2})$. Then we have

$$\partial_r{}^2 (r\,[1/r]) = \partial_r{}^2 (r \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2})) = \lim_{\varepsilon\to 0}[\,\partial_r{}^2 (r/\sqrt{r^2+\varepsilon^2})\,] \tag{1.2}$$

Maple tells us that

$$\partial_r{}^2 (r/\sqrt{r^2+\varepsilon^2}) = -3\, r\varepsilon^2 / (r^2+\varepsilon^2)^{5/2} \tag{1.3}$$

But we can write

$$r\varepsilon^2 = \varepsilon^3(r/\varepsilon)$$

$$\{(r^2+\varepsilon^2)\}^{5/2} = \{\varepsilon^2\,[(r/\varepsilon)^2+1\,]\}^{5/2} = \varepsilon^5\,[(r/\varepsilon)^2+1]^{5/2}$$

$$\Rightarrow \quad -3\, r\varepsilon^2 / (r^2+\varepsilon^2)^{5/2} = -3\,\varepsilon^3(r/\varepsilon) / (\varepsilon^5\,[(r/\varepsilon)^2+1]^{5/2}) = \varepsilon^{-2}\,\{\,-3(r/\varepsilon)/\,[(r/\varepsilon)^2+1]^{5/2}\,\} \tag{1.4}$$

Therefore, adding a factor of K to both sides, we have shown that

$$K\,\partial_r{}^2 (r/\sqrt{r^2+\varepsilon^2}) = \varepsilon^{-2}\,\{\,-3K(r/\varepsilon)/\,[(r/\varepsilon)^2+1]^{5/2}\,\} \tag{1.5}$$

If we now define

$$g(x) = -3Kx/[(x^2+1)^{5/2}] \tag{1.6}$$

Then the above becomes

$$K\,\partial_r{}^2 (r/\sqrt{r^2+\varepsilon^2}) = \varepsilon^{-2}\, g(r/\varepsilon) \tag{1.7}$$

We now want to find the value of K which provides our desired normalization $\int_0^\infty dx\, x\, g(x) = 1/2\pi$. So

$$(1/2\pi) = \int_0^\infty dx\, x \{ -3Kx/[(x^2+1)^{5/2}] \} = -3K \{ \int_0^\infty dx\, x^2/(x^2+1)^{5/2} \} = -3K \{1/3\} = -K \quad (1.8)$$

where Maple evaluates the integral as shown. Therefore, $K = -(1/2\pi)$. We then have

$$-(1/2\pi)\, \partial_r^2 (r/\sqrt{r^2+\varepsilon^2}) = \varepsilon^{-2}\, g(r/\varepsilon) \quad (1.9)$$

and then according to Stakgold's theorem with n = 2,

$$\lim_{\varepsilon\to 0} [-(1/2\pi)\, \partial_r^2 (r/\sqrt{r^2+\varepsilon^2})] = \lim_{\varepsilon\to 0} [\varepsilon^{-2}\, g(r/\varepsilon)] = \delta^{(2)}(\mathbf{r}) \quad (1.10)$$

which says

$$\lim_{\varepsilon\to 0} [\partial_r^2 (r/\sqrt{r^2+\varepsilon^2})] = -2\pi\, \delta^{(2)}(\mathbf{r}) \quad (1.11)$$

---

2. Showing that $(1/r)\, \partial_r(r\, [1/r]) = +2\pi\, \delta^{(2)}(\mathbf{r})$.

Consider $(1/r)\, \partial_r(r\, [1/r])$, where we define as a shorthand notation

$$[1/r] \equiv \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2}) \quad (2.1)$$

so the operator is $(1/r)\, \partial_r(r..)$ and the thing operated on is $[1/r] \equiv \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2})$. Then we have

$$(1/r)\, \partial_r(r\, [1/r]) = (1/r)\partial_r(r \lim_{\varepsilon\to 0} (1/\sqrt{r^2+\varepsilon^2})) = \lim_{\varepsilon\to 0}[\, (1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2})\,] \quad (2.2)$$

Maple tells us that

$$(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}) = (1/r)\, \varepsilon^2/(r^2+\varepsilon^2)^{3/2} \quad (2.3)$$

