Sneddon Chap 2 META notes : Supporting Math PhL 8.16.10

This math support chapter is a monster 36-page grab bag with perhaps 200 results, mostly derived. My raw notes are 51 pages, here including the brief Overview we have 11 pages.

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Overview (2.5 pages, added 12.10.10)

Section 2.1 concerns "integrals involving Bessel functions" , meaning J_ν Bessel functions. The reason these integrals are of interest is that the $J_\nu(z)$ functions are part of the atomic form for cylindrical coordinates, and therefore dual integral equations arising from mixed BC problems in cylindricals involve such integrals, the disk being an example. In particular, azisym problems will involve $J_0(z)$. Sned discusses the "Weber discontinuous" integrals involving J_0 with sine and cosine, and the general Weber and Schafheitlin integral involving two J_ν functions and a power, of which the trig integrals just noted are a special case. So far, we are talking specific integrals. The Hankel Transform family involves integrals of $J_\nu(\mu x)$ times an *arbitrary* second function f(x). Consider $F_\nu(\mu) = \int_0^\infty dx\, xJ_\nu(\mu x)\, f(x)$. On one hand, this can be regarded as a Hankel transform (projection). On the other hand, if $F_\nu(\mu)$ is given, one can regard this as an integral equation for unknown function f(x), and the solution is $f(x) = \int_0^\infty d\mu\, \mu J_\nu(\mu x)\, F_\nu(\mu)$. The general idea to keep in mind that any "integral transform" can be thought of an integral equation and its solution. Sned discusses three different "forms" of the **Hankel transform** which I summarize in the meta notes below (one is called "modified"). Sned's last gasps on Bessel integrals are the Sonine integrals which involve J_ν and more complicated functions.

Section 2.2 concerns *sums* involving Bessel functions and included here is the Fourier-Bessel transform. Here you would have the sum analog of an "integral equation" and its solution.

Section 2.3 continues on the topic of integral equations. A big contribution comes from Srivastav, R.P. (an Indian coworker with Sned) which describes a very general form for a transform where the integrand kernel has the form $[h(x)-h(t)]^{-\alpha}$ with h being ANY monotonic function. This seems an amazing result. I verified it in the raw notes, and in a sense you can regard it as a "re-speeding" of the simpler case of the factor being $[x-t]^{-\alpha}$ which has the historical name "generalized (ie, $\alpha \neq 1/2$) Abel integral equation". You in fact get exactly the Abel equation if you pick h(x) = x, and a variant if $h = x^2$. Another common case is to pick h(x) = 1 - cos(x). The exponent α has a limited range beyond which you have to do parts.

Another doable case is a kernel of the form k(x,t) = K(x-t) and we then get Laplace diagonalization which then provides a method of solution. Here our integral equation is not a transform per se, it is a convolution equation, but we use a transform to solve the integral equation. Here is a comparison. First we have the Srivastav general case, then the Abel special case, and then the convolution case:

S1: $\int_a^x dt\, f(t) / [h(x)-h(t)]^{\alpha} = g(x)$ interval for x,t is [a,b] interval for α is (0,1)

=> $f(t) = \pi^{-1}\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, h'(u)g(u) / [h(t)-h(u)]^{1-\alpha} \}$ // which is 2.3.2

7. $\int_a^x \frac{y(t)\,dt}{(x-t)^{\lambda}} = f(x), \qquad 0 < \lambda < 1.$

Generalized Abel equation (generalized Abel integral equation).

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}} = \frac{\sin(\pi\lambda)}{\pi}\left[\frac{f(a)}{(x-a)^{1-\lambda}} + \int_a^x \frac{f_t'(t)\,dt}{(x-t)^{1-\lambda}}\right].$$

$\int_0^x dt\, K(x-t)f(t) = g(x)$

=> f(t) = inverse Laplace of $\overline{F}(s)$ where $\overline{F}(s) = \overline{G}(s)/\overline{K}(s)$

The Abel situation is a special case of both the Srivastav situation and the convolution situation, but the solution seen here from Polyanin seems to be the Srivastav form solution. The Laplace solution is symbolic in that adjustments might have to be made with powers of s as shown in the text. I think these adjustments are analogous to the parts integrations which arise in the Srivastav approach.

