Jim,

1. I think you proposed by phone that a solution to $L_{xy}(1/r) = 2\pi\,\delta^{(2)}(\mathbf{r})$ might be this

$$L_{xy}([1/r]) = (1/\sqrt{x^2+y^2})\ (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

But dimensionally this says $L^{-3} = L^{-2}$, so I think what you intended was:

$$L_{xy}([1/r]) = (\partial_x^2+\partial_y^2)(\sqrt{x^2+y^2}\,([1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

2. Assuming this is what you meant, you are then claiming this (perhaps apart from a constant)

$$\nabla^2_{2D}(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

where $\nabla^2_{2D} = \partial_x^2+\partial_y^2$ is the 2D Laplacian. In *cylindrical* coordinates we know that

$$\nabla_{3D}^2 = r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2 + \partial_z^2$$

So for *polar* coordinates r and φ it must be true that

$$\nabla_{2D}^2 = \partial_x^2+\partial_y^2 = r^{-1}\partial_r(r\,\partial_r) + r^{-2}\partial_\varphi^2$$

Therefore

$$\nabla^2_{2D}F(r) = r^{-1}\partial_r(r\,\partial_r F(r)) = r^{-1}(r\,\partial_r^2F(r) + \partial_rF(r)) = \partial_r^2F(r) + r^{-1}\,\partial_rF(r)$$

Let's take F(r) = r[1/r] so this then says

$$\nabla^2_{2D}(r[1/r]) = \partial_r^2(r[1/r]) + (1/r)\,\partial_r(r[1/r])$$

Thus, a new term has been added to your original proposal which was just $\partial_r^2(r[1/r])$.

3. Using my Part 2 methods of the last doc, I was able to show these interesting facts:

$$\lim_{\varepsilon\to 0}\ \partial_r^2(r[1/\sqrt{r^2+\varepsilon^2}\,]) = -2\pi\,\delta^{(2)}(\mathbf{r})$$
$$\lim_{\varepsilon\to 0}\ \{(1/r)\partial_r(r\,[1/\sqrt{r^2+\varepsilon^2}\,])\} = +2\pi\,\delta^{(2)}(\mathbf{r})$$

After doing this, I realized that it was obvious from the divergence theorem applied to $\nabla\psi$,

$$\int_V dV\,\nabla^2\psi = \int_S dS\,\partial_n\psi$$

which applied to a 2D disk of radius a says

$$\int_0^a r dr \int_0^{2\pi} d\theta\, \nabla^2\psi = a \int_0^{\infty} d\theta\, \partial_r\psi$$

Setting $\psi = r\,[1/r]$ we get

$$\int_0^a r dr \int_0^{2\pi} d\theta\, \nabla^2(r\,[1/r]) = a \int_0^{\infty} d\theta\, \partial_r(r\,[1/r])|^{r=a} = a \int_0^{\infty} d\theta\, \partial_r(1) = 0$$

So that

$$\nabla^2(r\,[1/r]) = 0$$

4. So the bad news is that your two terms exactly cancel so that

$$L_{xy}\,(1/r) = \nabla^2_{2D}\,(r[1/r]) = -2\pi\,\delta^{(2)}(\mathbf{r}) + 2\pi\,\delta^{(2)}(\mathbf{r}) = 0$$

The good news is that we now have *two* different solutions to the problem which are these

$$-\partial_r^2(r\,[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$
$$+(1/r)\partial_r(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) = \delta(r)/r$$

which correspond to (Maple) // this stuff is wrong, see Question #2 doc review

$$L_{xy}^{(1)} = -(1/\sqrt{x^2+y^2})\,[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\partial_x + 2y\partial_y\,]$$
$$L_{xy}^{(2)} = +(1/\sqrt{x^2+y^2}\,)[\,x^2(\partial_x^2+\partial_y^2) + y^2(\partial_x^2+\partial_y^2) + 2x\partial_x + 2y\partial_y]$$

and any linear combination must also be viable. If we add these two we get, for example,

$$L_{xy}^{(3)} = +(1/\sqrt{x^2+y^2}\,)[\,x^2\partial_y^2 + y^2\partial_x^2 - 2xy\partial_x\partial_y] = (1/\sqrt{x^2+y^2}\,)\;(x\partial_y - y\partial_x)^2$$

But since $\partial_\varphi = (\partial_\varphi y)\partial_y + (\partial_\varphi x)\partial_x = (x\partial_y - y\partial_x)$, we get $L_{xy}^{(3)} = (1/r)\,\partial_\varphi^2$ and this is just the kind of term we know we can add in since $L_{xy}^{(3}\,F(r) = 0$.