If this Word doc is a problem for you (I did embed the fonts) let me know and I will make it a PDF and resend it. I like Word because perhaps you can just edit in your responses in-line, something like this

> And therefore, we obtain: $L_x\ g(x) = [\ \Sigma_{n=0}^{\infty}\ \varphi_n(x)^*\varphi_n(0)[\partial_x^n g(x)]^{-1}\partial_x^n\ ]\ g(x)$
> are you outta yer forkin' mind?

It might be good to do a full read through before inserting comments.

Richard,

This is all certainly not a rush matter, so please ignore me until you have enough time to fully enjoy my confusion on the various issues discussed below. For that reason, I have felt at ease in rambling on a bit. I know that if you have some free time, you will at least find this entertaining. I imagine there might be some integral equations floating around in black holes where issues like these might arise.

1. Motivation. **1**
2. The 1D situation. **2**
3. The 3D situation. **4**
4. A particular problem in 2D **4**

1. Motivation. Maybe I should explain the motivation behind my strange set of questions. Suppose you have a first-kind Fredholm integral equation f = Kg which you want to solve for g, given f,

$$f(x) = \int dy\ k(x,y)g(y) \qquad\qquad f = Kg$$

where we are in n dimensional R_n space, say. Then you might wonder if, given that kernel k(x,y), there might be some differential operator L_x such that

$$L_x\ k(x,y) = \delta(x-y) \qquad \text{// meaning } \delta^{(n)}(\mathbf{x}-\mathbf{y})$$

so that you could then perhaps have an associated differential equation

$$L_x f(x) = \int dy\ [\ L_x\ k(x,y)\ ]\ g(y) = \int dy\ \delta(x-y)\ g(y) = g(x) \qquad\qquad g = Lf$$

where I sweep technical issues under the rug, such as integration endpoints and order interchange. In the operator sense then you are saying f = Kg with solution g = Lf and then LK = 1 and maybe even $L = K^{-1}$ with all the usual caveats about domains and ranges and concerns about operators in infinite dimensional Hilbert Spaces.

[By the way, my knowledge on this subject (such as it is) comes from an old 2 volume book set by Ivar Stakgold called Boundary Value Problems of Mathematical Physics, 1967. It is pretty good, but only after you stare at it for a while.]

So, I was wondering: for which kernels k(x,y) can I find L_x such that $L_x\, k(x,y) = \delta(x-y)$? Linguistically, I guess that is the same as saying which kernels are Green's Functions of some differential operator? More on this below.

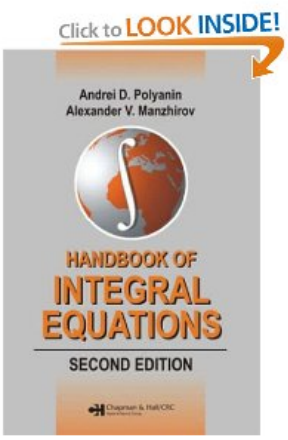
The motivation of course is that differential equations seem a lot easier to solve than integral equations. For the simple 1D string on (0,1) the integral equation with the usual Green's Function is this

$$\int_0^1 d\xi\, u(\xi)\, [\, \theta(x<\xi)x(1-\xi) + \theta(x>\xi)\xi(1-x)\,] \;= \mu\, u(x) \qquad\qquad Ku = \mu u$$

but it is not at all obvious (to me at least) that the solutions are $u(x) = \varphi_n(x) = \sin(n\pi x)$ with $1/\mu = \lambda_n = (n\pi)^2$, although it is easy to verify that this is correct. If you apply $L_x = \partial_x^2$ to make this be an ODE then the solutions are obvious because we are trained to recognize differential equations, not integral equations. And I guess a differential equation is "local" and is therefore intrinsically simpler to solve than an integral equation which is "global" over your interval or region of R^n. On the other hand, the integral equation is probably better in terms of operator properties and numerical work.

Aside: On this last subject, here is an unusual 2008 Russian book (1143 pages) which I luckily found as a pdf file floating around somewhere. It is organized just like Gradshteyn and Ryzhik, but it has (1D) integral equations instead of integrals, and even has some dual integral equation pairs. It has Fredholm first-kind, second-kind, and each with fixed endpoints or one-variable endpoint. And it has lots of nonlinear integral equations as well. You can see the entire table of contents on "look inside" here:

http://www.amazon.com/Handbook-Integral-Equations-Handbooks-Mathematical/dp/1584885076



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2. The 1D situation. In 1D and for a second-order differential operator one would argue as you did that a candidate kernel has to have a slope discontinuity at x = y. The usual argument might be this:

$$\{ a(x)\partial_x^2 + b(x)\partial_x + c(x) \}\ k(x,y) = \delta(x-y)$$

Integrate from y-ε to y+ε and assume that all coefficient functions of L_x are continuous at y, and that k and ∂_x k(x,y) are also continuous at y, so then

$$a(y) \int_{y-\varepsilon}^{y+\varepsilon} dx\, \partial_x[\partial_x g] + 0 + 0 = \int_{y-\varepsilon}^{y+\varepsilon} dx\, \delta(x-y) = 1$$

$$\Rightarrow\ [\partial_x g]|^{y+\varepsilon}_{y-\varepsilon} = 1/a(y)$$ = the jump in the first derivative.