In all these cases we have one endpoint of the integral being a variable. In the Laplace diagonalization case, both endpoints can be fixed if you redefine the kernel (I guess this is true in all cases).

Finally, if you cannot use one of these transform tricks to solve your integral equation (these tricks including Hankel, Abel type forms, and Laplace diagonalizable forms), Williams 1962a provides a method of getting the integral equation into Fred 2 form for numerical solution.

Section 2.4 concerns what seems to me is just an alternate form of the Abel integral equation transform situation. Sned first shows how these forms arise from doing multiple integrations of a function f(x), and this builds up the second factor $[x-t]^{\alpha-1}$ where α is an integer, but then you continue off the integers and for this reason we have the name "**fractional integration operators**". [Recall from the opening note in the Chap 1 notes that fractional calculus is Sned's favorite subject.] If x is the upper endpoint, we have

the **Riemann-Liouville 1850** transform, and if x is the lower endpoint we have the **Weyl 1917** transform. In this section, Sned is only defining the "projections", and is not claiming any inverse transform yet, though we know such a thing will exist. The **Erdelyi-Kober (EK) operators** are nothing more than fancied up versions of our fractional transforms; I goes with variable upper, and K with variable lower. Fancied up just means the function F(t) being projected is redefined by adding a power inside the integral, and by adding some factors outside the integral. These operators have two numbers as indices, such as $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ and they provide a convenient compact form to represent certain integrals we need to work with. There are certain "properties" of these things such as $K_{\eta,\alpha}\ K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$ as you fiddle the indices. Since these I and K operators are really just adjusted generalized Abel Transforms, we know they must be invertible. Since $K_{\eta,0} = 1$, our relation above with $\beta = -\alpha$ says $K_{\eta,\alpha}\ K_{\eta+\alpha,-\alpha} = 1$ so we "formally" can conclude that $(K_{\eta,\alpha})^{-1} = K_{\eta+\alpha,-\alpha}$, with the understanding of parts integration adjustments, and similarly for I. Regarded as matrices, both I and K are triangular, but in opposite senses, just due to the Volterra endpoint on their definitions. It is good to emphasize this fact: the I and K operators are nothing more than the Abel transforms for general α with different Volterra endpoints.

Section 2.5 derives many relationships between the E-K operators and the modified Hankel transform $S_{\eta,\alpha}$. To obtain these relations, Sned uses that second Sonine Bessel integral which involved "more complicated arguments" which I mentioned above. An example is $S_{\eta,\alpha}\ S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$. It is not clear at this point *why* we are finding these relationships, but you can be sure they will play a role soon in solving integral equations. It is also not clear why there are such *simple* relationships between our Abel transform operators I and K, and our Hankel transform operator S. Sned does not really comment on this, he just presents the facts. As usual, I suspect some kind of group theory underpinning.

Section 2.6 discusses Jacobi Polynomials which are generalizations of the associated Legendre P functions. We know such functions appear in other atomic forms, so maybe that is why they are discussed here.

2.1 Integrals involving Bessel functions.

We start on p 26-27 with a GR-like catalog of various integrals involving J_ν functions. Some of these are in the class of "discontinuous" integrals of Weber (1873), and I have lots of notes on this detail. These integrals are

$$\int_0^\infty dx\ J_0(bx)\cos(ax) = 1/\sqrt{b^2-a^2} \qquad |a| < |b| \qquad \text{// which are 2.1.13}$$

$$= 0 \qquad |a| > |b|$$

$$\int_0^\infty dx\ J_0(bx)\sin(ax) = 0 \qquad |a| < |b| \qquad \text{// which are 2.1.14}$$

$$= 1/\sqrt{a^2-b^2} \qquad |a| > |b|$$

$$\int_0^\infty dx\ J_0(ax)\sin(bx)\ 1/x\ =\ \sin^{-1}(b/a) \qquad b < a \qquad \text{// which are 2.1.15}$$

