Sneddon Chap 1 notes : Motivation PhL 8.10.10

FCAA is a journal: Fractional Calculus and Applied Analysis. Ian almost reached his 81st birthday.

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Fields of interest:

fractional calculus

Integral transforms, special functions

- integral and differential equations, potential theory

wave mechanics, theoretical physics

Books and Surveys in FCAA area:

I.N. Sneddon:
Fourier Transforms.
McGraw-Hill, New York, 1951.

I.N. Sneddon:
Special Functions of Mathematical Physics and Chemistry.
Wiley-Interscience, New York, 1956.

I.N. Sneddon:
Elements of Partial Differential Equations.
McGraw Hill 1957, Dover 2006.

I.N. Sneddon:
Mixed Boundary Value Problem in Potential Theory.
North-Holand Publ. Co., Amsterdam, 1966.

I.N. Sneddon:
The Use of Integral Transforms.
McGraw-Hill, New York, 1972.

I.N. Sneddon:
The use in mathematical analysis of EK operators and some of their applications.
In: *Fractional Calculus and Its Applications* (Proc. Int. Conf. New Haven, 1974), L.N.M. 457, 1975, 37-79.

I.N. Sneddon:
The Use of Operators of Fractional Integration in Applied Mathematics.
RWN - Polish Sci. Publishers, Warszawa-Poznan, 1979.

The book I have is the 4th one down the above list: **Mixed BV Problems in Potential Theory.** I have a complete binder hard copy of the book which I checked out from Marriott Library.

OVERVIEW (WRITTEN 12.10.10, 1/2 PAGE) 3

1.1 ELECTROSTATIC PROBLEMS 3

1.2 DIFFUSION PROBLEMS (STEADY STATE) 4

1.3 ELASTOSTATIC PROBLEMS 5

(a) punch problems. 8

(b) crack problems. 8

(c) Problems involving cones and spheres. 8

1.4 HYDRODYNAMIC PROBLEMS 8

1.5 THE BASIC ELEMENTARY PROBLEMS 9

1.6 GENERALIZED AZISYM POTENTIAL THEORY. 9

Overview (written 12.10.10, 1/2 page)

In **Section 1.1** Sneddon states the canonical mixed boundary value problems in electrostatics. The first is Dirichlet for the disk with some prescribed $f(\rho,\theta)$ on the disk. He notes that oblates works for this, but not other cases. Example 2 is same with a grounded surrounding cylinder. Example 3 is Dirichlet with two parallel disks. Example 4 is an annulus, Example 5 is two parallel annuli. Example 6 is the spherical cap, with various surrounding grounded objects like cylinder or sphere. No mention of inversion, but I am reminded that under inversion, an annulus becomes a spherical barrel, and that I have never tried to solve either of these problems. In general a mixed BV problem involves specifying the potential on one set of surfaces, and charge density on another set. Either set might have surfaces with φ or σ being 0.

Section 1.2 discusses diffusion including diffusion of heat (where temperature is the potential).

In **Section 1.3** Sned is off into his specialty area which is called "elastostatics", and that discussion occupies much of this chapter. There are multiple potentials which are somehow related to the displacement vector **u** in a material such as steel under stress. He talks about such things as punch problems and crack problems. I try to give myself a quick review of continuum mechanics here including the stress tensor σ_{ij}, but it is not much help understanding the "Duhamel-Neumann equations."

Section 1.4 is the fluid flow application where the potential's gradient is the velocity field.

Section 1.5 talks about two "basic" problems. The first is the Dirichlet disk. The second is new to me and I pay it no attention. Sort of a "curvature Neumann" where instead of $\partial_z\varphi$ on the disk (charge) you worry about $\partial_z{}^2\varphi$, and of course this relates to Sned's crack world.

Section 1.6 mentions a few oddball potential theory applications, one in 5 dimensions relating to torsion in shafts.

1.1 Electrostatic Problems

We open of course talking about the Dirichlet disk with some $f(\rho,\theta)$ on it. This is quickly reduced to solving the $z \geq 0$ half-space problem with the BC's shown in 1.1.8, and this is indeed a "mixed" boundary value problem.