But we can write

$$(1/r)\, \varepsilon^2 = \varepsilon\, (\varepsilon/r) = \varepsilon / [\, (r/\varepsilon)\,]$$

$$\{(r^2+\varepsilon^2)\}^{3/2} = \{\varepsilon^2\, [(r/\varepsilon)^2+1\,]\}^{3/2} = \varepsilon^3\, [(r/\varepsilon)^2+1]^{3/2}$$

$$\Rightarrow (1/r)\, \varepsilon^2/(r^2+\varepsilon^2)^{3/2} = \varepsilon / (\, (r/\varepsilon)\, \varepsilon^3\, [(r/\varepsilon)^2+1]^{3/2}) = \varepsilon^{-2}\, \{1/(\, (r/\varepsilon)\, [(r/\varepsilon)^2+1]^{3/2})\, \} \quad (2.4)$$

Therefore, adding a factor of K to both sides, we have shown that

$$K\, (1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}) = \varepsilon^{-2}\, \{ K/[\, (r/\varepsilon)\, [(r/\varepsilon)^2+1]^{3/2}] \} \quad (2.5)$$

If we now define

$$g(x) = K/[x(x^2+1)^{3/2}] \tag{2.6}$$

Then the above becomes

$$K\,(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^{-2}\, g(r/\varepsilon) \tag{2.7}$$

We now want to find the value of K which provides our desired normalization $\int_0^\infty dx\; x\; g(x) = 1/2\pi$. So

$$(1/2\pi) = \int_0^\infty dx\; x\; \{\, K/[x(x^2+1)^{3/2}]\} = K\,\{\int_0^\infty dx\,/\,(x^2+1)^{3/2}\,\} = K\,\{\,1\,\} \tag{2.8}$$

where Maple evaluates the integral as shown. Therefore, $K = (1/2\pi)$. We then have

$$(1/2\pi)\;\; (1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^{-2}\, g(r/\varepsilon) \tag{2.9}$$

and then according to our theorem with n = 2,

$$\lim_{\varepsilon\to 0}\,[(1/2\pi)\;\; (1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] = \lim_{\varepsilon\to 0}\,[\varepsilon^{-2}\, g(r/\varepsilon)] = \delta^{(2)}(\mathbf{r}) \tag{2.10}$$

which says

$$\lim_{\varepsilon\to 0}\,[(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] = +\,2\pi\,\delta^{(2)}(\mathbf{r}) \tag{2.11}$$

---

3. Confirmation from the 2D Divergence Theorem.

In the last two sections we have argued that,

$$\lim_{\varepsilon\to 0}\,[\partial_r^2\,(r/\sqrt{r^2+\varepsilon^2}\,)\,] = -\,2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (3.1) = (1.11)$$

$$\lim_{\varepsilon\to 0}\,[(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] = +\,2\pi\,\delta^{(2)}(\mathbf{r}) \qquad (3.2) = (2.11)$$

Now let's consider the sum of the two objects on the left and see if we can show that they add up to 0 by some other method. In our compact notation we have for this sum

$$\partial_r^2(r\,[1/r]) + (1/r)\partial_r(r\,[1/r]) = [\,\partial_r^2 + (1/r)\partial_r]\,(r\,[1/r]) \tag{3.3}$$

We can write

$$[\,\partial_r^2 + (1/r)\partial_r]\,F(r) = (1/r)\partial_r(r\,\partial_r\,F(r)) = \nabla^2_{2D,\mathrm{radial}}\;F(r) = \nabla^2_{2D}\;F(r) \tag{3.4}$$

Thus, the sum of the two objects on the left above is

$$\partial_r^2(r\,[1/r]) + (1/r)\partial_r(r\,[1/r]) \;=\; \nabla^2(r\,[1/r]) \tag{3.5}$$

where by ∇^2 we mean the full 2D Laplacian

$$\nabla^2 = \;(1/r)\partial_r(r\,\partial_r) + (1/r^2)\partial_\phi^2 \;= \partial_x^2 + \partial_y^2 \tag{3.6}$$