$= (\pi/2)$ $\quad\quad\quad

$\mathcal{AB}\ \{ f(r);\ \mu\} \equiv \mathcal{A}\{ \mathcal{B}\{ f(r);\ s \} ;\ \mu\}$
$(AB)f = A\ (Bf)$

The *Symmetric* Regular Hankel Transform : (my doc uses K in place of H, Sned uses H)

$$F_\nu(\mu) = \int_0^\infty dx\ (x\mu)^{1/2} J_\nu(\mu x) f(x) \qquad f(x) = \int_0^\infty d\mu\ (x\mu)^{1/2} J_\nu(\mu x)\ F_\nu(\mu)$$

$$F_\nu(\mu) = \int_0^\infty dx\ H_\nu(\mu,x)\ f(x) \qquad f(x) = \int_0^\infty d\mu\ H_\nu(x,\mu) F_\nu(\mu)$$

$$H_\nu(\mu,x) = (x\mu)^{1/2} J_\nu(\mu x) \qquad H_\nu(x,\mu) = (x\mu)^{1/2} J_\nu(\mu x)$$

$F_\nu = H_\nu f \qquad f = H_\nu G_\nu \qquad H_\nu$ = a matrix op

$H_\nu^2 = 1 \qquad H_\nu = H_\nu^{-1}$ and now also: $H_\nu = H_\nu^{T}$
$\mathcal{H}_\nu^2 = \mathcal{1} \qquad \mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$

$F_\nu = H_\nu f \qquad f = H_\nu F_\nu$
$F_\nu(\mu) = \mathcal{H}_\nu\{f(s);\ \mu\} \qquad f(x) = \mathcal{H}_\nu\{F_\nu(s); x\ \}$ s = "dummy variable"

The Modified Hankel Transform: $S_{\eta,\alpha}\ f(\mu)$: // can write $S_{\eta,\alpha}\ f(u) = [S_{\eta,\alpha}\ f](u)$

$$F_{\eta,\alpha}(\mu) = 2^\alpha \mu^{-\alpha} \int_0^\infty dx\ x^{1-\alpha} J_{2\eta+\alpha}(\mu x)\ f(x) = \mathcal{S}_{\eta,\alpha}\ \{ f(r);\ \mu \} = S_{\eta,\alpha}\ f(u)$$

$$f(x) = 2^{-\alpha} x^\alpha \int_0^\infty d\mu\ \mu^{1+\alpha} J_{2\eta+\alpha}(\mu x)\ F_{\eta,\alpha}(\mu) = \mathcal{S}_{\eta+\alpha,-\alpha}\ \{ F_{\eta,\alpha}(r);\ x \} = S_{\eta+\alpha,-\alpha}\ F_{\eta,\alpha}(x)$$

$1 = S_{\eta+\alpha,-\alpha}\ S_{\eta,\alpha}$ // for matrix ops
$\mathcal{1} = \mathcal{S}_{\eta+\alpha,-\alpha}\ \mathcal{S}_{\eta,\alpha}$ // for script ops

$\mathcal{S}_{\eta,\alpha}\ \{ f(s);\ \mu \} = 2^\alpha \mu^{-1/2-\alpha}\ \mathcal{H}_{2\eta+\alpha}\{ s^{1/2-\alpha} f(s);\ \mu \}$ // how $\mathcal{S}$ and $\mathcal{H}$ are related

or

$\mathcal{H}_{2\eta+\alpha}\{ s^{1/2-\alpha} f(s);\ \mu \} = 2^{-\alpha} \mu^{1/2+\alpha}\ \mathcal{S}_{\eta,\alpha}\ \{ f(s);\ \mu \}$