He notes on page 3 that you could attack the disk problem as the potential between two spheroids (disk and great sphere), but says that that method does not generalize very well to other problems.

In any event, the disk is "the classical" mixed boundary value problem!

You could approach the problem by putting a finite grounded cylinder around the disk, and he in fact sets all this up on page 4. This problem seems to have "more" boundary conditions in its mixed BC set, but don't forget that for the isolated disk you do required $V \to 0$ at some minimal rate at ∞. So this is his second example of "an electrostatics problem."

A third example is two disks as shown top of page 5, each with its own prescribed $f(\rho,\theta)$. Now we have three Smythian "regions" to worry about. This is regarded as another "classical problem" and he claims that Riemann solved it to some extent in 1855.

Example four is the circular annulus, that is, the disk with a hole in it. You could then place this in a cylinder, or you could have two of them as you had two disks (examples 5a and 5b lets call these). Each annulus is sort of triple interval BC, and we will see later in the book that we get triple integral equations instead of dual ones!

Example 6 is the famous spherical cap with $f(\theta,\varphi)$ prescribed. Secondary problems would be the cap in a cylinder, or two caps on the same sphere.

The last example is what I call a barrel on a sphere, but barrel is to spherical cap as annulus is to disk. I might mention in passing that the "method of inversion" is not even mentioned in this entire book!

So in this little section, we get some good examples of mixed BV problems in electrostatics. He has just written the BC's, no integral equations or anything like that yet. Some vocabulary;

Sneddon	me
electrified	charged
earthed	grounded
prescribed	prescribed
screen	iris

Thankfully, Sneddon does have little pictures in his 1966 book.

1.2 Diffusion Problems (Steady State)

He sort of states that diffusion of "heat" is more or less the same as "diffusion" more generally, and this book will stick with heat. The basic equation is $\kappa\nabla^2\theta = \Theta$ where θ (~ V) is "temperature" and Θ (~ρ) is a heat source. On page 8 he draw a little heat situation. We have a closed boundary around some region. Suppose on part of this boundary you have some prescribed heat flow, so $\partial_{\mathbf{n}}\theta = \kappa g(\mathbf{r})$, so that is your Neumann part

[This is conduction theory, let $\mathbf{r}$ = some point on the boundary: "heat flow" = heat current density = $(\Delta Q/dt)/dA = J = \text{const}(\mathbf{r}) * \Delta T$. It is ΔT that "drives" the heat flow J. Maybe T =

So imagine a heat reservoir at T_0 just outside the boundary, and $T(\mathbf{r})$ just inside the boundary. Then $J(r) = \text{const}(\mathbf{r})$ ($T-T_0$) and then $\partial_{\mathbf{T}}J = \text{const}(r)$.

And maybe the rest of the boundary S_1 has $\theta = 0$, so that is your Dirichlet part. We get our usual mixed BC's as shown in 1.2.2. People have studied the case where $\theta = 0$ on a strip or on a circle.

The second example brings up the "radiation business". Newton's law of cooling says that when a surface at temperature θ_1 radiates into a "space" of θ_2, with $\theta_1 > \theta_2$, the rate of heat flow is ~ $\partial_{\mathbf{n}}\theta$ and is proportional to $(\theta_1 - \theta_2)$, so generally we have $\partial_{\mathbf{n}}\theta = k(\theta-\theta_2)$. One usually takes $\theta_2 = 0$ so we then get a BC which says $\partial_{\mathbf{n}}\theta = k\theta$ which we write as $\partial_{\mathbf{n}}\theta + h\theta = 0$. Now Sneddon says this has to do with "radiation", but wiki says it is really a heat convection law. Wolfram says this law applies when you are cooling something with a fan, which is to say, "forced convection". Regardless of the heat loss mechanism, the heat flowing out of a boundary (per second, per area) is given by $k\,\partial_{\mathbf{n}}\theta$, see Stak appendix A page 327), where k is the "thermal conductivity". Forced convection is then a linear thing

like heat conduction. In fact, radiation is a fourth power law on absolute temperature, but we shall forgive Sned for this radiation oversight. The point is that Newton's law of cooling gives us a "mixed" BC that is mixed at every point on a region of our boundary!

