**Sneddon Chap 5 Notes : Dual Series Equations** PhL 8.4.10

Chapter 5 is 44 p long ( second in length only to Chapter 4 at 54 p) with huge detail. My raw notes below were only 7 pages, to which I added a 4 page Overview. There is no meta notes doc.


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# Overview (4 pages, written 12.15.10)

**Section 5.1** is general remarks for this Chapter.

**Section 5.2** (**Fourier-Bessel Dual Series Equations** ) :

The most general Bessel case we did in Chapter 4 looked like this ( I have changed u to k)

$$\int_0^\infty dk\, k^{-2\alpha} A(k) J_\mu(k\rho) = f_1(\rho) = F(\rho) \qquad x \text{ in } (0,1) = I_1$$

$$\int_0^\infty dk\, k^{-2\beta} A(k) J_\nu(k\rho) = g_2(\rho) = G(\rho) \qquad x \text{ in } (1,\infty) = I_2$$

and of course we then considered subcases where $\mu = \nu$, and/or where the exponents took certain special values. In this Chap 5 the most general Bessel case we shall consider is this: [ $J_\nu(k_n a) = 0$ ]

$$\Sigma_{n=1}^{\infty} k_n^{-2p} a_n J_\nu(\rho k_n) = f_1(\rho) \qquad f_1 = 0 \text{ for "problem (b) "} \qquad I_1 = (0,1)$$

$\Sigma_{n=1}^{\infty}$ $a_n\ J_\nu(\rho k_n) = g_2(\rho)$ $g_2 = 0$ for "problem (a) " $I_2 = (1,a)$

so we have $\mu=\nu$ and $\alpha = p$ and $\beta = 0$, the Titchmarsh type case. These duals would arise if you worked with a disk-like object inside a grounded (or perhaps chargeless) cylinder of radius a. Here, the Fourier-Bessel Transform plays the role of the Hankel Transform in the dual integral equations. We are always thinking cylindrical coordinates and atoms here. Note definitions of problems (a) and (b) above, and note that you can construct a solution to the general problem by simply adding these two solutions. As a reminder, here is the F-B transform from my transforms.doc,

$f(\rho) = \Sigma_{n=1}^{\infty} B_{n,\nu}\ J_\nu(k_n\rho)$ // expansion

$B_{n,\nu} = 2/[b^2 J_{\nu+1}(bk_n)^2] \int_0^b d\rho\ \rho\ f(\rho)\ J_\nu(k_n\rho)$ // projection

In **Subsection 5.2.1** we attack Problem (a) [Cooke and Tranter 1959]. We first expand

$a_n = C(k_n, \nu, a) \Sigma_{m=0}^{\infty} b_m\ J_{\nu+2m+1-p}(k_n)$

This ansatz is made because this form satisfies the second equation of the dual pair for any $b_m$. The price paid for dismissing the second equation so quickly is that finding the $b_m$ becomes a huge and ugly $\infty$ x $\infty$ Cramer's Rule problem $\Sigma_{m=0}^{\infty} b_m B_m = E$ where E and $B_m$ fully determined but ugly integrals. So unlike the dual integral case where we could obtain A(u) at least as a few integrals of known functions, here our result $a_n$ is an $\infty$ series of $b_m$ which we need an $\infty$ **Cramer** to solve. So, there is no "write down able" solution, at least with this approach. The section goes on to describe a perturbation theory method which uses some tabulated integrals in which Smythe and Tranter played a role.

In **Subsection 5.2.2** Sned describes what is probably a better method invented by him and Srivastav in 1964. This time $a_n$ is represented not as a sum, but as an integral of the unknown $g_1$ partner function just by a Fourier-Bessel transform. This $g_1$ in turn is represented as an Abel transform ($\alpha = p$) of another function h, and they then come up with a Fred 2 integral equation for h which is 5.2.23, in which the kernel involves a Bessel I function, a power, and the K-object defined on page 32 Chap 2 which is an integral involving four I/K type Bessel functions. This K-object is similar to what was tabulated in the previous solution method. A **Fred 2** integral equation of course can be regarded as an $\infty$ Cramer problem, but one which we perhaps know more about solving. In either $\infty$ Cramer case, we are going to numerically turn it into an NxN matrix problem where N is hopefully fairly large but not too large!

**Subsection 5.2.3** takes the same approach to the (b) problem $f_1 = 0$ .

**Section 5.3. (Dini Dual Series)** In the previous section, our quantization situation was $J_\nu(k_n a) = 0$ which is useful for making a potential vanish on a surrounding cylindrical metal boundary. If we threw in $\exp(-k_n z)$ as part of that atomic form, then $\partial_z$ makes a power of $k_n$ and we can think of $\sigma = 0$ or some prescribed value in the z = 0 plane. But we don't have a way to make $\sigma = 0$ on the surrounding cylinder, which involves $\partial_\rho$ and hence involves $J_\nu'(k_n a)$. Instead of considering this situation, we can consider the more general situation where we want $(k_n a)J_\nu'(k_n a) + H\ J_\nu(k_n a) = 0$ on the cylindrical boundary, where H is just some positive number. This boundary condition is sometimes called the "radiation" one for historical reasons, and we might call it a "locally mixed boundary condition". Then H=0 gives our case

just mentioned. Just as the case $J_\nu(k_n a) = 0$ leads to what we usually call the Fourier-Bessel series expansion (just part of that transform), the H condition leads to what is called a **Dini Series** expansion, which is part of the **Hankel-Schwartz Transform**. [ Remember: every homo BC leads to a different Sturm-Liouville problem.] For any value of H, you get some spectrum for your $k_n$ and you expand on your eigenfunctions, whatever they might be. Sned expends 6 long pages showing how you can solve a Dini Dual Series equation pair by grinding it down to a Fred 2 integral equation. Obviously the Dini case is not going to be any simpler that the regular Fourier-Bessel case we treated above, and which we found has no closed-form solution.

**Section 5.4 (Trig Dual Series)** is the analog of our dual integral equations discussion where we considered trig functions replacing the Bessel ones. A typical dual pair here is (now $k_n = n\pi/a$)

| | | |
|---|---|---|
| $\Sigma_{n=1}^{\infty} k_n^{+p} a_n \sin(k_n x) = f_1(x)$ | $f_1 = 0$ for "problem (b) " | $I_1 = (0,d)$ |
| $\Sigma_{n=1}^{\infty} a_n \sin(k_n x) = g_2(x)$ | $g_2 = 0$ for "problem (a) " | $I_2 = (d,a)$ |

Our geometries of interest are now more Cartesian in nature, and we want $\sin(k_n a) = 0$ as our boundary condition with the spectrum $k_n$ as shown. Sned simplifies by setting $a = \pi$ so the above becomes

| | | |
|---|---|---|
| $\Sigma_{n=1}^{\infty} n^p a_n \sin(nx) = f_1(x)$ | $f_1 = 0$ for "problem (b) " | $I_1 = (0,c)$ |
| $\Sigma_{n=1}^{\infty} a_n \sin(nx) = g_2(x)$ | $g_2 = 0$ for "problem (a) " | $I_2 = (c,\pi)$ |

In **Subsection 5.4.1** he attacks this "sine problem" for p = +1 and $g_2 = 0$. As described in detail below, Sned is able to solve this problem in closed form, just as we solved certain dual integral equations. The answer involves doing two integrals of known objects. We get $h_1$, then $g_1$, then $a_n$. Sned next does the case with p = +1 but now $f_1 = 0$, and gets a similar result. He then does the two p = -1 cases by reflecting the interval and making use of the p = +1 solutions. Sned concludes by noting that Williams 1964 got the same results by a different path.

**Subsection 5.4.2**. **(Trig Dini Dual Series)** The analogy with the Dini series in this trig world would be to have the locally mixed boundary condition be something like $\sin'(k_n a) + H\sin(k_n a) = 0$. In this case, the spectrum for $k_n$ will be different. I guess there is some H you can select so the spectrum comes out being $k_n = (n-1/2)$ and eigenfunctions are then $\sin[(n-1/2)x]$. So the connection to "Dini" is perhaps a little strained here, what he wants to solve are the duals shown above with $n \rightarrow (n-1/2)$. I have no motivation to know why this is interesting. He treats the same set of cases $p = \pm 1$.

**Subsections 5.4.3 and 5.4.4.** Here we repeat the last two subsections with sin → cos.

**Subsection 5.4.5.** Tranter solved the trig problems by a different method which is messier, but the answers are sometimes simpler, so Sned comments on this stuff and quotes those simpler results.

**Section 5.5 (Jacobi Poly Dual Series)** is totally new to this book. Above we first did Bessel $J_\nu$ duals, then trig duals, and now we are going to do Jacobi Polynomial duals. This is important to me because a special case of Jacobi functions are the Legendre functions and this will relate to the bowl. The duals of interest are these

$\Sigma_{n=0}^{\infty} K_n(\alpha,\beta)\, A_n\, P_n^{(\alpha,\beta)}(\cos\theta) = f_1(\theta)$ $\quad\theta$ in $(0,\varphi)$

$\Sigma_{n=0}^{\infty} K'_n(\alpha,\beta)\, A_n\, P_n^{(\alpha,\beta)}(\cos\theta) = g_2(\theta)$ $\quad\theta$ in $(\varphi,\pi)$

where K and K' are certain 1/ΓΓ as shown page 166 and replace the "powers" we had for Bessel and Trig. The big mystery of course is why these strange K factors are appearing here, they must arise in some natural way he is not saying. His buddy Srivastav 1964 came up with a solution. As usual, he considers the two cases where one or the other driver function vanishes, and we know we can add these solutions to get a general result. For $g_2 = 0$, his result is this: $h_1$ = 5.5.11 and then $A_n$ = 5.5.8, all closed form! Similarly for $f_1 = 0$ his result is $h_2$ = 5.5.13 and $A_n$ = 5.5.12.

On page 168 he then offers another solution by Noble 1963. The duals are now these:

$\Sigma_{n=0}^{\infty} p_n(\gamma,\beta)\, a_n\, \mathcal{F}_n(\alpha,\beta;x) = f_1(\theta)$ $\quad x$ in $(0,a)$

$\Sigma_{n=0}^{\infty} \quad a_n\, \mathcal{F}_n(\alpha,\beta;x) = g_2(\theta)$ $\quad x$ in $(a,1)$

where the function is that "shifted Jacobi" object I discuss in the Chap 2 notes. Again we get closed form results. A final section adds a weight function to the first equation above and this leads to Fred 2 stuff.

I am not aware of any curvilinear coordinate systems which have general Jacobi's as atomic forms, but they are of course famous orthogonal polynomials.

In **Section 5.6 (Legendre Dual Series)** Sned takes the general Jacobi result and just sets $\alpha = \beta = m$ and does some other fiddling to get a special case of the Jacobi result, where m is a bystander parameter, as in 5.6.1 and 2. I am only interested in the m=0 case which is the unit charged bowl situation with $V = F(\theta)$,

$\Sigma_{n=0}^{\infty} \quad a_n\, P_n(\cos\theta) = F(\theta)$ $\quad\theta$ in $(0,\varphi)$

$\Sigma_{n=0}^{\infty} (2n+1) a_n\, P_n(\cos\theta) = 0$ $\quad\theta$ in $(\varphi,\pi)$

Here is how the bowl works. Atomic form would be

$V_i = \Sigma_{n=0}^{\infty} a_n\, r^n\, P_n(z)$ $\qquad \partial_r V_i = \Sigma_{n=0}^{\infty} \; n \quad a_n\, P_n(z)$ $\quad$ at r = 1

$V_o = \Sigma_{n=0}^{\infty} a_n\, r^{-n-1}\, P_n(z)$ $\qquad \partial_r V_o = \Sigma_{n=0}^{\infty} (-n-1)\, a_n\, P_n(z)$ $\quad$ at r = 1

If there is no charge on the cap, we must have $\partial_r V_i = \partial_r V_o$ on the cap which gives the second of our dual sums. The solution to this problem is the $a_n$ as in 5.6.6 where $F_0^*$ is the integral in 5.6.4. For F = 1 we get this explicit and I suspect famous result

$a_n = (1/\pi)\, \{ \sin[(n+1)\varphi]/(n+1) + \sin[n\varphi]/n \}$ $\quad n \geq 1$ $\qquad$ 5.6.7

$a_0 = (1/\pi)\, \{ \quad \sin\varphi \quad + \quad \varphi \quad \}$ $\quad n = 0$ $\quad$ // just the $n \to 0$ limit of the above

From this you could compute σ on the two sides of the bowl and you would magically find that the difference is a constant. So this gives the potential that I never figured out from the Smythe inversion method Problem 42. Sned goes on to do the other case where we have G(θ) in the second equation. He then gives a different solution by Collins, then throws in a weight thing to get a Fred 2

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## 5.1 Introduction [134]

Sned gets right to the point with an example. Suppose you have a potential problem inside a cylinder with V = 0 on the cylinder wall (which forces a certain *summation* Smythian $J_0$ form for the potential) and then in addition you have V = 0 on an internal disk at z = 0, and then $\partial_z V = 0$ outside the disk, also at z = 0. This would be the charged disk in a cylinder problem. The $\partial_z$ brings down a factor of k, so you get a dual pair of *sums* one of which has an extra k power relative to the other, which would then be covered by the so called Titchmarsh form where you have both powers and $J_\nu$ functions in the potential form. Sned also mentions that you might alternatively have $\partial_\rho V = 0$ on the cylinder wall, or even a combination of the form $V + H\,\partial_\rho V = 0$ on the wall (we saw such a mixture in Stak I recall. He calls this mixture the "radiation condition" but I don't know why right now.) [ See Section 1.2, it has to do with Newton's Law of Cooling. ]

As I well know, the cylinder causes quantization of the values of k in the radial SL problem involving the zeros of Bessel functions, so our Smythian form is no longer an integral but a *sum*. In fact, I would say the sum was the most common Smythian form. The immediate upshot is that you get **dual series equations** in place of dual integral equations. The motivation could not be more clearly stated. Toward the end below, we see exactly the dual series equations which appear in the charged bowl problem!

## 5.2 Dual Series involving Fourier-Bessel Series [135]

I think the term "dual relations" means "dual series" -- the sum version of "dual integral equations". I wonder why he didn't just call these things "dual series"? I will call them that.

I am not covering this in detail, just looking at the overall flow of things. Sned sets things up with the same approach, the same definitions. We are doing the dual *sum* pair with $J_\nu(\rho k_n)$ in each sum as our opening problem. The first dual (only) has a power that used to be $u^{-2\alpha}$ in Titchmarsh for dual integrals, and here is $\lambda_n^{-2p}$. One must remember that the earlier u as in A(u) was really my k, and the current $\lambda_n$ is really the quantized k which I would have called $k_n$. $I_1$ is ρ in (0,1) which might be on a disk inside a cylinder, while $I_2$ is then ρ in (1,a) where a is the radius of the bounding outer cylinder. The full-range driving functions are again called f(ρ) and g(ρ) and the prescribed are $f_1(\rho) = F(\rho)$ and $g_2(\rho) = G(\rho)$, his usual notation.

### 5.2.1 Problem (a) reduced to an infinite Cramer's Rule problem. [136]

In the so-called Problem (a), the second dual driver function is 0 [ that is to say, $G = g_2(\rho) = 0$ ]. In this case, if we expand our candidate solution for A(u), now called $a_n$ which is the summation coefficient, as shown in 5.2.7, then we automatically satisfy the second dual due to some fancy summation identity involving Bessel functions which I did not study in Chapter 2, sort of a "discontinuous sum". This expansion takes the form $a_n = \Sigma_m\, b_m\, J_{\nu+2m+1-p}$ .

This expansion idea is, I am sure, completely analogous to what we did on page 99 with trig kernels, where we could construct an expansion for A(u) which automatically satisfied the second dual based on the fact that integrals of even Bessel functions vanish in the $I_2$ region (a discontinuous integral). This method of getting one of the equations "out of the way" is good. Sned did these elsewhere as well. [

Remember, if in doing this you are able to come up with a non-zero solution, that must be the right solution by uniqueness theorems.]

The math carries on and we arrive at 5.2.10 which says $\Sigma_m b_m B_m(s) = E(s)$, an infinite sum, where $B_m$ is an infinite Bessel sum, and E is a certain nasty looking integral of $F(\rho)$ which includes a Jacobi poly in its integrand. The integer s magically appears as an index s = 1,2,3... and thus we have an infinite number of equations each in an infinite number of variables $b_m$ (infinite Cramer problem). Assuming we can somehow solve for the $b_m$, we can then compute the $a_n$ from our series already shown. Ouch.

On page 138 Sned outlines a possible "iterative solution" which is some kind of perturbation theory perhaps where 1/a (ratio of radii) is the smallness parameter. In this iterative method, one sees a certain new quantity called L (as in 5.2.14) which involves an integral of K I / I t with various orders on these Bessel functions. We are now in the realm of numerical work, and Sned notes that authors computed tables for this L integral (or pieces of it) in 1959, and even our friend **Smythe** was involved in 1961 except poor Smythe's tables had some errors that were fixed up by **Roberts** in 1964. So having this numerical data helps you do the iteration step 5.2.18. **Tranter** did earlier work here.

The computation of the integral E(s) is also painful and Sned displays it in the case $F(\rho)$ = a poly.

So this is very painful and ugly stuff, but if you want to know about a disk in a cylinder, you have to do it to some extent. This is where I suspect a direct computer numerical job on Laplace would get you the desired answer faster than all this series stuff, but I am not sure of that.

### 5.2.2 Problem (a) [$g_2$=0] reduced to a Fred 2 integral equation. [139]

Here **Sned** is showing off his own recent work of 1964 (book = 1966). He offers three different ranges for the power exponent p (recall how we used to do ranges of α and/or β). For each range, he does in fact come up with a Fred 2 integral equation for a function h(t) from which one can compute the $a_n$ . For example:

$0 < p \leq 1/2, \nu > 0$ integral equation is 5.2.23 for h $a_n$ expansion is 5.2.21

This is one of four cases he considers.

### 5.2.3 Problem (b) [$f_1$=0]reduced to a Fred 2 integral equation. [142]

He did so well in the last section, that here he attacks the (b) problem with the same method, this being the problem where the *first* driving function is F=0.

## 5.3 Dual Series involving Dini Series [144]

This section is a pretty special case of dual sums as you shall see. It's just another thing Sned felt he should throw in because it is something different.

This "thing" seems to be a generalization of the Fourier-Bessel Transform where the $\lambda_n$ , instead of being the zeros of $J_\nu(\lambda)$, are instead the zeros of the function $\lambda J_\nu'(\lambda) + H J_\nu(\lambda)$ where H is some positive constant (this is that "radiation" deal again). This would become the F-B transform for large H, and would become an unnamed Neumann transform if H = 0. The expansion part of the corresponding transform is

called "a **Dini series**", while the projection part is given in 5.3.3. Some people I think call this thing a **finite Hankel-Schwartz Transform**. [ Dini also has a theorem about uniform convergence that I ran into once. ] I supposed there might be an infinite version of this transform as well, but maybe not. This whole thing is what you get in your Stak ODE study if you put "mixed" inhomo BC's at the high end of your interval, so not rocket science.

**Comment on Mixed BC's.** For our charged disk problem, we had Dirichlet BC on part of ρ's range, and we had Neumann BC on the residual range. That is one thing people mean when they talk about "mixed BC". But this Dini thing is a different meaning of "mixed". You get a mixture of Dirichlet and Neumann at every point on your boundary! Stak did a little with this on page 145 in his Chapter 6, Exercise 6.34

Sned then makes some special case comments which remind me of some Stak cases where for some values of a parameter we have a purely continuous spectrum for $\lambda = k^2$, while for other ranges of that parameter we pick up a finite number of discrete spectral points on the negative real axis. Here Sned is talking about adding terms to the projection in certain parameter cases.

So now that we have said what Dini is, we consider a NEW pair of duals shown page 145 (Sned) as 5.3.7 and 8. These are exactly as our previous dual sums except (1) coefficients are called $c_n$ instead of $a_n$ ; (2) the sums are over these new Dini $\lambda_n$ spectral values; (2) first has $\lambda^{+2p}$ instead of $\lambda^{-1p}$. Thus, H is a hidden parameter in everything here. Sned then proceeds to expend 6 pages talking about the solution for 6 different ranges of the parameters p, ν and H. I think for most of the cases he produces a Fred 2 integral equation for a function h(t) from which he can find the solution of the problem. [ I think this is nowadays the industry's "preferred method" of dealing with a problem requiring a numerical solution. ]

## 5.4 Dual Series involving Trig Series [150]

The motivation here would be perhaps a square disk inside a square waveguide (ie, replace round things with square things relative to the previous discussion)? Now you want V=0 on the waveguide surface. The Smythian form is going to be a *sum* with atoms of this form (from my Cartesian atoms doc)

$$u(x,y,z) = \Sigma_{nx,ny} A_{nx,ny} \sin(k_x x) \sin(k_y y) \exp(-\kappa_z |z|)$$

with $k_x^2 + k_y^2 = \kappa_z^2$. This problem I now see does not really fit our mold here because if you require that $\partial_z V = 0$ on the square plate exterior area, you pull down not a factor of $k_x$ (say), but a factor $\kappa_z$ which is not a simple power. So I guess I will not have a motivating potential problem for this section. Perhaps Sned will think of some example in his electrostatics chapter.

So, the dual series considered here have $J_\nu(k_n \rho) \rightarrow \sin(k_n x)$ where now $k_n = n\pi/a$ where a is the place you want potential to vanish. But he quietly sets $a = \pi$, so we get then just sin(nx) and then things are pretty simple, the sum just being over positive integers n as in 5.4.1. This time $n^p$ is the "power" which is only in the first sum, and we have the same driving functions F and G.

He in fact writes down four pairs of dual series that one might be interested in, and we have the usual division into the worlds of even and odd functions. He does not use the $\varepsilon_n$ Neumann symbol here, so we have the extra term as seen in 5.4.4.

Various people worked on this problem going back to 1937, including our friend **Tranter** who got closed form results which are very messy. Sned is going to do only the cases p = ±1 since these are usually the ones of interest.

We know that if we take a $J_\nu$ result and let $\nu = \pm 1/2$, we get the sine and cosine results, provided they are allowed by the restrictions. As happened earlier in the book, it turns out that the "general results" we got in Section 5.2 are restricted so as to *not* be useful for our trig cases, so these problems have to be solved separately, and I guess that is what Tranter did. These are the first pair of duals 5.4.1 and 5.4.2.

Similarly, it turns out that the two duals given in 5.4.3 and 5.4.4 are special cases of the Dini series results we got in Section 5.3, where we set ν and H to certain values (H = 1/2). But again, it turns out that the solutions are restricted away so he has to do them from scratch, see below.

**Detail concerning H = 1/2.**

Our Dini mixed BC was $(k_n a)J_\nu'(k_n a) + H\, J_\nu(k_n a) = 0$. If we take $\nu = +1/2$, Maple tells us that

$$J_{1/2}(x) = \sqrt{2/\pi}\ \sin(x)/\sqrt{x} \qquad // \ J_{-1/2}(x) = \sqrt{2/\pi}\ \cos(x)/\sqrt{x}$$

so we then have, letting $x_n = k_n a$,

$$(x_n)\ \partial_x[\sin(x)/\sqrt{x}\ ]|^{x=x_n} + H\ \sin(x_n)/\sqrt{x_n} = 0$$

Doing the derivative gives

$$\sqrt{x_n}\cos(x_n) - (1/2)\ \sin(x_n)/\sqrt{x_n} + H\ \sin(x_n)/\sqrt{x_n} = 0$$

If we select H = 1/2, the last two terms cancel and we get just

$$\sqrt{x_n}\cos(x_n) = 0 \quad => \quad \cos(x_n) = 0 \quad => \sin(x_n+\pi/2) = 0 \quad => \quad x_n = (n-1/2)\pi$$

But since $x_n = a k_n$, and we set $a = \pi$ as usual, then we have $k_n = x_n/\pi = (n-1/2)$.

Now, in Bessel terms our Dini series will be this, but with our new $k_n$ values.

$$\Sigma_{kn}\ k_n^{+p}\ b_n\ J_\nu(\rho k_n) = f_1(\rho)$$

which becomes

$$\Sigma_n\ (n-1/2)^p\ b_n\ \sqrt{2/\pi}\ \sin(x(n-1/2))/\sqrt{x(n-1/2)} = f_1(x)$$

$$\Sigma_n\ (n-1/2)^p\ [b_n\ \sqrt{2/\pi}\ /\sqrt{(n-1/2)}\ ]\ \sin(x(n-1/2)) = \sqrt{x}\ f_1(x)$$

$$\Sigma_n\ (n-1/2)^p\ a_n\ \sin(x(n-1/2)) = F(x)$$

and this is what you see in 5.4.3, where we have just redefined that $b_n$ and $f_1$ top be an and F. So this explains why this series is called "the Dini series with ν = 1/2 and H = 1/2". The second equation is

similar with p = 0. As for the spectrum of n values, I think we need $k_n \geq 0$ because a Stak spectrum is on the non-negative real k axis, so that rules out n = 0 and n < 0. Just my guess on this.

Now what happens for ν = -1/2? We replicate the above steps:

$$J_{-1/2}(x) = \sqrt{2/\pi}\ \cos(x)/\sqrt{x}$$

$$(x_n)\ \partial_x[\cos(x)/\sqrt{x}\ ]|^{x=x_n} + H\cos(x_n)/\sqrt{x_n} = 0$$

$$-\sqrt{x_n}\sin(x_n) - (1/2)\cos(x_n)/\sqrt{x_n} + H\cos(x_n)/\sqrt{x_n} = 0$$

Again we take H = +1/2 so last two terms cancel, and we then have

$$\sqrt{x_n}\sin(x_n) = 0 \qquad \text{as our BC}$$

We have $x_n = ak_n = \pi k_n$ so $\sin(\pi k_n) = 0 \Rightarrow k_n = 0,1,2,3... = n$.

Now, in Bessel terms our Dini series will be this, but with our new $k_n$ values.

$$\Sigma_{kn}\ k_n^{+p}\ b_n\ J_\nu(\rho k_n) = f_1(\rho)$$

which becomes

$$\Sigma_n\ n^p\ b_n\sqrt{2/\pi}\ \cos(nx)/\sqrt{nx} = f_1(x)$$

$$\Sigma_n\ n^p\ [\ b_n\sqrt{2/\pi}\ /\sqrt{n}\ ]\cos(nx) = \sqrt{x}\ f_1(x)$$

$$\Sigma_n\ n^p\ a_n\cos(nx) = F(x)$$

which is Sned's first equation in 5.4.4. The spectrum I suppose now is n = 0,1,2,3. You can see there is an "issue" with n = 0 in redefining the coefficient. Sned breaks off the n = 0 term in 5.4.4 in a strange way where instead of writing just $a_0$ he write $(\alpha/2)a_0$ further redefining this n=0 coefficient, but he says nothing about this α. Later on page 161 this α is called λ. The factor is significant because $a_0$ also appears in the second equation, but I shall ignore this detail.

**5.4.1 Dual Series involving the sine, dual 5.4.1 [152]**

He first does p = +1 as case (i), then p = -1 as case (ii). Our duals are 5.4.6/7 where $g_2 = 0$ (no second driving function). There will of course be some partner function $g_1(x)$ which he sine-Fourier-series expands with some coefficients $a_n$ which we would like to learn. I think everything in this problem has to be an odd function, and this is why he can do as in 5.4.10. What follows is a huge trick/ansatz, but we have seen it before. We know certain facts which lead us to represent $g_1(x)$ as an integral of another unknown function $h_1(t)$ as shown in 5.4.11. Only when we are done can this be justified! Since $g_2 = 0$, we have a "full range" in effect for $g = (g_1,g_2)$ and we can "invert" our expansion 5.4.10 to get p 152A. Into

these we install our integral rep for $g_1$ and we end up with the double integral p 152 B for $a_n$. We then recognize each of the two terms in the inner integral as Mehler's integral for a P function, hence 5.4.12. The next trick is a strange one. We are supposed to integrate our first dual from 0 to x which trivially gives p 153A. Into this thing we sub $a_n$ from 5.4.12 getting 5.4.13 where we have isolated a certain infinite sum in curly brackets. This sum can be done using 2.6.32 (Note: back in Chapter 2, Sned considered expansions (with projections) on Jacobi polynomials page 58. This general formula has many special cases which he considers there, shown on page 59, and one of those is our sum shown here. The quantity $\sqrt{\text{cosx-cost}}$ seems to naturally occur in this Jacobi stuff. ) Thus, we get {...} = p 153 B and this is one of those Volterra equations we know how to invert from our Chapter 2 work, and this gives p 153 C for $h_1$ as an integral of the driving function F. We then know $h_1$, hence we know $g_1$ from 5.4.11 so we know the entire g function over full range. Put this $g_1$ into p 152 A to get your $a_n$ and problem is solved!

So the method here is highly contrived or directed, but you can verify each step, so good for the record. Note that this result does not appear in Polyanin because that book does not talk about dual series, only dual integrals.

So in the above he was doing p = 1 with G = $g_2$ = 0. Now he does the F = $f_1$ = 0 case instead. Things proceed in similar fashion and we end up with $h_2$ in 5.4.18 which then gives $a_n$ as in 5.4.17.

Now look back at p 152 top. Suppose $a_n$ solved the G = 0 case, and suppose $b_n$ solved the F = 0 case. If you define $c_n = a_n + b_n$ , you will have the solution to the general problem! He of course points this out. I guess this concept applies to all duals.

Now he starts into case (ii) with p = -1. But if you take the p = +1 duals, if you were to define $A_n = n^{-1}a_n$, you can cause the $n^{-1}$ in the first dual to go away, and you get an $n^{+1}$ in the second dual. But you need to also cause the ranges to switch, and that is done by using x = π-y ! Doing this causes our duals of interest to become p 154 B and we just solved these duals! Yes, the driving functions have shifted arguments which is easily accounted for, so Sned then just edits the previous results to get $a_n$ as in 5.4.21 for our two subcases which are again F = 0 and G = 0. He then adds the results as above and we are done. That is to say, with p = -1, the answer is 5.4.21 for $a_n$ with k(t) as in 5.4.22.

[ 155-157] We next consider a Williams 1964 alternate solution to the G=0 case, and I only scanned this quickly, but it has interesting facts in it. We know 5.4.23 is the Fourier $a_n$ expansion, and if we put that into our first dual, we do get the integral 5.4.24 where the kernel is the interesting sum in 5.4.25 which I may have met elsewhere. Continuing on, Williams was able to recast this integral equation 5.4.24 into a form which can be solved by inspection using our usual Chapter 2 formulas. His first step is to rewrite the kernel as in 5.4.26, install this expansion into 5.24, reorder the integration region, and boom: you get a certain h as an integral of F, then g as integral of h. Sned then does a little extra to show that Williams' result agrees with what we got by our previous method.

**5.4.2 Dual Series involving the sine analog of the Dini Series, dual 5.4.3 [158]**

This looks similar to what went before, he does the two cases p = ± 1.

**5.4.3 Dual Series involving the cosine, dual 5.4.4 [ p 161]**

Again, this looks similar to what went before, he does the two cases p = ± 1.

### 5.4.4 Dual Series involving the cosine analog of the Dini Series, dual 5.4.3 [ p 162]

More of the same!

### 5.4.5 Tranter's Formulas [163]

Sned earlier mentioned that Tranter did all this stuff but it was messy development so he did not want to transcribe that into this book. Here, however, he quotes the key Tranter results since in some cases the formulas are better than ones he has gotten above.

## 5.5 Dual Series involving Jacobi Series [165]

The duals here are stated in 5.5.1/2 and things are really quite different here. There is no "power" as we have had in other forms -- instead there are gamma functions of n. The "function" which used to be $J_\nu(k_n\rho)$ or sin(nx) is now $P_n^{(\alpha,\beta)}(\cos\theta)$ and thus does not have the same kind of two-variable "argument", but the sum is on "n" as in the trig cases. The two equations of this dual pair are the same apart from different gamma functions and as usual different driving functions. Sned expends 8 long pages on this problem, but I have no motivation to learn it, so this is a simple "skip" decision. At least I know it is here. The work is credited to **Srivastav 1964c**.

BUT: I now realize that a special case of this stuff will be the Legendre stuff that I need for the bowl, so I will now sit up and pay more attention:

The duals of interest are these

$$\Sigma_{n=0}^{\infty} K_n(\alpha,\beta)\, A_n\, P_n^{(\alpha,\beta)}(\cos\theta) = f_1(\theta) \qquad \theta \text{ in } (0,\varphi)$$
$$\Sigma_{n=0}^{\infty} K'_n(\alpha,\beta)\, A_n\, P_n^{(\alpha,\beta)}(\cos\theta) = g_2(\theta) \qquad \theta \text{ in } (\varphi,\pi)$$

where K and K' are certain $1/\Gamma\Gamma$ as shown page 166 and replace the "powers" we had for Bessel and Trig. The big mystery of course is why these strange K factors are appearing here, they must arise in some natural way he is not saying. His buddy Srivastav 1964 came up with a solution. As usual, he considers the two cases where one or the other driver function vanishes, and we know we can add these solutions to get a general result. For $g_2$=0, his result is this: $h_1$ = 5.5.11 and then $A_n$ = 5.5.8, all closed form! Similarly for $f_1$ = 0 his result is $h_2$ = 5.5.13 and $A_n$ = 5.5.12.

On page 168 he then offers another solution by Noble 1963. The duals are now these:

$$\Sigma_{n=0}^{\infty} p_n(\gamma,\beta)\, a_n\, \mathcal{F}_n(\alpha,\beta;x) = f_1(\theta) \qquad x \text{ in } (0,a)$$
$$\Sigma_{n=0}^{\infty} \qquad a_n\, \mathcal{F}_n(\alpha,\beta;x) = g_2(\theta) \qquad x \text{ in } (a,1)$$

where the function is that "shifted Jacobi" object I discuss in the Chap 2 notes. The object p is like the K guys above, but we have a third free parameter called $\gamma$. Noble comes up with some pretty straightforward solutions, $a_n$ in 5.5.23 in terms of $A_n$ which in turn are given in 5.5.24 where F* and G* are other simple integrals. I think $\gamma$ is a range limiter thing, since he needs $\gamma < \beta < \gamma+1$, like we had for $\alpha$. He then writes the results for f(x) and g(x) including the partner functions. So this stuff all has closed-form results.

On page 172 Noble goes on to add a weight function, and of course this leads to some kind of Fred 2 situation.

## 5.6 Dual Series involving Associated Legendre Functions [173]

Now *this* gets more of my attention! Really though it is just a special case of the Jacobi stuff just done in the previous section. He uses the T form of the P functions, and again there are no powers. So he quotes the special case results, and here we have G = 0. He notes that **Collins 1961b** also worked on this stuff.

If I compare 2.6.14 to my Leg Prop Doc for on-the-cut P functions, I conclude that

$$T_n^m(x) = P_n^m(x) \qquad \text{// exactly}$$

so I think Sned's P function has a different phase as he shows on the right on 2.6.14. He associates this T with a guy he calls Ferrer but whose name I think is Ferrers who did stuff in 1882. If I want to cast Collin's equations in a form I like, the sums would start at n = -m which seems a bit odd. I think people only use this result in the m=0 case where things take a familiar form

$$\Sigma_{n=0}^{\infty} \quad a_n P_n(\cos\theta) = F(\theta) \qquad \theta \text{ in } (0,\varphi)$$
$$\Sigma_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta) = 0 \qquad \theta \text{ in } (\varphi,\pi)$$

This sure looks like the **charged bowl** problem (angle φ) with radius r = 1. Yes! And later on page 270 he is going to use the result we get here, he calls it a spherical cap! The first term will be the potential on the bowl, the second with driving function 0 is the charge density of 0 on the rest of the sphere. Since this is a subcase of our subcase above, we know the result, the $a_n$ are as in 5.6.6 where $F_0^*$ is the integral in 5.6.4. So this solves the bowl problem in the quite general case that the bowl has a symmetric but varying potential F(θ)! In the special case F = 1 which is our metal charged bowl, the solution for the $a_n$ are as in 5.6.7 which is very simple indeed (and I think appears in Canonical). I quote it here since this problem has been important to me:

$$a_n = (1/\pi) \{ \sin[(n+1)\varphi] /(n+1) + \sin[n\varphi]/n \} \qquad n \geq 1 \qquad 5.6.7$$
$$a_0 = (1/\pi) \{ \quad \sin\varphi \quad + \quad \varphi \quad \} \qquad n = 0$$

So pay attention: this is a solution of the bowl problem by solving the dual series equations that said problem implies!!! I will have to look back soon at my "half spherical shell" mysteries and see if these clears things up a bit!

We now repeat the above for the same duals but this time with only G = $g_2$ driving in the second equation. The special subcase is then 5.6.12 which would be a bowl that is grounded but has some azisym charge distribution G(θ) on the complementary sphere. In Smythe problem 42 we had this situation with a constant σ on the complementary sphere. Sned states the solution here, but does not compute $G^*_0$ when G = constant.

Then on page 175 Sned gives a different solution of this F = 0 problem due to Collins.

The last gasp is another piece of Collins' work where we install a "weight function" $H_n$ into the duals, as we did back in the dual integral world. This of course leads to a Fred 2 integral equation.

Done! This chapter had more payoff than I expected. This is simply because Smythian forms for boundary value problems often are sums.