Now since we have ∇^2 floating around, we try to apply the divergence theorem. In N dimensions this says

$$\int_V dV\; \nabla\bullet\mathbf{A} \;=\; \int_S \; d\mathbf{S}\bullet\mathbf{A} \tag{3.7}$$

where V is an N-dimensional volume, and S is an N-1 dimensional surface enclosing that volume. We can apply this theorem to $\mathbf{A} = \nabla\psi(\mathbf{r})$ where we are still in N dimensions. We find that

$$\int_V dV\; \nabla\bullet\nabla\psi = \int_S \; d\mathbf{S}\bullet\nabla\psi \;=\; \int_S (dS\;\hat{\mathbf{n}})\bullet\;\nabla\psi \;=\; \int_S dS\;\partial_n\psi \tag{3.8}$$

where $\hat{\mathbf{n}}$ is normal to the surface patch $d\mathbf{S}$ and ∂_n is the normal derivative. So far then we have

$$\int_V dV\;\nabla^2\psi \;=\; \int_S dS\;\partial_n\psi \qquad\qquad \text{valid in N dimensions} \quad N = 2,3,4..... \tag{3.9}$$

If the volume is a hypersphere in N dimensions this says

$$\int_{sphere}\; dV\;\nabla^2\psi \;=\; \int_S dS\;\partial_r\psi \tag{3.10}$$

since $\hat{\mathbf{r}}$ is now the normal direction. Applying this version of the divergence theorem for N = 2, we find

$$\int_{disk}\; dV\;\nabla^2\psi \;=\; \int_{circle}\; d\boldsymbol{\ell}\;\partial_r\psi \tag{3.11}$$

If the disk has radius "a", then $d\boldsymbol{\ell} = a\,d\theta$ and our theorem says

$$\int_{disk}\; dV\;\nabla^2\psi = a\int d\theta\;\;(\partial_r\psi)|^{r=a} \tag{3.12}$$

Now we apply this to our special function, using our same compact notation,

$$\psi = (r\,[1/r]) \tag{3.13}$$

We know that at r = a, [1/r] = (1/r) since we are away from the problem point r = 0. Therefore

$$(\partial_{\mathbf{r}}\psi)|^{\mathbf{r=a}} = (\partial_{\mathbf{r}}\{\ r\ [1/r]\}\ )|^{\mathbf{r=a}} = (\partial_{\mathbf{r}}\{\ r\ (1/r)\}\ )|^{\mathbf{r=a}} = (\partial_{\mathbf{r}}\{\ 1\}\ )|^{\mathbf{r=a}} = 0 \qquad (3.14)$$

Therefore we have shown that

$$\int_{\mathbf{disk}} dV\ \nabla^2(r\ [1/r]) = 0 \qquad (3.15)$$

We then take the limit of this result as $a \rightarrow 0$ and conclude that

$$\nabla^2(r\ [1/r]) = 0 \qquad (3.16)$$

This is consistent with the sum of equations (3.1) and (3.2) above,

$$\nabla^2(r\ [1/r]) = \partial_{\mathbf{r}}{}^2(r\ [1/r]) + (1/r)\partial_{\mathbf{r}}(r\ [1/r]) = -\ 2\pi\ \delta^{(2)}(\mathbf{r}) + 2\pi\ \delta^{(2)}(\mathbf{r}) = 0$$

---

4. 3D Divergence Theorem. Why does this give a different kind of result?

The claim in 3D is this

$$\nabla_{\mathbf{3D}}([1/r]) = -\ 4\pi\delta^{(3)}(\mathbf{r})$$

The divergence theorem above in N dimensions stated

$$\int dV\ \nabla^2\psi = \int_{\mathbf{S}} dS\ \partial_{\mathbf{n}}\psi$$

For N = 3 dimensions, and for the volume being a sphere of radius "a", this becomes

$$\int dV\ \nabla^2\psi = \int_{\mathbf{S}} a^2 d\Omega\ (\partial_{\mathbf{r}}\psi)|^{\mathbf{r=a}}$$

If $\psi = \psi(r)$ only, this becomes

$$\int dV\ \nabla^2\psi = a^2 \int d\Omega\ (\partial_{\mathbf{r}}\psi)|^{\mathbf{r=a}} = 4\pi\ a^2\ (\partial_{\mathbf{r}}\psi)|^{\mathbf{r=a}}$$

We apply this now to $\psi = [1/r]$ to get

$$\int dV\ \nabla^2[1/r] = 4\pi\ a^2\ (\partial_{\mathbf{r}}[1/r])|^{\mathbf{r=a}} = 4\pi\ a^2\ (\partial_{\mathbf{r}}(1/r))|^{\mathbf{r=a}} = 4\pi a^2(-1/r^2)\ |^{\mathbf{r=a}} = -4\pi$$