In the last section Sned imagines a cylinder into which we inject some prescribed heat function on a circle centered at one end of a "bar", and he talks about what BC's we might have on the other end and on the curved sides of the cylinder. He is just giving general problem examples here.

1.3 Elastostatic Problems

It turns out this is Sned's pet interest. I would like to read this section carefully because it is an area I know nothing about, and one which seldom is mentioned in potential theory discussions. I don't recall Stak mentioning it, for example.

Aside: I cannot find *any* PL notes on stress and strain stuff. A search for "Young's modulus" turns up nothing in my math and physics areas! (nothing in all in "work") This is pretty amazing. I know my "continuum mechanics" was bad, but I thought I had at least something.

A quick web tour:

• **stress** σ is force/area, like pressure, and exists on a surface. So $\sigma = F/A$.

• **Stress causes strain**. Putting pressure on an object causes it to stretch. Strain is a measure of how much an object is being stretched. $\varepsilon = \Delta L/L$.

• **Young's Modulus** E is a measure of the stiffness of a material. It states how much a material will stretch (i.e., how much strain it will undergo) as a result of a given amount of stress. $E = \sigma/\varepsilon$. If the stretch is very small, $\varepsilon = .0001$ say, then E is very large, material is very "stiff".

• if a material does simple stretching in response to strain which will self-restore, then it is doing a **Hooke's Law** action. This is called **elastic behavior**.

• If you try this with steel, pull it for example, there is an elastic region, but then the material "gives" and becomes **plastic**, and finally it breaks.

• stress exists at a point in a material and is really a **tensor** thing:

1.1). According to Cauchy, the *stress at any point* in an object, assumed to be a continuum, is completely defined by the nine components σ_{ij} of a second order tensor of type (2,0) known as the Cauchy stress tensor, $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

The Cauchy stress tensor obeys the tensor transformation law under a change in the system of coordinates. A graphical representation of this transformation law is the Mohr's circle for stress.

The Cauchy stress tensor is used for stress analysis of material bodies experiencing small deformations. For large deformations, also called finite deformations, other measures of stress are required, such as the first and second Piola-Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor.

I have no books at all on this subject as far as I know. Well, Feynman II-31-9 has a few pages! This is a good little presentation and here are the main ideas:

(1) Select an infinitesimal area inside a solid, aligned with axis i, so call it dS_i . Call the two sides of the material abutting this area side A and side B.

(2) There is some force $\mathbf{F}_i$(A on B) across this area of the material which side A exerts on side B, and it is likely NOT just a normal force. If the material is a gas or liquid, then the force must be normal and is identified with the pressure. The equal and opposite force is of course $\mathbf{F}_i$(B on A). In **jello** one suspects that the normal component of the force is such that the two sides are attracted to each other and this is what holds the jello together. You can intuit that as you twist a piece of jello, you are creating some shear stress in there, but it does not rip apart if you are gentle about it.

(3) The force $\mathbf{F}_i$(A on B) of course has x,y,z components, and we could call these $[\mathbf{F}_i(\text{A on B})]_j$. By tradition, this component of the force F_i is denoted σ_{ji} so the second index tells you the direction of the area dS_i and the first is indicates the component of the force. This set of 9 force components is called the **stress tensor**. Components like σ_{xx} are the **normal** force components, while components like σ_{xy} are tangential or **shear** components. We thus have: stress, normal stress, shear stress. For a fluid there is no shear stress in a static situation, so σ_{ij} is diagonal, and moreover, the three diagonal elements are equal.

(4) Another argument shows that σ_{ij} is a **symmetric** rank 2 tensor, so the good news is you don't have to remember the meaning of the index positions, but of course you should.

(5) If you select an area that is in some arbitrary direction $\hat{\mathbf{n}}$, there will be some $\mathbf{F}_n$(A on B) across the boundary **n** defines. By a brief argument one can show (Feynman a special case, my downloaded intro book the general case) that you can compute $\mathbf{F}_n$ from the matrix σ_{ij} according to : $\mathbf{F}_n = \sigma\ \mathbf{n}$ where we regard σ here as a matrix and n is the unit vector of interest. As we know, there are various ways to write this matrix equation such as dyadic with double over arrow and so on.

(6) We know that a symmetric tensor (real, so Hermitian) can be diagonalized by a similarity, and that means you can find three perpendicular axes x_i' which will make σ be diagonal. Think inertia tensor and a tennis racket. In these primed coordinates, there are only normal forces in the three primed directions.

(7) The stress tensor σ_{ij} can vary with position in a material, so it is a tensor field $\sigma_{ij}(\mathbf{r})$.

(8) The stress tensor's elements are forces (per unit area).

(9) One can define a vector field **u**(**r**) called the **displacement** vector field. I think if there are no forces inside a material, you might say that **u**(**r**) = 0 everywhere, the sort of "rest position". Think jello again. As you apply some small force, the jello that at point **r** moves by amount δ**u**(**r**) and has thus been "displaced". A "soft material" might displace a lot, whereas a "stiff" material might have a very tiny displacement, for the same forces. For a mass and spring we have $\delta x = F/k$ where δx would be a normal strain. In a general material, an internal force in the i axis direction F_i can cause a stress displacement in all three directions (I think), so $\delta u_i = (\text{matrix})_{ij} F_j$. Somehow I think this matrix is the **strain tensor** , or is related to it, but I think I have had enough for now!

(10) **Strain** describes displacement of a point in a material in response to the stress there. In a simple linear case with only normal strain in response to a stress, we have strain/stress = **Young's Modulus**. I don't know you this thing is related to the above two tensors. This is directly like $\delta x/F = k$ for a spring. Large k means **stiff** material or stiff spring.

OK, I downloaded some books on this subject for some future rainy day. Back to Sned:

So the Duhamel-Neumann equations (p 11 top) relate the σ_{xy} forces to certain derivatives of the displacement vector **u**, where certain constants called b and β appear, as in 1.3.1. These are given in 1.3.3 in terms of temperature T and η = Poisson ratio (ratio of shear strain to axial strain under an axial stress) and α = linear expansion coefficient (vs temperature I presume). So this is some kind of temperature-involving model of elasticity. He then introduces not one but **three harmonic functions** χ,φ and ψ whose derivatives in a fancy way give you the displacement **u**, then way the gradient of V gives you **E**. This really is horrible stuff because you have those three potentials floating around all the time, as on page 12, even in the simple case of axial symmetry!

So what have we got here? You treat a physical object as an elastic entity which is characterized by various parameters. Both the internal stress tensor of forces, σ_{ij}, and the internal displacement **u** are (tensor and vector) fields which are *unknown*s when you start into an elastostatic problem. Both these unknowns appear in the D-N equations just noted. It seems that the three components of **u** can be replaced by three harmonic potentials as shown in 1.3.4,5,6, those being χ,φ and ψ, so then your coupled problem is to solve for the three potentials and the σ_{ij} in a problem which has some "mixed boundary conditions" on the potentials. I must admit that I don't really follow any of this except in the most general sense.

Presumably any situation where an elastic object is in a static situation subject to various BC's is a candidate problem to look at. Think again of a piece of jello in some situation where each point is displaced from the rest position by **u**, and there is some internal σ_{ij}, and the problem is to solve for those two objects. In the next few sections, Sned talks about certain "classes" of elastostatic problems.

(a) punch problems.

In the first problem, you take an axially symmetric heated infinitely stiff "punch" and press it into the surface of some metal. Picture page 13 top. The Neumann like condition is that there is no radial shear force on the surface because the surfaces are "smooth", which says $\sigma_{\rho z} = 0$, and this is a full range condition. Next, the z displacement has to match the punch shape for $\rho < 1$ and that is the first of 1.3.12. Beyond the punch, there is no z force so $\sigma_{zz} = 0$ for $\rho > 1$, Next, we assume we have our potential $\psi(\rho,z)$ which meets conditions 1.3.13. This leads to some conditions on one of the other potentials φ as shown in 1.3.14. The first applies $\rho < 1$ and the second $\rho > 1$. So if you were to assume some Smythian form for φ, this would give you a pair of dual integral equations! I understand zero of the physics here, but accept that in this punch problem you might end up with a BV problem for φ in this way.