Lest this seem vague, let's just write things out one more time

$$\mathcal{S}_{\eta,\alpha}\ \{ f(r);\ \mu \} = 2^\alpha \mu^{-\alpha} \int_0^\infty dx\ x^{1-\alpha} J_{2\eta+\alpha}(\mu x)\ f(x)$$

$$= 2^\alpha \mu^{-\alpha} \int_0^\infty dx\ x\ J_{2\eta+\alpha}(\mu x)\ [\ f(x)\ x^{-\alpha}]$$

which shows that this modified thing is just a multiple of the non symmetric Hankel where we redefine the function we are transforming, and $\nu = 2\eta+\alpha$. Of course this then can be related to the symmetric

Hankel as shown above. So the "modified Hankel" is nothing new, it is just a way to write the Hankel transform that "fits" with needs which shall arise later.

On page 30-31 we get some famous **Sonine's** integrals involving Bessel functions, and other integrals involving other kinds of Bessel functions of the I and K variety.

2.2 Infinite Series involving Bessel functions. [33]

In our work to come on dual integral equations, we will be using the integral of Bessel functions stuff of the previous section. But we are also going to look at "dual series equations" (such as for the bowl) and these are going to make use of *sums* of Bessel functions and that is why this section is here. I did not do this section in detail, but it involves sums of this general form

$$S_{\nu,\alpha,\beta,\gamma}(\rho,t;a) = (2/a^2)\,\Sigma_n\,\lambda_n^{\gamma}\;J_\alpha(\rho\lambda_n)\;J_\beta(t\lambda_n)\,/\,J_{\nu+1}^2(a\lambda_n)$$

where the λ_n are such that $J_\nu(a\lambda_n) = 0$. Along the way Sned develops the "finite Hankel transform" which of course is a series instead of an integral, and which we usually call the **Fourier-Bessel transform**.

2.3 Some Remarks on Integral Equations. [40]

(a) (b) Srivastav's First and Second Results (x is upper, lower endpoint)

This relatively late 1963 and "broad class" transform is very general in that you can install any monotonic function h(x) you want. Here are the two resulting transform pairs

S1: $\int_a^x dt\, f(t) / [h(x)-h(t)]^\alpha = g(x)$ interval for x,t is [a,b] interval for α is (0,1)

=> $f(t) = \pi^{-1}\sin(\pi\alpha)\,\partial_t \{ \int_a^t du\, h'(u)g(u) / [h(t)-h(u)]^{1-\alpha} \}$ // which is 2.3.2

S2: $\int_x^b dt\, f(t) / [h(t)-h(x)]^\alpha = g(x)$ interval for x,t is [a,b] interval for α is (0,1)

=> $f(t) = -\pi^{-1}\sin(\pi\alpha)\,\partial_t \{ \int_t^b du\, h'(u)g(u) / [h(u)-h(t)]^{1-\alpha} \}$ // which is 2.3.4

where admittedly α is restricted to $0 < \alpha < 1$. Notice that both involve derivatives in the solution f(t) of the integral equation which is the first line in each case. As shown later for a special case of h(x), you can extend the range of α by doing parts integrations to define analytic continuation.

(c) Special cases of Srivastav's two results

A. Let $h(u) = 1 - \cos(u)$ on $(0,\pi)$

This gives results of a certain characteristic form which I see from time to time:

S1: $\int_a^x dt\, f(t) / [\cos(t) - \cos(x)]^{\alpha} = g(x)$ $\quad 0 < \alpha < 1$

$\Rightarrow$ $f(t) = \pi^{-1}\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, \sin(u) g(u) / [\cos(u) - \cos(t)]^{1-\alpha} \}$

S2: $\int_x^b dt\, f(t) / [\cos(x) - \cos(t)]^{\alpha} = g(x)$ $\quad 0 < \alpha < 1$

$\Rightarrow$ $f(t) = -\pi^{-1}\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, \sin(u) g(u) / [\cos(t) - \cos(u)]^{1-\alpha} \}$

B. Let $h(u) = u^2$ on the positive real axis

This gives results

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^{\alpha} = g(x)$ $\quad 0 < \alpha < 1$ $\quad$ interval for x,t is [a,b]

$\Rightarrow$ $f(t) = (2/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1-\alpha} \}$ $\quad$ // which are 2.3.7

S2: $\int_x^b dt\, f(t) / [t^2 - x^2]^{\alpha} = g(x)$ $\quad 0 < \alpha < 1$ $\quad$ interval for x,t is [a,b]

$\Rightarrow$ $f(t) = -(2/\pi)\sin(\pi\alpha)\, \partial_t \{ \int_t^b du\, u\, g(u) / [u^2 - t^2]^{1-\alpha} \}$ $\quad$ // which are 2.3.8

I show in the raw notes how you can change from squared to non-squared variables and then the above transforms become the "**generalized Abel integral equation**" and its solution, for example, S1 becomes

7. $$\int_a^x \frac{y(t)\,dt}{(x-t)^{\lambda}} = f(x), \qquad 0 < \lambda < 1.$$

Generalized Abel equation (generalized Abel integral equation).

Solution:

$$y(x) = \frac{\sin(\pi\lambda)}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}} = \frac{\sin(\pi\lambda)}{\pi}\left[\frac{f(a)}{(x-a)^{1-\lambda}} + \int_a^x \frac{f_t'(t)\,dt}{(x-t)^{1-\lambda}}\right].$$

Historically, it was the case $\lambda = \alpha = 1/2$ that Niels Abel ran into for his tautochrome problem around 1825. Sned gives another special case where he just shuffles factors a bit, I won't bother with that here.

(d) The Laplace method with a general kernel [42 bot]

In general if one has this "convolution" integral equation

$G(u) = \int_0^u d\tau\, K(u-\tau)F(\tau)$ // K and G are known, what is F ?

where the kernel is a function of a single variable in which the difference $u-\tau$ appears, you can diagonalize with the Laplace transform to get

$\bar{G}(s) = \bar{F}(s)\,\bar{K}(s)$ => $\bar{F}(s) = \bar{G}(s)/\bar{K}(s) \equiv \bar{G}(s)\,\bar{R}(s)$

then you just look up $\bar{F}(s)$ in your Laplace table to obtain F(t) and you have solved the equation. Notice that the generalized Abel kernel is of this form with $K(u) = u^{-\lambda}$, whereas the more general Srivastav kernel does not quite fit this mold with its $k(x,t) = [h(x)-h(t)]^{\alpha}$.

Technical issues can arise, however, when doing this diagonalization. In the application to the Abel equation where $K(t) = t^{-\alpha}$, we find $\bar{K}(s) = \Gamma(1-\alpha)/\, s^{1-\alpha}$ for $\alpha < 1$, no problem. However, we need $\bar{R}(s)$ which goes as $s^{1-\alpha}$ and this function $\bar{R}(s)$ has no inverse Laplace transform for α in our range! We extract ourselves from this problem by doing this shuffle:

$\bar{F}(s) = \bar{G}(s)/\bar{K}(s) = s\,\bar{G}(s)/[s\bar{K}(s)] = s\,\bar{Q}(s)$ $\bar{Q}(s) = \bar{G}(s)/\bar{L}(s)$ with $\bar{L}(s) \equiv s\bar{K}(s)$

From $\bar{F}(s) = s\,\bar{Q}(s)$ we know that $F(t) = \partial_t Q(t) + Q(0)$. The object $\bar{L}(s)$ *has* an inverse transform for α in our desired range, and so the solution is well defined. The main point is that a ∂_t has magically appeared and this matches the ∂_t we see in the generalized Abel solution quoted above.

