Sneddon Chap 4 META notes: Dual Integral Equations PhL 8.17.10

This chapter is the CORE of this book, and is the longest chapter at 54 pages. My raw notes fill 32 pages, meta notes 15 pages, overview 5 pages.

The **first chapter** outlined how dual integral equations appear in physical problems with mixed BC's.

The **second chapter** showed how we could solve certain ***single*** **integral equations** using the 1963 general Srivastav formula and its special cases which include the Abel cases. We also looked at possible Laplace diagonalization methods and the fancy Williams numerical method. We wrapped up with certain single integral equations which we could solve which were really just fractional integral transforms involving the Kober I and K operators. These are really just Abel transforms with powers added which in turn are tied in with the Hankel transform, another set of single "integral equations" we then know how to solve. There was no mention of "dual" integral equations in this chapter. All the math background for the rest of the book is jammed into this very long 37 page chock-a-block chapter.

The **third chapter** reviewed solutions of the Dirichlet disk problem, the "classic dual integral equation problem". This was our introduction to such problems.

In the current (**fourth) chapter** we look more generally at two classes of dual integral equations, one with powers and J functions for kernels, the other with powers and trig functions for kernels. Sometimes we can write a solution in "closed form", but in other cases we try to reduce things to an inhomo Fred 2 integral equation which can then be numerically solved.

In passing, I note that both Jackson applications (disk and hole in plate) used duals of the Titchmarsh type reviewed below. For example, in blue Jackson we find:

$$\int_0^\infty dy\ yg(y)J_n(yx) = x^n \qquad \text{for } 0 \le x < 1$$
$$\int_0^\infty dy\ g(y)J_n(yx) = 0 \qquad \text{for } 1 \le x < \infty \qquad (3.179)$$

Examination of the formula of Sonine and Schafheitlin for the integral of $J_\mu(at)J_\nu(bt)t^{-\lambda}$ (see *Watson*, pp. 398 ff, or *Magnus* et al., p. 99) shows that the solution for $g(y)$ is

$$g(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\, j_{n+1}(y) = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\,\frac{J_{n+\frac{3}{2}}(y)}{(2y)^{1/2}} \qquad (3.180)$$

The green Jackson equations are exactly the same, except the y^1 power is in the second equation. I have not verified Jackson's results at this point, but know I could with the data in this chapter!

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Overview (5 pages, written 12.14.10)

Section 4.1 says nothing useful, so we happily skip it.

Section 4.2 gets us warmed up with the **Titchmarsh-type** dual form (1948), which is this:

$$\int_0^\infty du\, u^{-2\alpha} A(u) J_\mu(xu) = F(x) \qquad x \text{ in } I_1 \qquad \text{// Titchmarsh type duals}$$

$$\int_0^\infty du\, u^{-2\beta} A(u) J_\nu(xu) = G(x) \qquad x \text{ in } I_2 \qquad (0,\infty) = I_1 \cup I_2$$

Right off the bat we do a certain rescaling which lasts us through many methods,

$$A(u) = u\,\Psi(u) \qquad f_1(x) = 2^{2\alpha}\, x^{-2\alpha}\, F(x) \qquad g_2(x) = 2^{2\beta}\, x^{-2\beta}\, G(x)$$

I refer to f_1 and g_2 as the prescribed functions, and f_2 and g_1 are the partner functions. The idea of course is that $f(x) = f_1(x)$ on I_1 and $f(x) = f_2(x)$ on I_2. Our problem is always this: given f_1 and g_2, solve for the three functions Ψ, f and g on $(0,\infty)$. When we apply the scalings shown above, our dual integral equations can be written as simple modified-Hankel transforms as were studied in Chapter 2,

$S_1\Psi = f_1$ x in I_1 $S_1 = S_{\mu/2-\alpha,2\alpha}$

$S_2\Psi = g_2$ x in I_2 $S_2 = S_{\nu/2-\beta,2\beta}$

where from now on we are thinking of integral operators like S_i as ∞ x ∞ matrices, and functions as ∞ component vectors. I hemmed and hawed about notation in the raw and even the meta notes, using Ψ and ψ to mean the same thing, and F and f, etc. I originally intended to use capital letters for matrix notation objects and lower case for functions, but later decided against that, for among other reasons, we already have F and G. So I ended up using Ψ for $\psi(x)$ and f_1 for $f_1(x)$, and f for the vector $(f_1(x), f_2(x))$ etc. It is important for the reader to clearly understand that f is a Hilbert Space vector with an infinite number of components $f(x) = \langle x|f\rangle$, but we break it into two sub-vectors f_1 and f_2 and use a 2x2 matrix notation to track this idea.

Subsection 4.2.1 describes "the main act" of this chapter, while later subsections are just variations on the theme. This main act is called **The Peters Solution [1961]**, which I now describe. We assume that the dual equations shown above are portions of a pair of full interval equations which read

$S_1 \Psi = f$ or $[S_1\psi](x) = f(x)$, x in $I_1 \cup I_2$

$S_2 \Psi = g$ or $[S_2\psi](x) = g(x)$, x in $I_1 \cup I_2$

We then learn, both theoretically from matrix theory, and in practice in terms of the EK operators, that we can find triangular matrices I and K which have the property $IS_1 = KS_2$ and in fact this product which we call S is another modified Hankel transform. So $IS_1 = KS_2 = S$. Upon left multiplication by I and K, our equations above become

$S \Psi = I f$ $I = I_{\mu/2+\alpha,\lambda-\mu}$ $\lambda = (\mu+\nu)/2 + \beta - \alpha$

$S \Psi = Kg$ $K = K_{\lambda-\nu/2-\beta,\nu-\lambda}$ $S = S_{\mu/2-\alpha,\lambda-\mu+2\alpha}$

so we have accomplished the very major step of getting the same operator S to the left of Ψ in both our equations (each equation has the same LHS). The triangular matrices I and K are nothing more than adjusted Abel transforms (= fractional integral transforms) which are invertible, as are any transforms, and they have the fancy name Erdelyi-Kober operators. In our matrix sense we can think of the I and K matrices in this way

$$I = \begin{pmatrix} I_{11} & 0 \\ I_{21} & I_{22} \end{pmatrix} \qquad K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}$$

It is then child's play to show that, if we take the top half of our first equation and the bottom half of the second equation, we get

$$S\Psi = \begin{pmatrix} I_{11} f_1 \\ K_{22} g_2 \end{pmatrix} \quad => \quad \Psi = S^{-1} \begin{pmatrix} I_{11} f_1 \\ K_{22} g_2 \end{pmatrix}$$

so we have a complete result for Ψ which will be a double integral. To get f and g, we then use our initial equations $f = S_1\Psi$ and $g = S_2 \Psi$.

In **Subsection 4.2.2** Sned specializes the Peters Solution to the case $\beta = 0$ and $\mu = \nu$ and $I_1 = (0,1)$ which is the **Titchmarsh case** of 1948. You get a different answer for each integer range you allow for exponent α due to the parts integration issue, which is expressed generally using the $\mathcal{D}$ operator from Chapter 2. Sned shows the general result for $\alpha < 0$ and $\alpha > 0$.

In **Subsection 4.2.3** Sned does a variation of the Peter's solution known as the **Noble Solution [1958].** The solution is elementary as shown in meta notes below, and one ends up with these results:

$$g_1 = K_{11}^{-1}(I_{11}f_1) \qquad f_2 = I_{22}^{-1}(K_{22}g_2) \qquad \Psi = S^{-1}KF = S^{-1}\,KG$$

so we seem to get more quickly to the partner functions, and they are used to get Ψ. In the Peters method we got Ψ first and then from it we computed the partner functions. The Noble problem seems to be restricted to $\mu = \nu$, though I don't know if it fails otherwise. The matrices are these in that case:

$$K = K_{\nu/2-\alpha,\alpha-\beta} \qquad I = I_{\nu/2+\alpha,\beta-\alpha} \qquad S^{-1} = S' = S_{\nu/2+\beta,-\alpha-\beta}$$

In **Subsection 4.2.4** we get the **Gordon-Copson Solution** [**1954,1961**] which is a variation of the Noble method, where we again have $\mu = \nu$. In this method we compute an intermediate vector Φ (different from a certain Φ which appears in the middle of the Noble solution) and we find that

$$\Phi = \begin{pmatrix} I_{11}f_1 \\ K_{22}g_2 \end{pmatrix} \qquad \Psi = S'\Phi = S^{-1}\Phi \qquad f = I'\,\Phi = I^{-1}\Phi \qquad g = K'\,\Phi = K^{-1}\Phi$$

The $\Psi = S'\Phi$ is exactly what we got in the Peters case, but the partner function equations are expressed here more directly in terms of vector $\Phi = (\varphi_1, \varphi_2)$, instead of as $f = S_1\Psi$ and $g = S_2\,\Psi$. Sned then writes out the exact solution for A(u) (but not the partner functions) for two exponent ranges as follows:

Case (a) : $0 < \alpha-\beta < 1$ [92] $\varphi_1 = 4.2.48$ $\varphi_2 = 4.2.49$ A(u) = 4.2.51
Case (b) : $0 < \beta-\alpha < 1$ [92 bot] $\varphi_1 = 4.2.52$ $\varphi_2 = 4.2.53$ A(u) = 4.2.51

Sned always makes a point of finding confirmation of his results in the literature.

In **Section 4.3** Sned does some work directed toward getting the partner functions more directly if you don't care about the solution A(u). He does this only for the Titchmarsh special case noted earlier. I did not study this section very much, but the purpose seems clear.

In **Section 4.4** Sned further specializes to our **azisym case $\mu = \nu = 0$**, which is to say, J_0. He further specializes to certain interesting exponents. Here is a summary: (first is disk, second is crack thing)

$\int_0^\infty d\xi\, \xi^{-1}A(\xi)J_0(\xi\rho) = \varphi_1(\rho)$ $\rho < 1$ $\varphi_1 \equiv F$ $\alpha = 1/2, \beta = 0$

$\int_0^\infty d\xi\, A(\xi)J_0(\xi\rho) = \chi_2(\rho)$ $\rho > 1$ $\chi_2 \equiv G$

A(u) = 4.4.4 // double integrals of F and G

φ_2 = 4.4.5 // these are the partner functions
χ_1 = 4.4.6

$\int_0^\infty d\xi\, \xi^{+1} A(\xi) J_0(\xi\rho) = \varphi_1(\rho)$ $\rho < 1$ $\varphi_1 \equiv F$ $\alpha = -1/2, \beta = 0$

$\int_0^\infty d\xi\, A(\xi) J_0(\xi\rho) = \chi_2(\rho)$ $\rho > 1$ $\chi_2 \equiv G$

A(u) = 4.4.12 // double integrals of F and G
φ_2= 4.4.13 // these are the partner functions
χ_1 = 4.4.14

In **Section 4.5** Sned considers a completely different class of dual integral equations. In this class, we replace the $J_\mu(\xi\rho)$ with cos(ξρ) and sin(ξρ) [**trig duals**] . The solution for the potential comes out being a series instead of an integral. He treats the four cases sin/cos with ξ^{-1} /ξ^{+1} in the first equation. Here are my meta comments on one of those four cases: cos with ξ^{-1}. He considers the duals 4.5.3 and 4.5.4:

$\int_0^\infty d\xi\, \xi^{-1} A(\xi) \cos(\xi\rho) = \varphi_1(\rho) = F(x)$ $\rho < 1$

$\int_0^\infty d\xi\, A(\xi) \cos(\xi\rho) = \chi_2(\rho) = 0$ $\rho > 1$

"Sned rolls his own solution to this problem. His ansatz is the series solution 4.5.6 p 99 which automatically satisfies the second equation above. I will just summarize his results: the solution is given by A(u) = the series in 4.5.6 where the coefficients a_q are given by 4.5.8,9. Sned uses $D_u = \partial_u$. So we have a single infinite series where each coefficient is a single integral of F(x) but you have to do q fancy derivatives for a_q which is a little messy. The φ_2 partner function is very simple as in 4.5.12, while the other is a little more complicated. Sned then looks at a few special cases for F(x). "

In **Section 4.6** Sned considers the Titchmarsh **Bessel** situation where an **weight factor** [1 + k(u)] is added to the first dual, where k(u) is some known arbitrary function. Since k(u) is unspecified, and since the situation is now much more complicated than in the Titchmarsh case, Sned is just going for a **Fred 2** integral equation which could be **numerically solved**. See below for more details.

In **Section 4.7** Sned addresses what he calls **The General Problem**. Instead of having modified-Hankel S_1 and S_2 operators in our duals (or simple trigs times powers), suppose we have two *arbitrary* linear (perhaps integral) operators L_1 and L_2. There is no known closed form solution to this general problem.

Subsection 4.7.1. But if you assume that $L_1 L_2^{-1} = M_1 M_2(\infty)$ where the M_i have certain properties (a situation he claims is often the case in real world problems), then you can concoct a way to solve the problem numerically which I write this way in the meta notes below:

A = $L_2^{-1} G$ // solution of integral equation G = L_2A
g_1 = $[M_2(1)^{-1}]_{11}$ h* // solution of integral equation h* ≡ $[M_2(1)]_{11} g_1$
h* = $[M_1^{-1}]_{11}(f_1 - G^*)$ // solution of integral equation $[M_1]_{11}$ h* = f_1 - G*
G* = $[M_1 M_2(\infty)]_{12}\, g_2$ // something you just compute

You work from the last to the first. You compute G*, then you solve a certain Fred 2 integral equation for function h*, then you solve another Fred 2 equation for partner function g_1, and finally if you want the potential, you use the first line above. I did all this in full detail in my raw notes.

In **Subsection 4.7.2** he starts over and treats M_1^{-1} and $M_2(\infty)$ just the way we treated I and K in our Peters solution. Doing pre-multiplies he gets $M_1^{-1}\ L_1 = M_2(\infty)L_2 = L_3$ (analogy to earlier S). But all this is really just "formal" since any inverse operator implies an integral equation you have to go solve! This is his "**multiplying factor method**".

In **Subsection 4.7.3** he considers an auxiliary function H with the idea that $A = L_3^{-1}H$ and you then solve for H instead of A. He calls this an "**integral representation method**". Fine.

In **Subsection 4.7.4** he shows how his General Problem analysis gets specialized if you apply it to the Peter's solution situation.

In **Section 4.8** discusses two **numerical methods** for the [1 + k(u)] weighted version of Titchmarsh. The first is a perturbation theory idea, while the second is a variational method.

In **Section 4.9** he considers "**systems of dual integral equations**" where now we have n potentials and n pairs of duals. Perhaps this is what once faces in those crack problems he mentions in Chapter 1.

4.1 Introduction.

The dual integral equations tend to look like this:

$$L_1A(x) = f_1(x) \qquad \text{on } I_1$$
$$L_2A(x) = g_2(x) \qquad \text{on } I_2$$

where the L_i are integral operators. The RHS functions are prescribed only on the ranges shown. Once we solve these dual equations for A(x), we can evaluate each equation over the full integral I_1+I_2. The LHS of each equation is a function of x defined on the entire interval. Sned likes to *define* these functions as

$$f(x) \equiv L_1A(x) \qquad \text{on } I_1 + I_2$$
$$g(x) \equiv L_2A(x) \qquad \text{on } I_1 + I_2$$

Then we can think of $f = (f_1,f_2)$ and $g = (g_1,g_2)$ on (I_1,I_2). When we are at first presented with the dual equations, f_1 and g_2 are "prescribed" and what I call "the partner functions" f_2 and g_1 are unknown; but of course when the problem is solved, we know A(x) on the full interval and we know $L_iA(x)$ on the full interval and thus we know f and g on the full interval, so then we know the partner functions f_2 and g_1.

Sned ends this section with a sort of generalized solution method which does not do much for me and I don't think is ever used in the book, so I won't comment on it here.

4.2 Dual Integral Equations of the Titchmarsh Type

The kernels here are J_μ and J_ν, each with its own power function as a second factor, exponents are 2α and 2β, as shown in 4.2.1 and 4.2.2. Seems a reasonably broad class of kernels to ponder. Titchmarsh in 1948 only did the case $\mu = \nu$, but here we are being a little more general.

As our first step we write out the dual equations

$$\int_0^\infty du\, u^{-2\alpha} A(u) J_\mu(xu) = F(x) \qquad x < 1 \qquad \text{// Titchmarsh type duals}$$

$$\int_0^\infty du\, u^{-2\beta} A(u) J_\nu(xu) = G(x) \qquad x > 1$$

But Sned wants us to rescale things at this point and write

$$A(u) = u\psi(u) \qquad f_1(x) = 2^{2\alpha}\, x^{-2\alpha}\, F(x) \qquad g_2(x) = 2^{2\beta}\, x^{-2\beta}\, G(x)$$

then our duals take this form

$$2^{2\alpha}\, x^{-2\alpha} \int_0^\infty du\, u^{-2\alpha+1} \psi(u) J_\mu(xu) = f_1(x) \qquad x < 1$$

$$2^{2\beta}\, x^{-2\beta} \int_0^\infty du\, u^{-2\beta+1} \psi(u) J_\nu(xu) = g_2(x) \qquad x > 1$$

Using our modified Hankel operator definition, these equations become

$$\mathcal{S}_{\mu/2-\alpha,2\alpha}\{\psi(r); x\} = f_1(x) \qquad x < 1 \qquad 4.2.4a \qquad S_1$$
$$\mathcal{S}_{\nu/2-\beta,2\beta}\{\psi(r); x\} = g_2(x) \qquad x > 1 \qquad 4.2.4b \qquad S_2$$

So notice this important point: this very general class of duals can be written using nothing but the Modified Hankel operators as shown!! That is why we learned that stuff in Chapter 2.

4.2.1 Peter's Solution [1961] of Titchmarsh Type Duals [84]

In a sense, this Peters solution is "the main act" of this book, so I give lots of detail. I use my own 2x2 matrix notation to explain what happens here. I don't like writing all those subscripts, so I just call the above two operators S_1 and S_2. Then we start off with these prescribed dual integral equations (as shown for example page 84 4.2.4a,b) [Notice that f_1 and g_2 and Ψ are certain "scaled" versions of the functions F(x), G(x) and A(k) which actually appear in the dual integral equations, as shown in 4.2.1,2,3.]

$$[S_1\, \Psi]_1 = f_1 \quad \text{or} \quad [S_1\psi](x) = f_1(x), \ x \text{ in } I_1 \qquad (*)$$
$$[S_2\, \Psi]_2 = g_2 \quad \text{or} \quad [S_2\psi](x) = g_2(x), \ x \text{ in } I_2$$

while the other two equations are (we justify these a few lines below)

$[S_1\,\Psi]_2 = f_2$ or $[S_1\psi](x) = f_2(x),\ x \text{ in } I_2$

$[S_2\,\Psi]_1 = g_1$ or $[S_2\psi](x) = g_1(x),\ x \text{ in } I_1$

We know S_1 and S_2 for full range of both "indices", think $[S_1]_{xy}$ for example, which really means the kernel of the integral operator so $s_1(x,y)$ if you like. We know f_1 and g_2. What we DON'T know are these three items: Ψ, f_2 and g_1. In particular, we don't know Ψ on either I_1 or I_2. We know nothing about it! We know f on I_1 which is called f_1, and we know g only on I_2 which is called g_2.

Our general attack plan is as follows: First, we assume that each of our interval equations (*) is part of a larger full interval equation,

$S_1\,\Psi = f$ or $[S_1\psi](x) = f(x),\ x \text{ in } I_1 \cup I_2$

$S_2\,\Psi = g$ or $[S_2\psi](x) = g(x),\ x \text{ in } I_1 \cup I_2$

and this then justifies our equations (**), where f_2 refers to f on interval I_2, for example. If we can somehow find a solution that works, then our assumption will be justified. Our method will be to first find Ψ on $I_1 \cup I_2$. Then using that we will compute the full range f and g using the above two equations, and that of course tells us f_2 and g_1 so we will then have our three unknowns and the dual integral equations are then completely solved.

Here is how the solution goes. We *hope* to find two triangular matrices I and K such that

$IS_1 = KS_2 \equiv S,$ with S invertible,

so we then would have

$IS_1\,\Psi = If$ or $S\,\Psi = I\,f$

$KS_2\,\Psi = Kg$ $S\,\Psi = Kg$

If we can find matrices I and K which are triangular in the sense shown below, then we would have

$$S\,\Psi = If = \begin{pmatrix} I_{11} & 0 \\ I_{21} & I_{22} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} I_{11}\,f_1 \\ * \end{pmatrix} \quad \Rightarrow \quad [S\,\Psi]_1 = I_{11}\,f_1$$

$$S\,\Psi = Kg = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} * \\ K_{22}g_2 \end{pmatrix} \quad \Rightarrow \quad [S\,\Psi]_2 = K_{22}\,g_2$$

where * means "don't care". We can combine the useful parts of the last two equations to get

$$S\,\Psi = \begin{pmatrix} I_{11}\,f_1 \\ K_{22}\,g_2 \end{pmatrix} \equiv \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h$$

where we now know everything on the RHS (f_1 and g_2 were prescribed)!! So NOW we can invert S and the problem is solved: (S turns out to be a Hankel transform deal)

$\Psi = S^{-1}h$ // get Ψ ie $\psi = S^{-1}(h_1 \cup h_2)$

$[S_1 \Psi]_2 = f_2$ // get the two partner functions

$[S_2 \Psi]_1 = g_1$

This outlines the solution concept. It turns out that triangular I and K operators DO exist such that we can say $IS_1 = KS_2 \equiv S$. These operators are those Erdelyi-Kober things with carefully tuned subscripts which make them work right! So the idea is to compute the function h, then apply S^{-1}. Undoing our initial scaling regarding f and g, we can write

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} I_{11} f_1 \\ K_{22} g_2 \end{pmatrix} = \begin{pmatrix} I_{11} 2^{2\alpha} x^{-2\alpha} F(x) \\ K_{22} 2^{2\beta} x^{-2\beta} G(x) \end{pmatrix}$$

He is able to do a little combining of the external factor with the I and K here to write this as

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2^{2\alpha} x^{-2\alpha} I'_{11} F(x) \\ 2^{2\beta} x^{-2\beta} K'_{22} G(x) \end{pmatrix} \quad \text{or} \quad h = h_1 \cup h_2 = (h_1(x), h_2(x))$$

where I' and K' are a little simpler (different subscripts) than I and K and appear in 4.2.8, so we can write our solution in this same notation as

$$A(u) = u S^{-1}\{ 2^{2\alpha} s^{-2\alpha} I' F(s) \cup 2^{2\beta} s^{-2\beta} K'G(s); x \}$$

$$= u S^{-1} [2^{2\alpha} s^{-2\alpha} I' F(s)] \cup u S^{-1} [2^{2\beta} s^{-2\beta} K' G(s)]$$

So our Peters solution gives a complete answer for the duals shown above, each having a J and a power and each driven by a function. This is a fairly broad class of duals in one fell swoop. (I'll say!)

Sned says that Peters himself (1961 paper) did not do the problem in this full generality, but did the following special case $\alpha = -\omega/2, \beta = 0$ so his duals were these:

$$\int_0^\infty du\, u^\omega A(u) J_\mu(xu) = F(x) \qquad x < 1$$

$$\int_0^\infty du\, A(u) J_\nu(xu) = G(x) \qquad x > 1$$

so basically he was doing the problem of a power in the first equation only.

In my simplified presentation above, I did not show all those subscripts on S_1, S_2, I, K and S, but they appear in the text. Hidden in these subscripts is the issue illuminated in the next section where you must realize that if the second subscript on I or K is negative, you have to use the "forms" for I and K which were shown in Chapter 2 and which involve "parts integrations" which cause the special differential operator $\mathcal{D}^n$ to appear, as for example 2.4.31 on page 52 where α is assumed negative but $\alpha+n$ is positive.

Going back to our search for I and K such that $IS_1 = KS_2 \equiv S$, we can phrase this as a general matrix problem, which I study in "The Sneddon Matrix Problem.doc". I consider the matrix equation LA = UB, where A and B are invertible matrices, and L is lower triangular and U is upper triangular. The problem is this: given A and B, find L and U that make LA = UB. It turns out that, barring anomalous matrices A and B, you can in fact find many L,U pairs that work. In particular, you can find unique matrices L' and U such that $AB^{-1} = L'U$ and such that L' has all ones on its diagonal. This fact is known as the "LU decomposition theorem", and there is a condition on matrix AB^{-1} which requires it to be what I call a "reasonable" matrix: none of the first minors are 0. Assuming this condition is met, we can invert L' to get $L'^{-1} = L$, another lower triangular matrix. Then we have

$$AB^{-1} = L^{-1}U \qquad \Rightarrow \qquad LA = UB$$

Moreover, if we call this product C, we find that $C^{-1} = (LA)^{-1} = A^{-1}L'$, so C is invertible. The condition is the one which always arises when you do Gaussian Elimination you get a sub column of all zeros so you cannot "pivot". I have not studied how this condition is met in the continuous matrix case we are studying here, but someone probably talks about that somewhere.

Notice that the I and K matrices are triangular regardless of their labels, simply due to the Volterra endpoints in the two cases.

4.2.2 Titchmarsh's Special Case Solution [1948] [86]

He considered this special case

$$\int_0^\infty du\, u^{-2\alpha} A(u) J_\nu(xu) = F(x) \qquad x < 1$$

$$\int_0^\infty du\, A(u) J_\nu(xu) = G(x) \qquad x > 1$$

which is like the later Peters, but $\mu = \nu$ here so this is a simpler case. So Sned just applies our general solution to this special case. The outer integrals are shown in 4.2.17 (the union of two terms) The two inner integrals for h_1 and h_2 are shown in 4.2.18.

Now we see a technical issue arise. We have this general form

$$h_1 \sim I_{\eta 1, -\alpha} F \qquad h_2 \sim K_{\eta 2, +\alpha} G$$

Therefore, if both F and G driver functions are present, you are going to have to deal with the "parts stuff" in one or the other of these h_i integrals! If $\alpha < 0$ and G = 0, then you are clean with h_1 and you get the simple answer which is presented in 4.2.22 (ie, no parts action required). A special case occurs when F = 0 and $\alpha < 0$ but $\alpha > -1$, then you get 4.2.23 which has one derivative ∂_t sitting in it. But for general $\alpha < 0$, you have to do parts n times and the h_2 integral is then 4.2.20 with that $\mathcal{D}^n$ thing, and 4.2.21 is the total answer for A(u), all when $\alpha < 0$.

Similarly, if $\alpha > 0$, you have to do parts with the h_1 term as in 4.2.24 and the total answer is then given in 4.2.26.

Sned then looks at the special case $0 < \alpha < 1$. This brings in a single ∂_t as shown top page 88. He then does parts to remove this thing which results in a shift in the J order and we get 4.2.27. He then looks at

the two sub cases F = 0 and G = 0 and gives a result for each. One of these cases replicates a Busbridge 1938 solution (Beltrami was 1880's).

So in this section all we did was apply our general Peters solution to this special case, and then we learned how to do parts to handle negative second indices on the I and K operators.

4.2.3 Noble's Special Case Solution [1958] [88]

This solution is very similar to the Peters one above but has a different pathway. It is also a special case inasmuch as the Bessel functions have $\mu = \nu$, but it is more general that the strict Titchmarsh case of the last section. [I think the presentation below goes through if $\mu \neq \nu$.] We are still in a framework where

$$A(u) = u\psi(u) \qquad f_1(x) = 2^{2\alpha}\ x^{-2\alpha}\ F(x) \qquad g_2(x) = 2^{2\beta}\ x^{-2\beta}\ G(x)$$

Recall the Peter's solution where we said:

We hope to find two matrices I and K such that $IS_1 = KS_2 \equiv S$ so we have

$$IS_1\ \Psi = IF \qquad \text{or} \qquad S\ \Psi = I\ F$$
$$KS_2\ \Psi = KG \qquad\qquad S\ \Psi = KG$$

where recall that F and G are column vectors holding the two components

$$F_1 \equiv \begin{pmatrix} f_1 \\ 0 \end{pmatrix} \qquad G_2 \equiv \begin{pmatrix} 0 \\ g_2 \end{pmatrix} \qquad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Whereas in our Peters solution we focused on the *matrix* equations $IS_1 = KS_2 = S$, Noble focuses instead on this obviously related *vector* equation shown above (which is "closer" to the partner functions)

$$IF = KG \qquad \text{// which is p89 A}$$

He writes this as $I[\ F_1 + F_2\ ] = K[\ G_1 + G_2\ ]$, and then defines

$$\Phi \equiv I\ F_1 - K\ G_2$$
$$\Rightarrow\ \Phi = K\ G_1 - I\ F_2 \qquad \text{// a restatement of } I[\ F_1 + F_2\ ] = K[\ G_1 + G_2\ ]$$

So Φ is one of those intermediary "auxiliary objects" you often see invented as part of a solution of dual integral equation problems! If we write out the two expressions for Φ shown above, and use the triangularity of the I and K matrices, we find that

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} I_{11} f_1 \\ -K_{22} g_2 \end{pmatrix}$$
$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} K_{11} g_1 \\ -I_{22} f_2 \end{pmatrix}$$

Equating the two components we obtain (we are now done with Φ)

$I_{11}f_1 = K_{11}g_1 \quad \Rightarrow \quad g_1 = K_{11}^{-1}(I_{11}f_1)$

$K_{22}g_2 = I_{22}f_2 \quad \Rightarrow \quad f_2 = I_{22}^{-1}(K_{22}g_2)$

and we thus obtain the partner functions g_1 and f_2 directly as shown on the right. Now that we know all four functions, we know the full vectors F and G, so we can solve for the potential Ψ using either of the following: (on the right we are just defining various primed matrices)

$S\,\Psi = I\,F \quad \Rightarrow \quad \Psi = S^{-1}\,I\;\;F \;\equiv\; S'\,I\;\;F \;\equiv\; S''\,F$

$S\,\Psi = K\,G \quad \Rightarrow \quad \Psi = S^{-1}\,K\,G \;\equiv\; S'\,K\,G \;\equiv\; S'''\,G$

In the Peter's method, we compute Ψ first, and then from it we get the partner functions. In the Noble method, we compute the partner functions first and from them compute Ψ. It may be that the Nobel method is more "efficient" somehow in getting a full solution, and might be a clear winner if all you want are the partner functions.

So this is just a variation of the Peters method, but of course it came before Peters. I took the time to verify all the subscripts, and here they are:

$S_1 = S_{\nu/2-\alpha,2\alpha}$

$S_2 = S_{\nu/2-\beta,2\beta}$

$S = S_{\nu/2-\alpha,\alpha+\beta}$

$I = I_{\nu/2+\alpha,\beta-\alpha}$

$K = K_{\nu/2-\alpha,\alpha-\beta}$

$S' = S_{\nu/2+\beta,-\alpha-\beta}$

$S'' = S_{\nu/2+\alpha,-2\alpha}$

$S''' = S_{\nu/2+\beta,-2\beta}$

As before, you have to do things correctly when a second index is negative which of course depends on the value of α and β.

Sned goes on to work out the Noble solutions in the following special cases:

Case (a) $\beta = 0$ and $0 < \alpha < 1$ [89]
Case (b) $\alpha = 0$ and $0 < \beta < 1$ [90]

There is a bit of confusion in the results because (1) I have overloaded the symbols F and G so that each really has two completely different meanings; (2) Sned keeps using something he calls Φ_1 to mean the complete function I call Φ, but not in my vector sense. I might call it just φ. But I don't think there is any serious confusion about the final solutions. I could not find these two "cases" as such in Polyanin which really does not have that much when it comes to duals!

4.2.4 The Gordon-Copson Solution of the Noble problem [1954, 1961] [91]

I rather balled this up in the raw notes. Really this is just a variant on Nobel. We are still in a framework where (noted overloaded F and G symbols)

$A(u) = u\psi(u)$ $\quad f_1(x) = 2^{2\alpha}\ x^{-2\alpha}\ F(x)$ $\quad g_2(x) = 2^{2\beta}\ x^{-2\beta}\ G(x)$

Recall once again the Noble starting point:

$\Psi = S^{-1}\ I\ F\ =\ S'\ I\ F\ =\ S''\ F$ // which is 4.2.35
$\Psi = S^{-1}\ K\ G\ =\ S'\ K\ G\ =\ S'''\ G$ // which is 4.2.33

Let's focus just on the following pieces of the above (two vector=vector equations)

$\Psi = S'\ I\ F$ $\quad S' = S^{-1} = S_{\nu/2+\beta,-\alpha-\beta}$
$\Psi = S'\ K\ G$

Gordon-Copson suggests representing Ψ in terms of a vector Φ,

$\Psi = S'\Phi$ where Φ is unknown // not same Φ as Nobel Φ !

So we know ahead of time that $\Phi = I\ F$ and $\Phi = K\ G$ are the right solutions, but since we don't know both parts of F or G, we cannot yet compute Φ. But consider:

$$\Phi = I\ F \quad => \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix} I_{11} & 0\\ I_{21} & I_{22}\end{pmatrix}\begin{pmatrix} f_1\\ f_2\end{pmatrix} = \begin{pmatrix} I_{11}f_1\\ *\end{pmatrix}$$

$$\Phi = K\ G \quad => \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix} K_{11} & K_{12}\\ 0 & K_{22}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix} = \begin{pmatrix} *\\ K_{22}g_2\end{pmatrix}$$

Therefore we at once have a solution for Φ in terms of known quantities,

$$\Phi = \begin{pmatrix} I_{11}f_1\\ K_{22}g_2\end{pmatrix}$$

Meanwhile, we can invert our starting equations and define primed I and K's this way

$F = I^{-1}\Phi\ \equiv I'\ \Phi$ $\quad I'\ \equiv I^{-1}$
$G = K^{-1}\Phi\ \equiv K'\ \Phi$ $\quad K' \equiv K^{-1}$

Thus we have

$\Psi = S'\Phi$
$F = I'\ \Phi$
$G = K'\ \Phi$

and we have solved the problem. *For example*, the second line says

$$\begin{pmatrix} f_1\\ f_2\end{pmatrix} = \begin{pmatrix} I'_{11} & 0\\ I'_{21} & I'_{22}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix} I'_{11}\varphi_1\\ I'_{21}\varphi_1 + I'_{22}\varphi_2\end{pmatrix} \quad => \quad f_2 = I'_{21}\varphi_1 + I'_{22}\varphi_2$$

We could write this out in more detail as

$f_2 = I'_{21} I_{11} f_1 + I'_{22} K_{22} g_2$

Sned then applies this theory to the following cases. In each case, one of the functions has a single derivative and the other has no derivative. You then have to do $\Psi = S'\Phi$.

Case (a) : $0 < \alpha-\beta < 1$ [92] **φ_1 = 4.2.48** **φ_2 = 4.2.49** **A(u) = 4.2.51**

Case (b) : $0 < \beta-\alpha < 1$ [92 bot] **φ_1 = 4.2.52** **φ_2 = 4.2.53** **A(u) = 4.2.51**

4.3 Functions derived from solutions of Dual Integral Equations

I understand this section ahead of time. Sometimes you don't care about the dual solution, you care instead about the unknown "partner functions". Here he derives expressions for both partner functions in two cases (each with the proper restrictions), and here we work with the duals shown in 4.3.3 which is really the Titchmarsh special case of section (2) above. As usual, he shows all the details. I think I can let this section ride, since I know exactly what it is all about.

4.4 Azisym Special cases

If we go back to Chapter 3 on the Beltrami disk, our opening problem was this classic one,

$\int_0^\infty d\xi\, \xi^{-1} A(\xi) J_0(\xi\rho) = f(\rho)$ $\rho < 1$ $f(\rho) = 1$ for the charged disk

$\int_0^\infty d\xi\, A(\xi) J_0(\xi\rho) = 0$ $\rho > 1$

where J_0 is forced by azisym on the Smythian form. So Sned wants us to do these a little more generally:

$\int_0^\infty d\xi\, \xi^{-1} A(\xi) J_0(\xi\rho) = \varphi_1(\rho)$ $\rho < 1$

$\int_0^\infty d\xi\, A(\xi) J_0(\xi\rho) = \chi_2(\rho)$ $\rho > 1$

and he now uses the same idea that $\varphi(x) = [\, \varphi_1(x), \varphi_2(x)] = [\, F(x), G(x)]$ on the two regions I_1 and I_2. So all he really wants to do here is write all the results in the simplest form possible, using perhaps our Peters general formula, or the Noble one. Actually, we are within the Titchmarsh boundary for the rest of this section where only the first equation has a power. So we get

A(u) = 4.4.4 // double integrals of F and G
φ_2= 4.4.5 // these are the partner functions
χ_1 = 4.4.6

The latter two involve F** and G** which are the integrals of F and G shown in 4.4.7. He then ends this little section with 4.4.8 which gives an integral of χ_1 in terms of F and G, so maybe this is "the total charge on the disk".

Sned then does (starting bottom of page 97) the similar case which has ξ^{+1}

$$\int_0^\infty d\xi\, \xi^{+1} A(\xi) J_0(\xi\rho) = \varphi_1(\rho) \qquad \rho < 1$$

$$\int_0^\infty d\xi\, A(\xi) J_0(\xi\rho) \quad = \chi_2(\rho) \qquad \rho > 1$$

and his answers here are

A(u) = 4.4.12 // double integrals of F and G
φ_2= 4.4.13 // these are the partner functions
χ_1 = 4.4.14
integral of χ_1 = 4.4.16 where X appearing here is an F function as shown in 4.4.17.

Whereas the first problem aligned with the electrostatics of a disk with prescribed potential on the disk and prescribed σ outside the disk (the two functions φ_1 and χ_2), this second problem he claims arises in the study of "stress near penny-shaped cracks in infinite elastic bodies".

4.5 Dual Integral Equations with Trig Kernels [p 98]

Up to this point in this chapter our kernels have been powers with J_μ function. Now we shift scenery and consider some cases where we have "trig" kernels with special powers.

(a) cosine case with ξ^{-1}

We now replace $J_\mu(x\rho)$ in kernels with cos(xρ) and we consider the duals 4.5.3 and 4.5.4

$$\int_0^\infty d\xi\, \xi^{-1} A(\xi) \cos(\xi\rho) = \varphi_1(\rho) = F(x) \qquad \rho < 1$$

$$\int_0^\infty d\xi\, A(\xi) \cos(\xi\rho) \quad = \chi_2(\rho) = 0 \qquad \rho > 1$$

Sned rolls his own solution to this problem. His ansatz is the series solution 4.5.6 p 99 which automatically satisfies the second equation above. I will just summarize his results: the solution of the trig duals 4.5.3,4 is given by A(u) = the series in 4.5.6 where the coefficients a_q are given by 4.5.8,9. Sned uses $D_u = \partial_u$. So we have a single infinite series where each coefficient is a single integral of F(x) but you have to do q fancy derivatives which is a little messy. The φ_2 partner function is very simple as in 4.5.12, while the other is a little more complicated. Sned then looks at a few special cases for F(x).

(b) sine case with ξ^{-1}

Now we have these equations instead (4.5.22)

$$\int_0^\infty d\xi\, \xi^{-1} A(\xi)\sin(\xi\rho) = \varphi_1(\rho) = F(x) \qquad \rho < 1$$

$$\int_0^\infty d\xi\, A(\xi)\sin(\xi\rho) \qquad = \chi_2(\rho) = 0 \qquad \rho > 1$$

He again finds a series 4.5.23 which satisfies the second equation. He then solves for the coefficients and does one special case.

(c) cosine case with ξ^{+1} (p 103 bottom)

This is a crack problem situation and the solution to the duals is pretty simple. He shows the solution, and we get as that solution 4.5.28 where g(t) is given by 4.5.31. The partner functions are also given and each is another integral of g(t), and he wraps up with something for the integral of the χ_1 partner.

Then on page 105 Sned gives another derivation of this exact same solution using a 1963a method of Tranter. I suspect then that the derivation given above is Sned's own, and he is then using Tranter just as verification, the way I do things as well. The Tranter solution is 4.5.33.

4.6 Dual Integral Equations with extra weight function

This is the Titchmarsh situation where an extra factor [1 + k(u)] is added to the first dual, and where k(u) is some known arbitrary function. Since k(u) is unspecified, and since the situation is now much more complicated than in the Titchmarsh case, Sned is just going for a Fred 2 integral equation which could be numerically solved. He starts with $\Psi = S\ H$ and where $H = (h_1, h_2)$ which need to be found. The solution for h_2 is 4.6.6, and it is for h_1 that he obtains the inhomo Fred 2 equation 4.6.11 with kernel given by 4.6.10. The inhomo driving term in this Fred 2 is H(x) as developed in 4.6.9. Since the kernel integral better converge, we get now some restrictions on k(u) as shown top page 108. Once you solve the Fred 2 for h_1, you have 4.6.12 as your final solution.

Case (a) $-1 < \alpha < 0$.
Subcase $\nu = 0$ and $\alpha = -1/2$

Case (b) $0 < \alpha < 1$
Subcase $\nu = 0$ and $\alpha = 1/2$

All the time Sned makes contact with the work done by others in the 1955-1963 time frame.

Noble 1963a: the approach here is get a Fred 2 integral equation for A(u). He is always using those Kober operators. The result is 4.6.33 with L in 4.6.32 and driving term given in various ways.

Subcase $\nu = 0$ and $\alpha = 1/2$ p 112
Subcase $\nu = 0$ and $\alpha = -1/2$ p 112 lower down

Finally, on page 113, he considers what happens if you put the weight function now called [1 + n(u)] in the second equation instead of the first. He quotes all results, method is similar he says.

4.6.2 System of Algebraic Equations. [p 113] This is a Tranter 1954 method which provides an alternative to solving a Fred 2 equation. I am going to skip this, perhaps it is similar to doing a Neumann solution to that integral equation. Sned says Tranter stimulated lots of activity with his early papers.

4.6.3. Integral equation for $\chi 1(x)$. Logical approach if this partner is what you are after, and he does this with G = 0. I skip the details, and he treats the usual two cases

Case (a) $0 < \alpha < 1$

Case (b) $-1 < \alpha < 0$.

4.7 The General Problem

Now we no longer talk about specific kernels, just operators L_1 and L_2 in our duals.

4.7.1 Reduction to the Solution of Two integral equations

Sned's goal in this "general case" is to come up with system of two Fred 2 integral equations. The details are all in the raw notes, but here is a little summary of the "chain" you end up with:

So we end up now with this set of equations:

$A = L_2^{-1}G$ // solution of integral equation $G = L_2A$

$g_1 = [M_2(1)^{-1}]_{11}\, h^*$ // solution of integral equation $h^* \equiv [M_2(1)]_{11} g_1$

$h^* = [M_1^{-1}]_{11}(f_1 - G^*)$ // solution of integral equation $[M_1]_{11}\, h^* = f_1 - G^*$

$G^* = [M_1 M_2(\infty)]_{12}\, g_2$ // something you just compute

In words: We assumed up front that $L_1L_2^{-1} = M_1M_2(\infty)$ so we assume that we know all about M_1 and M_2 as defined in 4.7.4 and 5. This is "the big assumption". Finding the M_i and verifying that $L_1L_2^{-1} = M_1M_2(\infty)$ might take some work! Then since g_2 is prescribed (he called it G) we know G*. Then since f_1 is prescribed as well, we know h*. But then we know g_1 which we did not know at the start! But then we know all of G, so we compute A, and we are done! If you only want to know partner g_1, then you skip the last step and don't bother computing A. In this case, my comments show how the problem has been reduced to "solving two integral equations", which then explains the title of this section. Remember that if we know some matrix M, finding M^{-1} means solving an integral equation.

4.7.2 The Multiplying Factor Method

This is really just the Peters method which I explained in my 2x2 matrix notation. The idea is to apply one op to the first equation, another op to the second, and then end up with this

$L_3A = F' = (f_1', f_2')$
$L_3A = G' = (g_1', g_2')$

so you can then invert L_3 and you are done. Triangular is important in this method!

4.7.3 The Integral Representation Method

In the previous section we has $A = L_3^{-1}H$ and we had two ways to write L_3^{-1} in terms of our known operators. He now gives $h = (h_1, h_2)$ the new name g. The point is that we are now supposed to regard the equation $A = L_3^{-1}H$ as an "integral representation" of A, such that we replace the unknowns A_1 and A_2 by the unknowns I call h_1 and h_2. If we can find these two unknowns, then we know A. But we found these two unknowns in the last section, namely

$h_1 = [M_1^{-1}]_{11} f_1$
$h_2 = [M_2]_{22} g_2$

We are warned that one of our two forms of L_3^{-1} may "not exist" and the other might. So somehow this "method" seems pretty much the same as our multiplier method above. I suspect he means by the term "integral representation" a single integral. I guess if all operators exist, something like $L_3^{-1} = L_2^{-1} M_2(\infty)^{-1}$ can be regarded as such a single integral. It certainly was in the Peters situation where this equation read $S^{-1} = S_2^{-1}K$ and we could use Sned's Appendix A to get S^{-1} as a single integral operator.

4.7.4 Identification of the Operators

OK, here finally he shows how the Peters solution fits into the general methods just reviewed. He "identifies" the operators M_1 and M_2, and N_1 and N_2 which make them up. So at least Sned gives us closure on this connection. See my comment (2) above.

4.8 Approximate Solutions to Dual Integral Equations

This is the kind of material that the true practitioner must study, while the passing dabbler can just browse. Two methods are discussed here.

(1) This is attributed to King (1936). We noted earlier that if you have a weight function as in 4.8.1, it is likely that you will never find a closed form solution to your duals and approximation is required. Basically the solution outlined here is perturbation theory in terms of the weight function ε inside the kernel which is assumed to be "small". With $\varepsilon = 0$, the solution is $A^{(0)}$ which we assume we know exactly! (so it pays, as I think Polyanin said, to try to template yourself around a known problem). Then you make a perturbations series as in (4.8.5). Had Schiff done this, he would show a smallness parameter λ more explicitly. You then end up with an iterative method as shown p 124 C. At each level of the perturbation theory, you have a dual integral equation problem where the second equation driver function

is 0. This can be formally solved as shown in 4.8.6 so that each level $A^{(r+1)}$ of the solution can be computed from the previous level $A^{(r)}$ as an integral involving ε and known operators. The reader is told to go read King's paper for example.

(2) This variational method is that of Noble 1958b and is quite elaborate, I did not read it through. You start off with a dual pair like 4.8.8,9 where we allow some single-variable factors as part of the kernel. But this is a very narrow case because we have $L_1 = L_2$! Also, he is going to assume that $L_1^{-1} = L_1$. This is what allows him to somehow replace the duals with a single integral equation. The variational part comes in on page 126 where he defines J(e) as a functional of the function e and sets δJ = 0 and all that stuff. I skip all the rest, this is a very specialized method for a very specialized case. I suppose Sned felt it was an interesting and different method (being variational) so he threw it in.

4.9 Simultaneous Dual Integral Equations

What Sned means by this title is shown in 4.9.1. The first "dual equation" is actually a system of n equations (with label i), and each equation involves some $J_{\mu i}$ Bessel function and has some prescribed driving function $p_i(x)$. In each equation, n different unknown functions Ψ_i appear, weighted by some coefficients c_{ij} . The second "dual equation" is also a system of n equations, where each equation is for a specific Ψ_i unknown with its $J_{\mu i}$ Bessel function, and the driving function is 0. In other words, the second equation set is much simpler than the first set. I could come up with some hyper matrix notation for all this stuff if I were serious about it.

So don't confuse this with some kind of n-fold set of equations where you have n intervals which generalize (0,1) and (1,∞). Here we still have only those two intervals, and they are still I_1 and I_2.

Why you would care about such a problem is unclear, but Sned throws it in to flesh out his chapter on duals. It is another "different" problem so it warranted a home in his monograph. Erdogan and Bahar 1964 are the people who did this one. Sned produces 4 page of details. In 4.9.2 he seems to expand each Ψ_i in certain Bessel's with a set of coefficients A_{jm} which he then wants to solve for. He produces a horrible set of equations from which you are supposed to find these A_{ik} as shown in 4.9.5. The index sum goes to infinity, so it is some kind of infinite matrix equation for A_{ik} which no doubt is just a reflection of the integral equation nature of things. At this point Sned thankfully ceases and desists, mentioning several groups who have attacked such a matrix equation in various contexts.

He then starts over with a slightly different problem. Here, in place of $J_{\mu i}(xy)$ we have sin(xy) or cos(xy) so things would seem a bit simpler. This is shown in 4.9.6. This time, he assumes a solution for as shown in 4.9.7a for Ψ_i which is a sum of Bessel functions, motivated of course by our knowledge of the Weber discontinuous trig integrals, which he requotes. The coefficients are A_{ij}. We end up with 4.9.13 which is again an infinite matrix equation for the A_{ij} , albeit a much simpler one than we got in our first example.

I don't know if this system case is mentioned in Polyanin.

And so, finally, ends this very long and difficult Chapter 4.

Appendix A. Digression on K and I being triangular:

I show that K and I are each triangular in a certain manner, as are the inverses of these operators.