Therefore we have shown that

$$\int dV\ \nabla^2[1/r] = -4\pi$$

We conclude taking the limit a→ 0 that

$$\nabla^2[1/r] = -4\pi\ \delta^{(3)}(\mathbf{r})$$

Comparison:

$$\nabla^2_{3D}([1/r]) = -4\pi\ \delta^{(3)}(\mathbf{r})$$

$$\nabla^2_{2D}(r\ [1/r]) = 0$$

---

5. How do we apply the Stakgold theorem to the 3D case?

For n = 3 the theorem says

$$\int_0^\infty dr\ r^2\ g(r) = 1/4\pi \qquad \text{// normalization integral}$$

$$\lim_{\varepsilon\to 0}\ [\ \varepsilon^{-3}\ g(r/\varepsilon)\ ] = \delta^{(3)}(\mathbf{r}) \qquad \text{// delta function generator}$$

Consider then

$$\nabla^2_{3D}([1/r]) = (1/r^2)\partial_r(r^2\ \partial_r\ [1/r]\ )$$

As usual the function acted upon is this

$$[1/r] \equiv \lim_{\varepsilon\to 0}\ (1/\sqrt{r^2+\varepsilon^2}\ )$$

So we consider

$$\nabla^2_{3D}([1/r]) = \nabla^2[\lim_{\varepsilon\to 0}\ (1/\sqrt{r^2+\varepsilon^2}\ )] = \lim_{\varepsilon\to 0}\ (1/r^2)\partial_r[\ r^2\ \partial_r(1/\sqrt{r^2+\varepsilon^2}\ )\ ]$$

where for the rest of this section, ∇^2 refers to the full 3D Laplacian which in sphericals is this

$$\nabla^2 = (1/r^2)\partial_r(r^2\ \partial_r\ + (1/r^2)\{\ \csc\theta\partial_\theta(\sin\theta\partial_\theta)\ +\ \csc^2\theta\ \partial_\varphi{}^2\ \}$$

As usual, we call upon Maple to compute

$$(1/r^2)\partial_r[\ r^2\ \partial_r(1/\sqrt{r^2+\varepsilon^2}\ )\ ] = -3\varepsilon^2\ /\ (r^2+\varepsilon^2)^{5/2}$$

But we can write (as we did below equation 1.3 above)

$$\{(r^2+\varepsilon^2)\}^{5/2} = \{\varepsilon^2 [(r/\varepsilon)^2+1]\}^{5/2} = \varepsilon^5 [(r/\varepsilon)^2+1]^{5/2}$$

so we get (again, adding constant K to both sides)

$$K(1/r^2)\partial_r[r^2 \partial_r(1/\sqrt{r^2+\varepsilon^2})] = \varepsilon^{-3} \{ -3K/ [(r/\varepsilon)^2+1]^{5/2} \}$$

If we now define (now there is no numerator factor x in g(x) as there was in 1.6)

$$g(x) = -3K/[(x^2+1)^{5/2}]$$

then the above becomes

$$K(1/r^2)\partial_r[r^2 \partial_r(1/\sqrt{r^2+\varepsilon^2})] = \varepsilon^{-3} g(r/\varepsilon)$$

We now want to find the value of K which provides our desired normalization $\int_0^\infty dx\, x^2 g(x) = 1/4\pi$. So

$$(1/4\pi) = \int_0^\infty dx\, x^2 \{ -3K/[(x^2+1)^{5/2}] \} = -3K \{ \int_0^\infty dx\, x^2 / (x^2+1)^{5/2} \} = -3K \{1/3\} = -K$$

and this in fact is the same integral we encountered in (1.8) above. Therefore K = - (1/4π) and we have then shown that

$$-(1/4\pi)(1/r^2)\partial_r[r^2 \partial_r(1/\sqrt{r^2+\varepsilon^2})] = \varepsilon^{-3} g(r/\varepsilon)$$

and then according to our theorem with n = 3,

$$\lim_{\varepsilon\to 0} [-(1/4\pi)(1/r^2)\partial_r[r^2 \partial_r(1/\sqrt{r^2+\varepsilon^2})] = \lim_{\varepsilon\to 0} [\varepsilon^{-3} g(r/\varepsilon)] = \delta^{(3)}(\mathbf{r})$$