(e) The Williams 1962a Method

When the kernel cannot be written as a function of a single variable, the Laplace diagonalization fails, and in general the integral equation [notice the fixed endpoints now]

$G(u) = \int_0^a d\tau\, K_1(\tau,u)\, F(\tau)$

does not have an analytic solution and you are forced to do it numerically. This Williams method is a way to recast this integral equation into a Fred 2 integral equation where something is small so the Fred 2 can be solved iteratively. A feature of this method is that you first approximate your $K_1(\tau,u)$ kernel with some more tractable kernel called $K_0(\tau,u)$ and then $G \equiv K_1-K_0$ is "small" if your approximation is reasonable and this then makes the Fred 2 solution series be convergent.

Comment: I think you can recast the above integral equation into a differential equation, although some restrictions such as H-S or compact or completely continuous might have to be imposed on K_1. This ODE then has the same solution functions. But there are lots of ODE's which don't correspond to "popular" ODE's and maybe cannot even be recast into popular ODE form, so the solutions of those ODE's would then be "special functions" having no name. With our numerical work, we are just exploring certain solutions of such an ODE.

2.4 Fractional Integration Operators. [p 46]

Sned shows us formulas for repeated integrals with upper and lower endpoints being variable. In the second case he has some typos which caused me grief. I verified both his formulas for "multiple integrals". He then "extends" the multiple integral formula so it has meaning for non-integral α, so then we have "fractional" integral transforms. We recognize that these integrals are just the generalized Abel integral equations which we already know how to solve. In other words, we have a pair of transforms here which can be historically traced back to a meaning when you do an iterated integration. A Gamma function is added out front, and the integral operators appearing in the integral equations are given fancy historical names.

$$\mathcal{R}_\alpha\{ f(t); x \} = (1/\Gamma(\alpha)) \int_0^x dt' f(t') (x-t')^{\alpha-1} \qquad \text{Riemann-Liouville 1850}$$

$$\mathcal{W}_\alpha\{ f(t); x \} = (1/\Gamma(\alpha) \int_x^\infty dt' f(t') (t'-x)^{\alpha-1} \qquad \text{Weyl 1917}$$

These operators are redefined once again and we end up with the **Erdelyi-Kober operators**, it is all the same thing, just a shuffle of variable names (to quadratic) and a redefining of the function inside the integral, and perhaps a constant or two added.

$$I_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{-2\alpha-2\eta} \int_0^x dt\, t^{2\eta+1} F(t) [x^2 - t^2]^{\alpha-1}$$

$$K_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{2\eta} \int_x^\infty dt\, t^{-2\eta-2\alpha+1} F(t) [t^2 - x^2]^{\alpha-1}$$

Sned shows various relations between these I and K operators, such as

$$I_{\eta,\alpha}\, I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} \qquad 2.4.17$$

$$K_{\eta,\alpha}\, K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta} \qquad 2.4.19$$

$$I_{\eta,0} = K_{\eta,0} = 1$$

Whereas the original operators are defined only for $\alpha > 0$, it is possible to extend the definition to any real value of α. This basically involves doing parts integrations. A typical formula is 2.4.31 which shows that you can "raise" the second index on K by n if you are willing to do parts n times, and that is where the symbol $\mathcal{D}^n$ arises with $\mathcal{D}$ defined as in 2.4.24.

2.5 Connection between the above EK operators and the Hankel operator

Sned then shows various relations between these I and K operators, and the S operator that was our modified Hankel transform,

$$S_{\eta,\alpha} \{ f(r); \mu \} = 2^\alpha \mu^{-\alpha} \int_0^\infty dx\, x^{1-\alpha} J_{2\eta+\alpha}(\mu x) f(x)$$

He proves these relations using those Sonine integrals he reviewed earlier in the chapter. He collects all the relations in his book's only appendix, called **Appendix A**, and here are some of the relations:

$I_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = S_{\eta,\alpha+\beta}$	13A	2.5.1	
$K_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}$	14A	2.5.2	same RHS as previous
$S_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = I_{\eta,\alpha+\beta}$	15A	2.5.3	
$S_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$	16A	2.5.4	reversed LHS compared to previous
$S_{\eta+\alpha,\beta}\, I_{\eta,\alpha} = S_{\eta,\alpha+\beta}$	17A	2.5.5	
$S_{\eta,\alpha}\, K_{\eta,\alpha+\beta} = S_{\eta,\alpha+\beta}$	18A	2.5.6	same RHS as previous

There must be some underlying group theory going on here, but Sned does not mention it.

