the doc is: jim delta 1_14_11 11AM.doc

Jim,

Here is some more evidence for

$$\lim_{\varepsilon\to 0}\,[(1/r)\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] \;= +2\pi\,\delta^{(2)}(\mathbf{r}) \;= +2\pi[\,\delta(r)/2\pi r] \;= \delta(r)/r$$

Suppose we multiply both sides by r. Then this ought to be true

$$\lim_{\varepsilon\to 0}\,[\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)\,] \;= \delta(r) \qquad (1)$$

and this is then an n=1 case of Stakgold's theorem.

Let's see if we can verify equation (1) . We first do the usual Maple calculation to find that

$$\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^2/(r^2+\varepsilon^2)^{3/2} \;= \varepsilon^2/[\,\varepsilon^2\,((r/\varepsilon)^2+1)]^{3/2} \;= \varepsilon^2/[\,\varepsilon^3\,((r/\varepsilon)^2+1)^{3/2}]$$

$$= \varepsilon^{-1}/\,((r/\varepsilon)^2+1)^{3/2}$$

Therefore, adding a factor of K to both sides, we have shown that

$$K\,\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) \;= \varepsilon^{-1}\,\{\,K/\,((r/\varepsilon)^2+1)^{3/2}\,)$$

So define

$$g(x) = K/(x^2+1)^{3/2}$$

and we then have

$$K\,\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^{-1}\,g(r/\varepsilon)$$

Stakgold's theorem for general n=1 says this (see Appendix below)

$$\int_0^\infty dr\; g(r) = 1/2 \qquad S_n(1) = \text{"area of 1 dimensional sphere"} = 2$$

$$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(x/\varepsilon)\,] \;= \;\delta(x) \qquad \text{on } R_1 \text{ which is the real axis x in } (-\infty,\infty)$$

$$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(r/\varepsilon)\,] \;= \;(1/2)\delta(r) \qquad \text{for r in } (0,\infty) \qquad \text{// See Appendix}$$

We determine K from the above normalization condition,

$$1/2 = \int_0^\infty dr\; g(r) \;= \int_0^\infty dr\;\{\,K/(x^2+1)^{3/2}\} = \;K\;\int_0^\infty dr/\,(r^2+1)^{3/2} \;= K * \;1 = K$$

which then says K = 1/2, so we then have

$$(1/2)\,\partial_r(r/\sqrt{r^2+\varepsilon^2}\,) = \varepsilon^{-1}\, g(r/\varepsilon)$$

$$\lim_{\varepsilon\to 0}\,[(1/2)\,\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)] = \lim_{\varepsilon\to 0}[\varepsilon^{-1}\, g(r/\varepsilon)] \;=\; (1/2)\delta(r)$$

Therefore we have shown that

$$\lim_{\varepsilon\to 0}\,[\,\partial_r(r/\sqrt{r^2+\varepsilon^2}\,)] \;=\; \delta(r)$$

which is the claim of equation (1) above.

Appendix on Stakgold's Theorem for n = 1

Stakgold's theorem for n = n says

$\int_0^\infty dr\, r^{n-1}\, g(r) = 1/S_n(1)$ $\quad$ $S_n(1)$ = area of n dimensional sphere

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-n}\, g(r/\varepsilon)\,] \;=\; \delta^{(n)}(\mathbf{r})$ $\quad$ on Euclidean space R_n

Stakgold's theorem for n = 1 says

$\int_0^\infty dr\, g(r) = 1/2$ $\quad$ $S_n(1)$ = "area of 1 dimensional sphere" = 2

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\, g(x/\varepsilon)\,] \;=\; \delta(x)$ $\quad$ on R_1 which is the real axis x in $(-\infty,\infty)$

(a) First of all, the normalization 1/2 arises simply in this manner

$$\int_{R1} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

If f(x) = f(|x|) = g(r), this becomes

$$1 = \int_{-\infty}^{\infty} f(|x|)\,dx = \;= 2\int_0^\infty f(x)\,dx \;= 2\int_0^\infty g(r)\,dr$$

so we get

$$\int_0^\infty g(r)\,dr = 1/2$$

Thus, we don't have to worry about interpreting "the area of a 1 dimensional sphere".

(b) Second of all, we need to modify the conclusion of the n=1 theorem in this way

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(x/\varepsilon)\,]\;=\;\delta(x)$ x in the range $(-\infty,\infty)$

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(r/\varepsilon)\,]\;=\;(1/2)\,\delta(r)$ r in the range $(0,\infty)$

Here is an explanation. In the first case we have, for a test function $\varphi(x)$

$$\int_{-a}^{a} dx\;\varphi(x)\;\delta(x) = \varphi(0)$$

and we imagine this as (think of $\delta(x)$ as a thin gaussian with two halves of equal area)

$$\int_{-a}^{a} dx\;\varphi(x)\;\{\;\delta_{\mathtt{lefthalf}}(x)\theta(-x)\;+\delta_{\mathtt{righthalf}}(x)\theta(x)\;) = \varphi(0)$$

Then we would claim that in the following integral only the right half contributes, so

$$\int_{0}^{a} dx\;\varphi(x)\;\delta(x)\;=\;\int_{0}^{a} dx\;\varphi(x)\;\delta_{\mathtt{righthalf}}(x)\;=(1/2)\;\varphi(0) \qquad (2)$$

But people usually think of a radial delta function as doing this

$$\int_{0}^{a} dr\;\varphi(r)\;\text{"}\delta(r)\text{"}\;=\varphi(0) \qquad (3)$$

Multiply both sides of (3) by 1/2 to get

$$\int_{0}^{a} dr\;\varphi(r)\;(1/2)\;\text{"}\delta(r)\text{"}\;=(1/2)\;\varphi(0) \qquad (4)$$

Comparing (2) and (4) shows that

$$(1/2)\;\text{"}\delta(r)\text{"}\;=\delta(x)$$

So our Stakgold theorem says

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(r/\varepsilon)\,]\;=\;\delta(x) = (1/2)\;\text{"}\delta(r)\text{"}$

But in the radial coordinate context for r in $(0,\infty)$, nobody writes a double quote. They instead say

$\int_{0}^{a} dr\;\varphi(r)\;\delta(r)\;=\varphi(0)$ // defines what is meant by the $\delta(r)$ symbol in this context

and so the Stakgold n=1 theorem says this

$\lim_{\varepsilon\to 0}\,[\,\varepsilon^{-1}\,g(r/\varepsilon)\,]\;=(1/2)\;\delta(r)$