which says

$$\lim_{\varepsilon\to 0} [(1/r^2)\partial_r[r^2 \partial_r(1/\sqrt{r^2+\varepsilon^2})] = \lim_{\varepsilon\to 0} [\varepsilon^{-3} g(r/\varepsilon)] = -4\pi \delta^{(3)}(\mathbf{r})$$

so we have then shown, using Stakgold's little theorem, that

$$\nabla^2_{3D}([1/r]) = (1/r^2)\partial_r[r^2 \partial_r([1/r])] = -4\pi \delta^{(3)}(\mathbf{r})$$

6. Attempt the same 3D problem using 1/(r+ε) regulation (this section added 1.17.11)

This will be a copy, paste and edit from the preceding section. For n = 3 the theorem says

$$\int_0^\infty dr\, r^2 g(r) = 1/4\pi \qquad \text{// normalization integral}$$

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-3}\,g(r/\varepsilon)\,]\;=\;\delta^{(3)}(\mathbf{r})$ // delta function generator

Consider then

$\nabla^2_{3D}([1/r]) = (1/r^2)\partial_r(r^2\,\partial_r\,[1/r]\,)$

Now assume the function acted upon is this

$[1/r]\equiv\lim_{\varepsilon\to 0}(1/(r+\varepsilon))$

So we consider

$\nabla^2_{3D}([1/r]) = \nabla^2[\lim_{\varepsilon\to 0}(1/(r+\varepsilon))] = \lim_{\varepsilon\to 0}\;(1/r^2)\partial_r[\,r^2\,\partial_r(1/(r+\varepsilon))\,]$

where for the rest of this section, ∇^2 refers to the full 3D Laplacian which in sphericals is this

$\nabla^2 = (1/r^2)\partial_r(r^2\,\partial_r\; + (1/r^2)\{\,\csc\theta\partial_\theta(\sin\theta\partial_\theta)\;+\;\csc^2\theta\,\partial_\varphi{}^2\,\}$

As usual, we call upon Maple to compute [for visual comparison: $(1/r^2)\partial_r(r^2\,\partial_r\,[1/r]\,)$]

```
(1/r^2)*Diff(r^2*Diff(1/(r+epsilon),r),r);
```

$$\frac{\frac{\partial}{\partial r}r^2\left(\frac{\partial}{\partial r}\frac{1}{r+\varepsilon}\right)}{r^2}$$

```
(1/r^2)*diff(r^2*diff(1/(r+epsilon),r),r): simplify(%),
```

$$-2\frac{\varepsilon}{r(r+\varepsilon)^3}$$

which tells us that

$(1/r^2)\partial_r[\,r^2\,\partial_r(1/(r+\varepsilon))\,]\;=\;-2\varepsilon\,/\,[r(r+\varepsilon)^3]\;=\;-2\varepsilon\,/\,[\,\{[r/\,\varepsilon]\varepsilon\}\{\;\;\varepsilon^3([r/\,\varepsilon]+1)^3\,\}]$

$=\;-2\varepsilon^{-3}\,/\,[\,\{[r/\,\varepsilon]\}\{\,([r/\,\varepsilon]+1)^3\,\}]\;=\;-2\varepsilon^{-3}\,/\,(\,[r/\,\varepsilon]\,([r/\,\varepsilon]+1)^3]\,)$

Therefore we have

$-K(1/r^2)\partial_r[\,r^2\,\partial_r(1/(r+\varepsilon))\,]\;\;=\;\varepsilon^{-3}\,\{\,2K\,g(r/\,\varepsilon)\,\}$ where $g(x) = 2K\,x/(x+1)^3 \sim 1/x^2$

$-K(1/r^2)\partial_r[\,r^2\,\partial_r(1/\sqrt{r^2+\varepsilon^2}\,)\,]\;\;=\varepsilon^{-3}\,\{\,3K/\,[(r/\varepsilon)^2+1]^{\,5/2}\,\}$ where $g(x) = 3K\,/(x^2+1)^{5/2} \sim 1/x^5$

where the second line in blue is the result from the preceding section, put there for comparison. Notice that the r^2 in $\sqrt{r^2+\varepsilon^2}$ causes a much greater denominator power at large x.