2.6 Jacobi Polynomials and Associated Legendre Functions

The Jacobi's are in effect those central functions which appear in the representations of the rotation group, functions I usually called $d^{j}_{mm'}(\theta)$, but of course there are various "shufflings" of the indices.

$$d^{j}_{m'm}(\phi) = \left[\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right]^{1/2} \left(\sin\frac{\phi}{2}\right)^{m-m'} \left(\cos\frac{\phi}{2}\right)^{m+m'} P_{j-m}^{(m-m',m+m')}(\cos\phi).$$

The Jacobis are orthogonal polynomials on (-1,1) and as such have orthogonality and completeness formulas and various other formulas. The associated Legendre functions are special cases of the Jacobis, here is some stuff from GR7

2.	$P_n(x) = P_n^{(0,0)}(x)$	CO, EH II 179(3)
3.	$T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$	CO, EH II 184(5)a
4.	$C_n^{\nu}(x) = \frac{\Gamma(n+2\nu)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{(\nu-1/2,\nu-1/2)}(x)$	MO 108a, EH II 174(4)

showing regular Legendre, Chubby and Gegenbauer (all having $\alpha = \beta$)

It's like pulling teeth to find a statement of the connection between associated Legendre and Jacobi, so I guess I have to do it manually.

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-z}{2}\right)$$

$$P_\lambda^{\mu}(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2})$$

So the second line can be written

$$P^{-m}_j(z) = 1/\Gamma(1+m) * [(1+z)/(1-z)]^{-m/2} F(-j, j+1; 1+m; [1-z]/2)$$

The first line is then, with n = j and writing out the Pochhammer symbol,

where $(m)_n$ is a Pochhammer symbol

$$(m)_n \equiv m\,(m+1)\cdots(m+n-1) = \frac{(m+n-1)!}{(m-1)!}.$$

$$= \Gamma(n+m)/\Gamma(m)$$

which says that

$$(\alpha+1)_n = \Gamma(n+\alpha+1)/\Gamma(\alpha+1)$$

$$P_j^{(\alpha,\beta)}(z) = \Gamma(j+\alpha+1)/\Gamma(\alpha+1) * 1/j! * F(-j, j+1+\alpha+\beta; 1+\alpha; [1-z]/2)$$

We set then α = m and β = -m to get

$$P_j^{(m,-m)}(z) = \Gamma(j+m+1)/\Gamma(m+1) * 1/j! * F(-j, j+1; 1+m; [1-z]/2)$$
$$P^{-m}_j(z) = 1/\Gamma(1+m) * [(1+z)/(1-z)]^{-m/2} F(-j, j+1; 1+m; [1-z]/2)$$

These now have the same F function, so conclude that

$$P_j^{(m,-m)}(z)\, j!\, \Gamma(m+1)/\Gamma(j+m+1) = P^{-m}_j(z)\, \Gamma(1+m)\, [(1+z)/(1-z)]^{+m/2}$$

$$P_j^{(m,-m)}(z)\, j!\, / \Gamma(j+m+1) = P^{-m}_j(z)\, [(1+z)/(1-z)]^{+m/2}$$

$$P^{-m}_j(z) = [(1+z)/(1-z)]^{-m/2} \Gamma(j+1) / \Gamma(j+m+1) * P_j^{(m,-m)}(z)$$

So there is the connection, though not very enlightening. If we set m = 0 we get

$$P^0_j(z) = P_j^{(0,0)}(z)$$