Sneddon Chapter 3 Notes: The Charged Disk problem PhL 8.17.10

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Overview (1.5 pages, written 12.10.10)

Recall that if you assume the usual cylindrical atomic form for the charged disk potential, you end up at once with a pair of dual integral equations which involve J_0 and an unknown coefficient A(k).

In **Section 3.1** Sned shows that in **1873 Weber** solved for this coefficient **"by inspection"**, since he had just derived a pair of integrals which exactly matched the dual equations. A classroom winner!

In **Section 3.2**, we see that later in **1881**, **Beltrami** generalized this to f(ρ) on the disk. This gives the same dual integral equations except now you have f(ρ) on the RHS of the potential equation. Beltrami solved this in a certain way, but we will do this "our way" later, basically the same idea.

In **Section 3.3** Sned redoes the Beltrami f(ρ) case using **oblates** (expand f onto $P_n(\eta)$). He gets a result, but notes the oblates are "Cartesian-unfriendly", and I agree. But it is a solution and Sned had to mention it.

In **Section 3.4** we have a huge jump in time to 1947 (7 decades have passed) when **Copson** came up with I presume the first-ever solution for the general f(ρ,φ) case. The method used is the Stak "integral equation method" to find σ first, then use that to get the potential φ. The first task -- finding σ -- starts off by expanding both f and σ into f_n and σ_n on partial waves in φ. I did this same start-off in my doc " Iris Green's Function attempt using the Stak integral equation method.doc". That is a different problem (Green's, not Dirichlet; iris, not disk), but the same drdθ integral arises more or less, and the same dθ integral exactly. I quote from that doc

$$V(\mathbf{r}') = q/r_1 + \Sigma_{n=0}^{\infty} \int_a^{\infty} r\, dr\, \sigma_n(r) \int_0^{2\pi} d\theta \cos(n\theta) / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} = 0$$

I then computed the dθ integral to be $(1/\pi) \sqrt{r/r'}\, \varepsilon_n\, Q_{n-1/2}[(r^2 + r'^2)/(2rr')]$ and then I did not know what to do. Copson faces exactly the same dθ integral, but he uses what I would call "an integral identity" to write it as another single integral in which r and r' are completely "separated" AND, the form of the

separated result is "Abel-like" ($\alpha = 1/2$) which then leads to the Copson solution. Sned then shows that the Copson solution gives the right answer in four known special cases. [Eventually I derived this integral.]

In **Section v5** Sned sets up the **Beltrami f(ρ)** problem again as a dual integral equation, and this time he solves the thing **"his way"** and he ends up getting "the standard result" for this dual pair, and I quote the solution from Polyanin which agrees with the Sned result. Basically Sned uses one simple Bessel integral and then one Abel transform with $\alpha = 1/2$. He does an opening parts integration which is what causes ∂_t to be in the answer.

Sned then considers the same dual pair, but changes the factor ξ^{-1} to ξ^{+1} in the first of the pair. I don't know what physical problem this corresponds to, though he suggests it might related to his crack "second basic" problem. Sned finds the solution to this problem, and this is the one that **Polyanin** has wrong in his book (I am pretty sure, have still heard nothing back from Polyanin. His errata website is dead.).

In **Section 3.6** Sned writes a Laplace-solving potential in a strange way. He starts with the cylindrical atomic form and messages that into what I call a **"z+it" form**. It seems strange to write the real potential as an integral of a **complex** integrand, but he will use this later in the book. The advantage is that, in this new form, there is no longer a J_0 function, you just have a sum of inverse square roots which reminds me a lot of the disk Green's function, and the charged disk solution, and my "ellipse string theorem", except things are complex.

He then gives another complex form which I am not sure I understood in the raw notes.

3.1 Weber's Charged Disk Solution (1873)

I have finally identified "H. Weber" as Heinrich Martin Weber 1842 - 1913, a pretty big shot guy in his time. Teacher of Hilbert, in the Konigsberg school with lots of other hot shots. This is not the Weber whose name is associated with magnetic fields! There is a third Weber of similar years to Martin who was "not good at math", taught Einstein who said his lectures were 50 years out of date!

Weber's 1873 solution to the charged disk dual integral equations is really a "solution by inspection", where you compare the dual integral equations to the famous discontinuous integrals that Weber (1873) and Schafheitlin figured out (1887 with more detail). There is not much more to say! Kelvin got the solution to the charged disk earlier using the oblate coordinates idea, years 1847 and 1872. The dual integral equations arise trivially when you assume the usual cylindrical atomic form for the disk potential. This seems a simpler method than green Jackson. Of course this solution requires $f(\rho,\varphi)$ = constant on the disk.

3.2 Beltrami's Symmetric Potential Solution (1881)

This is a disk Dirichlet solution with a given $f(\rho)$ on the disk, so is azisym only. Helper functions are called $F(\rho)$ and $Q(\rho)$. An integration order interchange is used, as is the Weber discontinuous integral. Things are cast into one of the simple "known integral equation forms", and we get $F'(\rho)$ in terms of $f(\rho)$ and then we get the A(k) in terms of $F'(\rho)$ and finally $V(\rho,z) = \int_0^\infty dk/k * A(k)\, e^{-kz}\, J_0(k\rho)$. The solution

has a certain very contrived feel to it, and Sned compacts it down in a later section. So this generalizes the 1873 $f(\rho) = 1$ solution of Weber and is perhaps the first non $f = 1$ solution ever done.

Beltrami was Italian and worked on surface geometry stuff, non-Euclidean, with Klein and others.

3.3 Doing the Problem in Oblates (Kelvin 1847/1872?)

This is the Kelvin azisym solution which I have read in the original, enough said.