His second problem uses a finite layer of elastic stuff over a rigid base, instead of an entire half-space of elastic stuff, and this makes the problem more complicated. In this case you get similar boundary conditions, but you have the two potentials χ and φ now cross coupled, so this will give some kind of set of coupled dual integral equations for two potentials.

(b) crack problems.

Well, understanding this requires more knowledge than I have. I guess a "crack" means you have no normal force at a boundary so nothing is holding the two sides together. He imagines a penny-shaped crack in the middle of something. If you solve the potential BV problem here, I don't even know what you have done! There seems to be no time involved, so static situation. Perhaps we know the displacement at all points in our solid object that meet the BC's that there is a penny-shaped crack in there.

He goes on to ponder various other problems. Sometimes it is assumed that the crack is "opened up" by some internal pressure p(x), unspecified. Planar cracks, 2D cracks, external cracks as opposed to internal ones. Basically this is a pitch for Sneddon's other book of 1966 which he coauthored with Lowengrub "Crack problems in the classical theory of elasticity".

(c) Problems involving cones and spheres.

We are still in the realm of elastostatic problems. His main point here is that you can think of σ_{ij} in spherical coordinates and then write out those D-N equations in such coordinates, which is what 1.3.32 and following must be. As before, the three potentials and the σ are unknown. [Perhaps this is like an electrostatics problem where the potential V and the induced charge σ are both unknown, this is the case for the most basic problem of the charged disk.] Sned then talks about two problems here. The first is that of a "cone" (ie, a spherical cone) where you prescribed the displacement $u(\theta)$ on the curved surface of the cone, and the rest of the surface has none. I think there are some typos here, but OK, the point is that the elastic object is a circular cone and we have some BC's on it, and they are "mixed".

The second problem is a sphere pressed against a flat plate as shown in the picture page 19. The two parts of the crushed sphere's surface have certain mixed BC's.

1.4 Hydrodynamic Problems

I have read my little "divergence J and V" doc on this subject, so 1.4.1 is on the money. The problem of interest is flow through a hole of some shape in a plate. What you "prescribe" here is the potential in the

hole, so that is your Dirichlet part of the BC. So specify φ(x,y) in the hole of the z = 0 plane, which he calls region S. The simplest case to consider would be φ = C on the hole. The Neumann part of the BC is that $v_z = 0$ outside the hole, since no fluid can flow through the iris. so $\partial_z\varphi = 0$ for Z-S, and there is your mixed BC problem! Sned talks about two cases: hole is a thin slit, and hole is a disk.

All of the above involves just the $z \geq 0$ half space. But another class of problem considers both half spaces. For example, on the left ($z<0$) you have some pressurized liquid, and it is going to spray through the hole into the $z \geq 0$ half space! He calls this a jet. The jet comes out (no gravity) on the right and heads off to $z = \infty$ with some terminal velocity U. He writes the mixed BC's for this problem on page 22 top.

Another problem of interest (reason not specified) is that of a spherical cap moving radially in a fluid. What happens here? Again, he comes up with some mixed BC's. His buddy Collins did this one.

1.5 The Basic Elementary Problems

The **first basic** (azisym) problem is the Dirichlet disk with f(ρ) as in 1.5.1. He restates this in x,y,z coordinates in 1.5.2.

The **second basic** (azisym) problem is same as the first but has $\partial_z^2\varphi = p(\rho)$ instead, and it too has the x,y,z formulation He claims these apply to crack problems, I don't think to electrostatics. He then shows how this second basic problem type can be reduced to the first, I am not very surprised.

1.6 Generalized Azisym Potential Theory.

This final section is a bit strange. Sned claims that in the theory of torsion of shafts you can relate things to ideal flow potential theory in 5 dimensions! So he is just pointing out that sometimes you are interested in more than the 3D Laplace. Various references are given, and we have passing mention of Tricomi's Equation in this regard. Arndt 1916 was the first to do this kind of stuff.