Jim,

I'm sure you are pretty bored by now with the problem I proposed, but here is a simple way to verify the two solutions you came up with, where there is no need for Stakgold's Theorem and all that stuff.

If you don't want to look at equations, at least jump down to item 3.

1. Use this shorthand notation:

$$[1/r] \equiv \lim_{\varepsilon \to 0} 1/\sqrt{r^2 + \varepsilon^2} \ . \tag{0}$$

We have this 3D known fact that

$$-\nabla^2_{3D,radial}([1/r]) = -r^{-2}\partial_r(r^2\partial_r[1/r]) = 4\pi\,\delta^{(3)}(\mathbf{r}) \tag{1}$$

and we know that $\delta^{(3)}(\mathbf{r}) = \delta(r)/[4\pi r^2]$ so the above says

$$-r^{-2}\partial_r(r^2\partial_r[1/r]) = \delta(r)/r^2 \tag{2}$$

Multiply both sides by r

$$-r^{-1}\partial_r(r^2\partial_r[1/r]) = \delta(r)/r \tag{3}$$

Use the fact that $\delta^{(2)}(\mathbf{r}) = \delta(r)/[2\pi r]$ so the above says

$$-r^{-1}\partial_r(r^2\partial_r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{4}$$

Now use "Jim's Theorem" which says this

$$r^{-2}\partial_r(r^2\partial_r f) = r^{-1}\partial_r^2(rf) \qquad // \ = 2r^{-1}\partial_r f + \partial_r^2 f \tag{5}$$

and multiply it by r to get

$$r^{-1}\partial_r(r^2\partial_r f) = \partial_r^2(rf) \tag{6}$$

and install this into (4) to get

$$-\partial_r^2(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{7}$$

and this then gives **your first solution** to my posed problem $L([1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r})$

2. **Your second solution** is this:

$$(1/r)\partial_r(r[1/r]) = 2\pi\,\delta^{(2)}(\mathbf{r}) \tag{8}$$

If this is correct, then if we subtract (7) from (8) we should get zero,

$$(1/r)\partial_r(r[1/r]) + \partial_r^2(r[1/r]) = 0 \qquad (9)$$

We shall now show that this is in fact zero and in this way we verify your second solution.
For the first step we have Maple show this fact,

$$(1/r)\partial_r(rf(r)) + \partial_r^2(rf(r)) = (1/r)\,[\, r^2\partial_r^2 + 3r\partial_r + 1]f(r) \qquad (10)$$

If we take f(r) = (1/r), this thing is trivially true:

$$[\, r^2\partial_r^2 + 3r\partial_r + 1](1/r) = 2/r - 3/r + 1/r = 0 \qquad (11)$$

Not surprisingly, if we take f(r) = [1/r], it is still true. Maple gives

$$[\, r^2\partial_r^2 + 3r\partial_r + 1][1/r] = -\varepsilon^2\ (2r^2-\varepsilon^2)/(r^2+\varepsilon^2)^{5/2} \rightarrow \varepsilon^2\ 2/r^3 \rightarrow 0 \qquad (12)$$

Setting f(r) = [1/r] in (10) and using (12) we conclude that

$$(1/r)\partial_r(r[1/r]) + \partial_r^2(r[1/r]) = 0$$

which is the fact (9) we sought to show, and your second solution is then verified.

3. I wish to give **proper acknowledgement to Jim** for solving this problem and, in doing so, pointing out these useful facts:

(a) The fact that $\delta^{(n)}(\mathbf{r}) = \delta(r)/[S_n(1)r^{n-1}]$ where $S_n(1)$ is the surface area of a sphere in n dimensions.

(b) That the proper way to regulate 1/r is to say $[1/r] \equiv \lim_{\varepsilon\rightarrow 0} 1/\sqrt{r^2+\varepsilon^2}$. Other methods I tried did not work, such as $[1/r] \equiv \lim_{\varepsilon\rightarrow 0} 1/(r+\varepsilon)$ or $[1/r] \equiv \lim_{\varepsilon\rightarrow 0} 1/r^{1+\varepsilon}$.

(c) The fact that $\partial_r(r^2\partial_r f) = r^{-1}\partial_r^2(rf)$ which I used above.

(d) for coming up with both solutions noted above. When I asked Jim about this problem, I had no solutions at all.