3.4 Copson's Solution to the Disk Dirichlet Problem (1947)

I guess this is the first non-azisym full Dirichlet solution to this problem, the date is rather amazing. The development starts out Stak-like, as I did in my roll-my-own solution, by writing $V(\rho,\theta,z) = \int \sigma dA/R$. We evaluate this at $z = 0$ and force it to equal $f(\rho,\theta)$ on the disk for $\rho < 1$. We partial wave expand both σ and f in the obvious say to get 3.4.5. It was at this point that I, in my own shot at this problem, did the θ integration to get a conical Q function and then I was stuck. But Copson instead replaces with θ integration with another integral which "separates" the two radial denominators as in 3.4.6 and also extracts the $\cos(n\theta)$ factor so you then have the thing "diagonalized" into partial waves n, a shown p 70 A. We still have a double integral, but the integrands are perfect for using the Abel inversion twice. We end up then with S_n as an integral of the known f_n, and then σ_n as an integral of S_n. No simple form is given for the actual potential V, however, You would face the task of computing the LHS of 3.4.5 with σ_n installed. Again, this is the single-layer method outlined by Stakgold.

Sned then goes through a list of special Copson cases.

(1) Disk with constant potential. $f_n(\rho) = \delta_{n,0}$

(2) Earthed Disk in an External Field Parallel to the disk in the x direction. $f_n(\rho) = \delta_{n,1}\rho$.

(3) Gallop's Problem of 1886. $f_n(\rho) = \delta_{n,0}\, J_0(c\rho)$

(4) MacDonald's Problem of 1895 $f_n = J_n(c\rho)$

So Copson's nice solution gives a result which agrees with all these earlier solution cases.

3.5 Sneddon's Solutions of two Dual Integral Equation Pairs

The **first problem** Sned goes after is this one [these are 3.2.1,2 , this is the Beltrami problem]

$$\int_0^\infty d\xi\, \xi^{-1} A(\xi) J_0(\xi\rho) = f(\rho) \qquad \rho<1$$

$$\int_0^\infty d\xi\, A(\xi) J_0(\xi\rho) = 0 \qquad \rho>1$$

Sned's opening move is to write $\xi^{-1}A(\xi) = \int_0^1 dt\ \varphi(t)\cos(\xi t)$, and he immediately does parts on this and proceeds to fiddle away. He ends up with an Abel transform which he inverts to get φ in terms of f (which has a derivative see 3.5.9) and then of course we have A in terms of φ as in 3.5.1. The whole result is nicely summarized in Polyanin where we can take a = 1 to get the Sned result:

3.9-2. Kernels Containing Bessel Functions of the First Kind.

9. $$\int_0^\infty J_0(xt)y(t)\,dt = f(x) \quad \text{for} \quad 0 < x < a,$$
$$\int_0^\infty tJ_0(xt)y(t)\,dt = 0 \quad \text{for} \quad a < x < \infty.$$

Solution:

$$y(x) = \frac{2}{\pi}\int_0^a \cos(xt)\left[\frac{d}{dt}\int_0^t \frac{sf(s)\,ds}{\sqrt{t^2-s^2}}\right]dt.$$

The **second problem** Sned goes after is this one [these are 3.5.15,16 and to keep things clear he has written $\psi(\xi)$ instead of his usual $A(\xi)$, and he will use χ instead of φ] [difference is ξ^{+1} in place of ξ^{-1}]

$\int_0^\infty d\xi\ \xi^{+1}\psi(\xi)J_0(\xi\rho) = f(\rho)$ $\quad\quad \rho<1$

$\int_0^\infty d\xi\ \psi(\xi)J_0(\xi\rho) = 0$ $\quad\quad \rho>1$

Sned's opening move is to write $\psi(\xi) = \int_0^1 dt\ \chi(t)\sin(\xi t) = -\xi^{-1}\int_0^1 dt\ \chi(t)\,\partial_t\cos(\xi t)$ where we again start off by doing parts. I verified all the steps in the raw notes and the result is 3.5.19 for $\psi(\xi)$. There is no derivative in this result, which convinces me that Polyanin's result is wrong because Polyanin's result *shows* the derivative!

10. $$\int_0^\infty tJ_0(xt)y(t)\,dt = f(x) \quad \text{for} \quad 0 < x < a,$$
$$\int_0^\infty J_0(xt)y(t)\,dt = 0 \quad \text{for} \quad a < x < \infty.$$

Solution:

$$y(x) = \frac{2}{\pi}\int_0^a \sin(xt)\left[\frac{d}{dt}\int_0^t \frac{sf(s)\,ds}{\sqrt{t^2-s^2}}\right]dt.$$

WRONG!

I thought this was worth "writing in" for an errata, but sadly I got no response at all from Polyanin.

There is a popular 1984 Nasim-Aggarwala PDF which I have. It replicates Sned's and Polyanin's result (with derivative) for the first problem. NA did not do the second problem, but Mandal does the more general problem in a 1987 paper and I convinced myself that a special case of this result does confirm

Sned's result for the second problem. You can superpose variants of the above two solutions to get a general pair of duals with functions in both equations, but that will appear in the next chapter.

3.6 Methods based on integral representation of harmonic functions.

If you start off with the usual Smythian form for V but install Sned's $\xi^{-1}A(\xi) = \int_0^1 dt\ \varphi(t) \cos(\xi t)$, you write out the cos in two expo terms and you get the form p 78 C which I call " an z+it form". This does seem strange, why would you want to encourage a complex notation, but the form is used a lot later in the book!

This section ends with another "complex form" which ends up real, but I did not play much with it.