For order-n L_x with continuous coefficient functions I guess a viable candidate kernel has to have k and all the derivatives $\partial_x^k g(x)$ for k < n be continuous. Then $[\partial_x^{n-1} g]|^{y+\varepsilon}_{y-\varepsilon} = 1/a(y)$ is the jump condition. But there might be other ways to make this work.

For an infinite order L_x [example $L_x = \exp(\alpha\partial_x)$] it is perhaps less clear what conditions k(x,y) has to have. Maybe all derivatives of k(x,y) have to be continuous, so it would be C^∞. If it happens that we can write k(x,y) = g(x-y) then an example kernel would be $g(x) = (\kappa/\pi)\ /(\kappa^2+x^2)$ which is the kernel of the Love integral equation for the parallel disk capacitor with plate separation κ.

So this is what led me to my wacko mystical L_x . The problem is: given g(x), find L_x such that

$$L_x\ g(x) = \delta(x)$$

and here was my contrived solution,

$$L_x\ g(x) = [\ \Sigma_{n=0}^{\infty}\ \varphi_n(x)*\varphi_n(0)[\partial_x^n g(x)]^{-1}\partial_x^n\ ]\ g(x) = \Sigma_{n=0}^{\infty}\ \varphi_n(x)*\varphi_n(0) = \delta(x)$$

since

$$\Sigma_{n=0}^{\infty}\varphi_n(x)*\varphi_n(x') = \delta(x-x')$$

is the completeness relation for some complete set of orthonormal functions $\varphi_n(x)$ (perhaps $P_n(x)$). I agree such a thing is pretty useless, I was just wondering if "in theory" for a given g(x) there was an L_x such that $L_x\ g(x) = \delta(x)$. I agree, it is probably not easy to solve an infinite order ODE (but I am not completely sure).

So barring this kind of L_x operator, I guess I *cannot* convert the Love integral equation

$$f(x) - 1 = -\int_{-1}^{1} dt\, r(x-t)\, f(t) \qquad r(x) = (\kappa/\pi) / [\ \kappa^2 + x^2],$$

into some kind of eigenvalue differential equation.

In general I have been wondering "which Fredholm integral equations can, and which cannot, be associated with reasonable differential equations? "

3. The 3D situation. Well, we all know about

$$-\nabla^2(1/|\mathbf{r}-\mathbf{r}'|) = 4\pi\,\delta^3(\mathbf{r}-\mathbf{r}') \qquad \text{or with } \mathbf{r}'=0: \qquad -\nabla^2(1/r) = 4\pi\,\delta^3(\mathbf{r}) = \delta(r)/r^2$$

The jump condition integral is now an integral over a tiny sphere and one proves this trivially with the divergence theorem. It is a little hard for me to say what conditions $k(\mathbf{r},\mathbf{r}') = 1/|\mathbf{r}-\mathbf{r}'|$ is satisfying which makes it a viable candidate that there exists some $L_{\mathbf{r}}$ such that $L_{\mathbf{r}}\,k(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$. If I hand you some different $k(\mathbf{r},\mathbf{r}')$, can you tell by inspection whether some $L_{\mathbf{r}}$ exists such that $L_{\mathbf{r}}\,k(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$? You can see that I am a bit mystified. Is $\int d\Omega \int_0^{\varepsilon} dr\, r^2\; L_{\mathbf{r}}\,k(\mathbf{r},\mathbf{0}) = 1$ for some $L_{\mathbf{r}}$?

4. A particular problem in 2D. Finally, I was puzzling over the eigenvalues and eigenfunctions of this specific 2D integral equation:

$$\int\int_S dx'dy'\; 1/\left(\sqrt{(x-x')^2+(y-y')^2}\right)\,u(x',y') \;=\; \mu\,u(x,y) \qquad\qquad Ku = \mu u \qquad (4.1)$$

so you can see how the above discussion might be relevant. Here S is perhaps a disk of radius a. The thought was that you might use the eigenfunctions of the above equation to solve this Dirichlet problem

$$\int\int_S dx'dy'\; 1/\left(\sqrt{(x-x')^2+(y-y')^2}\right)\,\sigma(x',y') = f(x,y) \qquad (x,y \text{ on disk}) \qquad Ku = \mu u \qquad (4.2)$$

(V = Q/r units) where σ is the surface charge on a disk on which potential f(x,y) is prescribed. I spent a lot of time futzing with the above integral equation (4.1). The kernel is not Hilbert Schmidt but it is weak enough that the eigenfunctions form a complete set I think. I don't know what to use for the boundary conditions in (4.1), see below. I can rewrite (4.1) in polar coordinates this way

$$\int_0^a dr'\,r' \int_0^{2\pi} d\theta'\; u(r',\theta') \;/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\;u(r,\theta) \qquad\qquad Ku = \mu u$$

I can then expand u in partial waves like so

$$u(r,\theta) = \Sigma_{m=0}^{\infty}\,[\,A_m(r)\sin(m\theta) + B_m(r)\cos(m\theta)]$$

and I can then do the angular integration which results in this unpleasant single-variable integral equation

$$2\;\int_0^a dr'\,r'\,Q_{m-1/2}[(r^2+r'^2)/(2rr')]\;/\sqrt{rr'}\;*\;A_m(r') = \mu\,A_m(r)$$

I think this is a case of "integrating too far" and you end up with a twisted up torqued out kernel. The Q function arises, by the way, from this expansion theorem

$$1/\sqrt{a - b\cos(x)} \;=\; (1/\pi)\sqrt{2/b}\;\;\Sigma_{n=0}^{\infty}\,\varepsilon_n\;\;Q_{n-1/2}(a/b)\,\cos(nx) \qquad\qquad \text{// expansion}$$

$$\int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} \;=\; \sqrt{2/b}\;\; Q_{n-1/2}(a/b) \qquad \text{// projection}$$

The kernel can be "untwisted" in two different ways according to

$$Q_{m-1/2}[\,(r^2+r'^2)/(2rr')\,] / \sqrt{rr'} \;=\; \pi \int_0^{\infty} J_m(rx) J_m(r'x)\, dx$$

$$= \; 2\,(rr')^{-m} \int_0^{\min(r,r')} ds\, s^{2m} / \left[\sqrt{r^2-s^2}\, \sqrt{r'^2-s^2} \right]$$

A book by Sneddon in a different situation uses the second form which separates r and r' in a way that allows two sequential Abel transforms to solve some dual integral equation, I forget the details.

The upshot is that basically, I don't know how to solve (4.1). I am not even sure whether to think of (4.1) just with (x,y) only on the disk, or for any x,y in R_2. What would the boundary conditions be?

Choice 1: $u(a,\theta) = 0$, vanishes on the edge of the disk. But you know that the σ solution to (4.2) is likely to blow up at the disk edge, just as it does for a simple charged disk f = 1 (integrably so). So how are you going to represent σ as a sum of eigenfunctions which vanish at the edge of the disk?

Choice 2: $u(r,\theta) = 0$ for r> a. ?

So this got me to wondering about an L_{xy} such that

$$L_{xy}\left[\, 1/\sqrt{(x-x')^2+(y-y')^2}\, \right] = \delta(x-x')\delta(y-y')$$

$$L_{xy}\left[\, 1/\sqrt{x^2+y^2})\, \right] = \delta(x)\delta(y)$$

$$L_{xy}\left[\, 1/ r\right] = \delta(x)\delta(y) = \delta^2(\mathbf{r}) = \delta(r)/(2\pi r) \qquad \text{// polar coordinates } r,\varphi$$

Using some rather obscure Stakgold methods, I came up with the following candidate

$$L_{xy} = -\,(1/\sqrt{x^2+y^2})\,[\; x^2\,\partial_x^2 + y^2\partial_y^2 \; +2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y \;]$$

The result is not unique because any other L_{xy} which differs from this L_{xy} by ∂_φ^2 in polar coordinates is also viable, since $\partial_\varphi^2\,(1/r) = 0$. For example, consider

$$L_{xy}^{(3)} \;= +(1/\sqrt{x^2+y^2}\,)[\; x^2\partial_y^2 + y^2\partial_x^2 - 2xy\partial_x\partial_y] \;= (1/\sqrt{x^2+y^2}\,)\;\; (x\partial_y - y\partial_x)^2 = (1/r)\;\partial_\varphi^2$$

$$L_{xy}^{(3)} F(r) = 0$$

So this provides a 2D example of my general question. Here I am wondering, given $L(1/r) = \delta^2(\mathbf{r})$, how can I tell whether or not there is some differential operator L that makes this equation be true?

OK If you have gotten this far, I think you have gotten the gist of my wayward ways. My question is really a set of questions with the theme of how do you find L such that Lk = δ for some given k, and how do you know whether an L even exists, where L means a differential operator of some order, possibly

infinite, in n dimensions. I have assumed a *linear* operator since otherwise the application I make above to a Fredholm equation would be invalid. The motivation is to be able to convert an integral equation to a reasonable differential equation, and how to know when this is or is not possible.