We now want to find the value of K which provides our desired normalization $\int_0^\infty dx\, x^2\, g(x) = 1/4\pi$. So

$$(1/4\pi) = \int_0^\infty dx\, x^2 \{2K\, x/(x+1)^3] \} = 2K \int_0^\infty dx\, x^2\, x/(x+1)^3] \sim \int_0^\infty dx\, 1 = \text{linearly divergent}$$

$$(1/4\pi) = \int_0^\infty dx\, x^2 \{ -3K/[(x^2+1)^{5/2}] \} = -3K \{ \int_0^\infty dx\, x^2 / (x^2+1)^{5/2} \} = -3K \{1/3\} = -K$$

where on the second line I show in blue the convergent corresponding line from the previous section.

Conclusion: This method of regulation fails! Therefore

$$\text{constant} * \lim_{\varepsilon\to 0} [(1/r^2)\partial_r[\, r^2\, \partial_r(1/\sqrt{r^2+\varepsilon^2}\,)] = \delta^{(3)}(\mathbf{r}) = \delta(r)/[4\pi r^2]$$

$$\text{constant}' * \lim_{\varepsilon\to 0} [(1/r^2)\partial_r[\, r^2\, \partial_r(1/(r+\varepsilon))] >> \delta^{(3)}(\mathbf{r}) = \delta(r)/[4\pi r^2]$$

The second sequence is "too big" in that it produces something whose 3D integral instead of being 1 is in fact infinite. If we multiply the above by r^2 on both sides we restate as

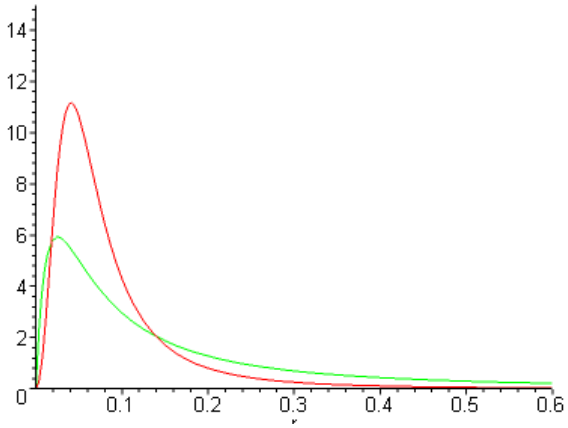
$$\text{constant}'' * \lim_{\varepsilon\to 0} [\partial_r[\, r^2\, \partial_r(1/\sqrt{r^2+\varepsilon^2}\,)] = \delta(r)$$

$$\text{constant}''' * \lim_{\varepsilon\to 0} [\partial_r[\, r^2\, \partial_r(1/(r+\varepsilon))] >> \delta(r)$$

Here is a plot of $\partial_r(r^2\, \partial_r\, [1/r]\,)$ for the three cases

	black	green	red
$[1/r]$ =	$(1/r)$	$1/\sqrt{r^2+\varepsilon^2}$	$1/(r+\varepsilon)$

```
epsilon := .05:
f1 := -diff(r^2*diff(1/r,r),r):
f2 := -diff(r^2*diff(1/sqrt(r^2+epsilon^2),r),r):
f3 := -diff(r^2*diff(1/(r+epsilon),r),r):
plot([f1,f2,f3],r=0..0.6, color = [black,red,green], view=[0..1,0..15]);
```



The black curve is just 0 everywhere since $\partial_r[\, r^2\, \partial_r(1/(r))] = 0$. The green curve is approaching a delta function, while the red curve is approaching something that looks like a delta function, but whose integral is becoming infinite in the limit $\varepsilon \to 0$. The green curve has a finite integral in this limit. Notice that the limit here is the "radial $\delta(r)$" discussed elsewhere which exists only for $r \geq 0$ since r is a radial coordinate, and for which

$$\int_0^\infty dr\ \delta(r)\ f(r) = f(0)$$

As $\varepsilon \to 0$, the red and green curves jam themselves against the wall r = 0. All three curves have the value 0 at the location r = 0:

```
r := 0; f1;f2;f3;
```

$$r := 0$$
$$0$$
$$0$$
$$0$$

Here is the same plot for $\varepsilon = .01$:

