Sneddon Chap 6 Notes: Triple Relations PhL 8.5.10

Sned's term "relations" is meant as a stand in for either "integral equations" or "series equations".

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Overview (2.5 pages, written 12.16.10)

In **Section 6.1** we get two examples of triple relations.

In the first example, imagine an infinite insulated plate (thickness c) of some heat-conducting material. On one side of the **plate**, in an *annular* region $a \le \rho \le b$ we strip off the heat insulation and press against the plate a special annular shaped heat reservoir which maintains the plate temperature at some $\theta(\rho,0) = f(\rho)$ but only in this annular region. In the static situation there will be some 3D distribution of temperature inside the plate. Sned shows all the math, and then takes the plate thickness to be infinite ($c \to \infty$). Our task is to find the 3D distribution of temperature $\theta(z,\rho)$ inside this half-space of plate (he uses θ for temperature, so he can continue to use u for k !) This limit results in the **triple *integral*** equations 6.1.8. Our problem is to find the coefficient function A(u) and then 6.1.7 gives us the resulting temperature distribution in the plate. This is the typical "triple situation": you have three regions, and in the first and third you have what we used to have in the second region of a "dual situation". It is as if that first region is opened up. The charged *annulus* is another obvious such problem in electrostatics.

The second example is that of a **spherical barrel** on which we have some Dirichlet potential $f(\theta)$ between the barrel angles β and α. On the rest of the spherical surface we have "nothing" so $\sigma = 0$ there and we end up with a the **triple *series*** equations 6.1.11. This barrel problem has been on "my list" for a long while now.

In **Section 6.2** (**Titchmarsh triple integral equations**) we consider a Titchmarsh-type set of triple integral equations. We assume three regions called $J_{1,2,3}$ and we assume that the LHS of the integral equations for regions 1 and 3 are the same, as they were in both our examples above. We refer to the function that is the LHS of the region 2 equation as f(x), and the function that is the LHS of the region 1 and 3 equations as g(x). We can then refer to f(x) in region i as being $f_i(x)$ and similarly we can talk about $g_i(x)$ for i = 1,2,3. Our Titchmarsh form has the usual power $u^{-2\alpha}$ in f(x), and no power in g(x), and J_ν in both. The driver functions are allowed to be E, F and G, which is to say: $g_1 = E$, $f_2 = F$, and $g_3 = G$. This is triple equation 6.2.1.

Cook in 1963 came up with an interesting way to solve these triple integral equations, and it sounds like something I would do. He first groups regions 1 and 2 into a single region J_{12} and then he investigates the dual integral equation pair which that implies. For example, in region J_{12} we have what we might call f_{12} which is f_1 in J_1 and f_2 in J_2. In the second region J_3 we have g_3. By doing the right stuff (ie, using our *dual* integration methods), we can solve this equation in a way that gives $f_3 = h(f_1)$. But then if we consider a second contrived set of duals by doing J_1 and J_{23} and we turn the same crank, we get $f_1 = w(f_3)$. He rescales the functions $f_{1,3}$ into functions $p_{1,3}$ and our two little equations are then shown in 6.2.15. In this work we assumed α lies in (0,1). For the range α = (-1,0) he gets 6.2.20 instead. In each case we have a pair of cross-coupled Fred 2 equations with the same kernel but different inhomo term. By then defining $p_{\pm}(x)$ in an obvious way, we end up with two decoupled Fred equations. This is the end of the line. To solve the problem, you have to solve each of these Fred 2 babies.

So, the upshot seems to be that even the fairly basic Titchmarsh triple integral equation set has no closed form solution! Don't lose sight of that main fact.

In **Section 6.3 [p184] (azisym Titchmarsh triple integral equations with weight).** Sned considers the azisym Titchmarsh situation with only the F driving function, and α = 1/2, but he throws in a weight function h(x). He ends up with a Fred 2 as shown in 6.3.15 that you have to solve for G_2 from which you can then get g_2 (6.3.9) and then A(u) (6.3.4) and basically solve the whole problem. He does not talk about the case h = 0 separately. In 6.3.15 this causes M = 0 but you still have the K part of the kernel, so you are still faced with a Fred 2 equation. So my tentative conclusion is that the potential of a charged annulus even with $f(\rho) = 1$ may require solving a Fred 2. But maybe he will come up with something better when he deals with this in Chapter 8 section 8.6.

Within Section 6.3 I have a very long digression dealing with certain integrals. Here are the conclusions of that section. First, here is an integral which appears in Chapter 3 as 3.4.6 (which I am just now getting around to deriving it correctly)

$$\int_0^{2\pi} d\psi \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr' \cos(\psi-\psi')} \quad \} \;=\; 2 \cos(n\psi') \, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

Then I show these related results:

$$\int_0^{\infty} J_\nu(ax) J_\nu(bx) \, dx \;=\; (2/\pi) \, (ab)^{-\nu} \int_0^{\min(a,b)} ds \, s^{2\nu} / [\sqrt{a^2-s^2} \sqrt{b^2-s^2}]$$

$$= (1/\pi) \, (ab)^{-1/2} \, Q_{\nu-1/2}[(a^2+b^2)/(2ab)]$$

so if we wanted, we could write our first integral LHS in two other ways.

In **Section 6.4 (convert Titchmarsh triple integral equations to dual series equations).** Sned quotes some Tranter work of 1960 where Tranter examines our Titchmarsh 6.2.1 *triple integral* situation with the added assumption that G = 0. He assumes a series form for the solution A(u) with some a_n coefficients. For general power α and Bessel ν, the shifted Jacobi thing $\mathcal{F}$ gets involved due to some Bessel integral, and he somehow manages to convert the *triple integral problem* into the *dual series problem* 6.4.7 in which appear Jacobi polynomials. But this he says is too hard to solve in the general case, so he tries some special cases. He sets ν = 1/2 so the J_ν becomes sine, and he also sets α = 1/2 in the power. The dual

series then becomes one of the four trig dual series we studied in Chapter 5, as shown brackets on page 188. Solve this dual trig series for a_n then use 6.4.10 and the problem is solved. In his work in Chap 5, he seems to get h_2 as in 5.4.45,and then the a_n are given in 5.4.44 and we do indeed seem to have a closed form solution! Now back to Chapter 6, if we have $\alpha = 1/2$, we get the result just noted. But if $\alpha = -1/2$, we get my pencil bracket on page 189 top, where we get a different trig dual series. Both these results require that $\nu = 1/2$ in the Titchmarsh's, so this does not apply to the azisym Bessel case. He has similar solutions for $\nu = -1/2$.

So here is the conclusion: For general ν, even if G = 0, you can convert the triple integral equations into dual series equations, but it is way to hard to solve. If we take four special cases $\nu = \pm 1/2$ and $\alpha = \pm 1/2$, we can get closed form solutions, and these don't apply to the azisym case which is $\nu = 0$.

In **Section 6.5 (Legendre azisym triple series equations)** In this section Sned has his only comments on solving "triple series". He uses Legendre series, so we are in some kind of azisym case here. Instead of calling the LHS's f(x) and g(x), he gives them names $\varphi(\theta)$ and $\chi(\theta)$ and he has thrown in now a series weight function H_n which must be better than 1/n at large n. He considers two solution cases. The first kind case is like a Dirichlet barrel problem, while the second kind case is a Dirichlet complement-of-barrel problem (two facing unequal Dirichlet bowls) (but both with H_n added). Both problems have $\sigma = 0$ off the selected surface. My notes below say that in the first problem he ends up with *four* Fred 2 equations you have to solve, and for the second case there are two Freds to solve. Of course I am going to be interested in $H_n = 0$ for my versions of these problems, and we wait till Chapter 8 for that. I presume things will simplify considerably.

6.1 The Origin of Triple Relations

Excellent motivation, right off the bat!

Example 1: Of a triple *integral* equation set.

First, as a reminder, there is an analogy between electrostatics and temperature stuff. Temperature is the potential, heat sources and sinks are positive and negative charges, heat flow vector is electric field vector. Specify temperature on a boundary is Dirichlet. Specify per unit area heat flow on a boundary is Neumann. Insulated boundary for example means no heat flow which is a Neumann = 0 boundary. I have a comment on heat flow in the notes below.

Unfortunately, this example is not clearly presented by Sneddon. I at first thought he was talking about a cylindrical plate of radius b, but I see now he is talking about an *infinite* plate of thickness c which runs from z = 0 to z = c. On the z = 0 side of this plate we somehow apply a Dirichlet temperature source of $f(\rho)$ but only in an annular region $a < \rho < b$ about a selected cylindrical origin on the z=0 plate surface. Apart from this annular region, the plate is completely insulated everywhere.

Consider now his proposed Smythian form 6.1.4 for temperature (potential), which by the way is consistent with our usual cylindrical atomic form (with m = 0)

expo osc osc

(1) $[\, e^{+kz}, e^{-kz} \,]\ [\, J_m(k\rho), N_m(k\rho)]\ [\, \sin(m\varphi), \cos(m\varphi)]$

Differentiation gives

$$\partial_z \theta|^{z=0} = -\int_0^\infty du\, A(u)\, J_0(\rho u)$$

so this thing will be 0 inside and outside the annulus, so these are two of our triple equations. The third is shown 6.1.56b with 6.1.6. When we take the plate $c \to \infty$, things simplify and our triple is 6.1.8. If the plate starts out cold, perhaps initially heat will flow in on the annular ring from our temperature reservoir source, but at some point the temperature pattern will stabilize and we will have our $\theta(\rho,z)$.

Comments on heat flow. At any insulated point on the plate surface, $\partial_n u = 0$. In electrostatics we would say that $\sigma = 0$ implies no E_n field at a *metal* boundary (at a math boundary, we know only that E_n is continuous through the boundary if no σ there). In heat flow, I think the thing that corresponds to a boundary against a solid piece of metal in electrostatics is a boundary that has a solid insulating medium on the other side (not metal). Then $\partial_n u = 0$ is like saying $F_n = 0$ where F is the heat flow vector component pointing right into the insulating boundary at some point. The thing that corresponds to σ is a heat source or sink per unit area of boundary, which we could just continue to call σ. If no σ, then heat flow is continuous across the boundary, but if the other side of the boundary cannot support heat flow (is an insulator), then $F_n = 0$ at the boundary. Since $F_n \sim \partial_n u$, as you approach an insulated boundary, the temperature curve u comes in for a flat landing on the boundary and has some well-defined value there. If we had a finite sized insulated heat-conducting object, I think you would end up with a stable temperature distribution such that no heat flows across the annulus. For an infinite object, I think this same thing is true, but it might take infinitely long from t = 0 to get to this stable temperature distribution.

Example 2: Of a triple *series* equation set.

This is the spherical barrel problem with the usual Smythian series inside and out. The potential is prescribed on the barrel portion, and there is no charge on the other two portions of the sphere which means that $\Delta\partial_r V = 0$ in these regions, which brings in the (2n+1) factor. Thus, you get the triple sum shown in 6.1.11. (You might do the barrel as a superposition of two bowls? Sorry Charlie.)

So it was not hard to find a simple electrostatic problem that leads to a triple series set!

6.2 Triple Integral Equations of the Titchmarsh Type

The form of the three integral equations is shown in 6.2.1. The " inner and outer" forms are the same on the LHS but have different driving functions E and G. The central form has a $u^{-2\alpha}$ power. As usual, we have $J_\nu(xu)$ in all three forms. This is an extension of Titchmarsh's problem, hence the name. So we have now three regions he calls $J_{1,2,3}$. I like to think of the first J_1 region as the "new region" compared to the dual integral equation scenario, the region that is now added into the soup. So he uses F and G as the driving functions for the last two, and E for the first. Notice now that when you define your "all range"

functions which he calls f(x) and g(x) [the J_2 and J_3 lines] , you only need *two* functions, not 3, because the inner and outer LHS's are the same! Then it seems clear that $g_3 = G$, $f_2 = F$, and $g_1 = E$ as in 6.2.4.

Sned is now going to write our triple relations as two pairs of dual relations. The first dual pair is 6.2.5. This is just slightly tricky. Each of the three relations in the triple exists in all three domains, but each is only prescribed in one of these domains. Now I find that using f(x) and g(x) as names for the *left* hand sides (over full range) is extremely useful. Then we use the notation $g.J_i$ meaning g object on domain J_i and we could then also talk about $g.J_{12}$ as the g object on the union of J_1 and J_2. We are going to only use J_{12} and J_{23} below, and each of these is a contiguous region of the x axis.

Consider then the triple:

Triple: $g.J_1 = g_1 = E = (g_1,0,0)$
$f.J_2 = f_2 = F = (0,f_2,0)$
$g.J_3 = g_3 = G = (0,0,g_3)$

Now we can consider these equations

$f.J_1 = f_1 = (f_1,0,0)$
$f.J_2 = f_2 = (0,f_2,0)$

We can of course add these to get

$f.J_{12} \equiv f.J_1 + f.J_2 = f_1 + f_2 = f_1 + F = (f_1,f_2,0)$

Then we could write the dual pair of equations (this is 6.2.5)

			1 2 3 12 3
First Dual:	$f.J_{12} = F'$	$F' = f_1 + F$	$= (f_1,f_2,0) = [F',0]$
	$g.J_3 = G$		$= (0,0,g_3) = [\ 0,G]$

We know how to solve a Titchmarsh dual pair. For this pair we might talk about $f = (f_{12}, f_3)$ and $g = (g_{12}, g_3)$. We can regard $f_{12} = F'$ and $g_3 = G$ as our two prescribed functions, and we could then solve the problem to find the partner pairs f_3 and g_{12}. We would obtain for example f_3 = function of (F',G) which is to say f_3 = function of (f_1+ F, G). Since in our triple problem we know F, this says f_3 = function of (f_1). So admittedly we don't know f_1 of our triple problem, but at least this gives us $f_3 = h(f_1)$. This function would be some complicated integral or nested set of same, and the domains appearing in this function would then be J_{12} (the contiguous union, so a sum of integrals) and J_3. So our solution will have the same "integral structure" we have always had, its just that we have J_{12} and J_3 as our two regions. In fact, we can be quite specific here. Back on page 95 Sned has written out the Tit solution for $0 < \alpha < 1$ which is what we are dealing with here. He gave there expressions for the two partner functions on page 96, and we can think of these as: $\varphi_2 = f_3$ and $\chi_1 = g_{12}$. It happens that the one we want is φ_2 which does not have the d/dx derivative.

Now we are going to start again and write a different dual pair. We can consider these equations

$f.J_3 = f_3$
$f.J_2 = f_2$

We can of course add these to get

$f.J_{23} \equiv f.J_3 + f.J_2 = f_3 + f_2 = f_3 + F$

Then we could write the dual pair of equations (this is 6.2.6)

Second Dual: $g.J_1 = g_1 = E$
$f.J_{23} = f_2 + f_3 = f_3 + F \equiv F''$

For this pair we might talk about $f = (f_1, f_{23})$ and $g = (g_1, g_{23})$. We can regard $f_{23} = F''$ and $g_1 = E$ as our two prescribed functions, and we could then solve the problem to find the partner pairs f_1 and g_{23}. We would obtain for example f_1 = function of (F",E) which is to say f_1 = function of (f_3+ F, G). Since in our triple problem we know F, this says f_1 = function of (f_3) of $f_1 = w(f_3)$.

So here is the upshot of all this. By examining the two dual Titchmarsh problems, we come up with

$f_3 = h(f_1)$
$f_1 = w(f_3)$ // agreed, 2.16.10

where h and w are known functions (but these involve integrals). Somehow we should be able to solve these for the two unknowns f_1 and f_3 . If we could do that, we would then know f_1, f_2 and f_3 so we know all of f, then we could use a Hankel transform to invert our second equation to get A(u).

In the next several paragraphs, Sned *writes out* the above two equations in detail, first for the case $0<\alpha<1$. In mid page 182 he then rescales f_1 and f_3 into new functions p_1 and p_3. Our two equations are now neatly stated as 6.2.15 which is a "system of two coupled Fred 2 integral equations" for the two unknown functions p_1 and p_3. They in fact have the same kernel, but different inhomo terms. Sned starts at this point regarding this as "the solution". We have reduced the triple integral equation set to a pair of coupled Fred 2 integral equations that I guess we have to solve using standard methods, like a Newton series for each of the unknown functions.

Now recall that the Titchmarsh solution changes its stripes for different ranges of the power exponent α. For $-1 < \alpha < 0$ the solutions are on page 95 (pencil bracket). So I am just saying we have to treat this as a separate case here with our triple, just as we did with the dual. Sned goes through all the wonderful details (he is very, very good about showing the paths!) and this time we end up with a system of two Fred 2's as shown in 6.2.20.

The whole method here is credited to **Cooke** 1963b.

Now the punch line at the end: look what happens if we add and subtract our coupled Fred 2 equations. We get

$p_1 \pm p_3 = M(p_1 \pm p_3) + K_\pm$

So this decouples the two equations, and we have then these two separate Fred 2 equations to solve

$p_{\pm} = M\, p_{\pm} + K_{\pm}$

and they differ only that they have different homo driving terms.

So to summarize, we have reduced our Titchmarsh triple integral equation to a pair of very similar Fred 2 integral equations. Not bad Mr. Cooke!

6.3 Triple Integral Equations in Azisym Situations

This means $J_\nu = J_0$. For his opening gambit here, Sned considers a special case of the above where now we have (E,F,G) = (g_1,f_2,g_3) = (0,F,0) so only the central driver function is non-zero. The power is u^{-1} so $\alpha = 1/2$. This is all set up then on page 184. But to complicate things just a bit, Sned throws in a weight function h(u) into the f(x) object, and we know this is going to lead us to a Fred 2!

Processing now begins, again we are following **Cooke**. Since $g_1 = g_3 = 0$, we have g = $(0,g_2,0)$ so we have g in "full range" (although g_2 is unknown) so we can Hankel invert our g(x) equation to get 6.3.4 which then gives A(u) as an integral of g_2. Using this A(u) expansion in the *first term* of 6.3.2 (the middle triple), he gets 6.3.5 which eventually becomes 6.3.8. A certain integral appears in this result which he calls G_2 = integral of g_2 and he can invert this from our chapter 2 work to get 6.3.9. He then works on the second term in 6.3.2 and gets p 186 A. He then mushes things for a while and ends up with a Fred 2 integral equation for G_2 in 6.3.15 which has a two-term kernel K + M, and driver function F*. If you could solve this for G_2, then you can find g_2 and then you know A(u). The K part of the kernel is the incredibly ugly but closed form expression 6.3.13. The M part is 6.3.14 which involves object I defined earlier, and F* is a similar object.

So I guess with all this stuff you could consider doing a numerical solution and things are boiled down to just a single Fred 2 to worry about.

I think this is the azisym annular disk Dirichlet problem with ~ $f_2(\rho)$ on the annulus. This problem would have that weight function h=0, which would make kernel M = 0, but you still have kernel K to deal with. So I think even in this simple h = 0 case, and even with only one driving function and the simplest of all powers and $\nu = 0$, we are still faced with a Fred 2 integral equation 6.3.15. Eventually I know he will attack this specific problem, so we shall see if maybe we don't somehow bypass this Fred 2.

_____ long digression _____

Comment on integrals 6.3.6 and 6.37.

Along the way, Sned uses the double J_0 integral 6.3.6 which has the interesting integral rep shown in 6.3.7. We saw this somewhere before! That was on page 70 in 3.4.6 but with n = 0 there. This suggests that the double J_o integral in 6.3.6 is the same as the integral in 3.4.6 which seems pretty amazing. This 3.4.6 integral is my old friend [see "cos(nx) over sqrt..." item New 2 near the start of that doc]

$$\int_0^{2\pi} d\psi \cos(n\psi) / \sqrt{r^2 + r'^2 - 2rr' \cos(\psi-\psi')} \quad \} \;=\; 2 \cos(n\psi') \, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

To get 3.4.6 to this form we take $\psi = \theta - \theta_n$ and $\psi - \psi' = \theta - \theta$. Subtract second from first to get that

$\psi' = \theta - \theta_n - (\theta - \theta) = \theta - \theta_n$. This makes the LHS of my equation above agree with the LHS of 3.4.6. If we then equate the RHS's and cancel $\cos(n\psi')$ we find that

$$4\,(rr')^{-n} \int_0^{\min(r,r')} dt\, t^{2n} / \left[\sqrt{r'^2-t^2}\,\sqrt{r^2-t^2} \right] = 2\, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

and a special case of this would be n = 0,

$$4 \int_0^{\min(r,r')} dt \,/ \left[\sqrt{r'^2-t^2}\,\sqrt{r^2-t^2} \right] = 2\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

Let's divide the above by 2π to get

$$(2/\pi) \int_0^{\min(r,r')} dt \,/ \left[\sqrt{r'^2-t^2}\,\sqrt{r^2-t^2} \right] = (1/\pi)\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

The LHS of my last line above is 6.3.7. Then looking at 6.3.6 it just be true that

$$\int_0^\infty dk\, J_0(kr)\, J_0(kr') = (1/\pi)\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

Is this really true? I cannot find the double J_0 per se, but I do have this:

6.512

$$1.^{11} \quad \int_0^\infty J_\mu(ax)\, J_\nu(bx)\, dx = b^\nu a^{-\nu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

$[a > 0, \quad b > 0, \quad \mathrm{Re}(\mu+\nu) > -1, \quad b < a.$
For $a > b$, the positions of μ and ν should be reversed.]

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which tells us that

$$\int_0^\infty J_o(ax) J_o(bx)\, dx = a^{-1}\, \Gamma(1/2)/[\,\Gamma(1)\Gamma(1/2)] * F(1/2;\, 1/2;\, 1;\, b^2/a^2)$$

$$= F(1/2;\, 1/2;\, 1;\, b^2/a^2) / a \qquad\qquad 0 < b < a$$

Now where do you look this up? I don't have a good list showing special cases, but I did find that (verified on page 318 Bateman HT 2)

The complete elliptic integrals K and E are given by

$$K(k) = \tfrac{1}{2}\pi {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right)$$
$$E(k) = \tfrac{1}{2}\pi {}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right)$$

so we must then have

$$\int_0^\infty J_0(ax)J_0(bx)\,dx = (1/a)\,(2/\pi)K(b/a) \qquad 0 < b < a$$

Strange that such a simple integral does not appear in any source I have, nor does Alpha know about it. Rewrite it this way using a = r' and b = r

$$\int_0^\infty J_0(r'x)J_0(rx)\,dx = (1/r')\,(2/\pi)K(r/r') \qquad 0 < r < r'$$

Now use GR p 908 which says

$$K[\,2\sqrt{k}\,/(1+k)] = (1+k)K(k)$$

and set k = r/r' to get

$$K[\,2\sqrt{r/r'}\,/(1+r/r')] = (1+r/r')K(r/r')$$

or

$$K[\,2\sqrt{rr'}\,/(r'+r)] = (1+r/r')K(r/r')$$

So this implies that

$$\int_0^\infty J_0(r'x)J_0(rx)\,dx = (1/r')\,(2/\pi)\,K(r/r') = (1/r')\,(2/\pi)\,K[\,2\sqrt{rr'}\,/(r'+r)]/\,(1+r/r')$$

$$= (2/\pi)\,K[\,2\sqrt{rr'}\,/(r'+r)]/\,(r'+r) \qquad (*)$$

MEANWHILE, A&S page 337 in conical functions section claims this in 8.13.3 for off the cut Q functions,

$$Q_{-1/2}(z) = \sqrt{2/(z+1)}\;K\left(\sqrt{2/(z+1)}\,\right)$$

which might be able to reconcile my two results above. Suppose we try

$$z = (r^2 + r'^2)/(2rr')$$

Then Maple tells us that

$\sqrt{2/(z+1)} = 2\sqrt{rr'}/(r+r')$

so that

$$Q_{-1/2}[(r^2 + r'^2)/(2rr')] = 2\sqrt{rr'}/(r+r') * K[2\sqrt{rr'}/(r+r')]$$

So things look promising at least. My first form for the JJ integral was this:

$$\int_0^\infty dk\, J_0(kr)\, J_0(kr') = (1/\pi)\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

and if we sub in from above this becomes

$$\int_0^\infty dk\, J_0(kr)\, J_0(kr') = (1/\pi)\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= (1/\pi)\, 2\sqrt{rr'}/(r+r') * K[2\sqrt{rr'}/(r+r')] / \sqrt{rr'}$$

$$= (2/\pi)\, K[2\sqrt{rr'}/(r+r')]/(r+r')$$

and lo and behold, this agrees with (*) above. So this shows that we can write the JJ integral in 3 ways:

$$\int_0^\infty dk\, J_0(kr)\, J_0(kr') = (1/\pi)\, Q_{-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$\int_0^\infty J_0(r'x) J_0(rx)\, dx = (1/r')\,(2/\pi) K(r/r') \qquad 0 < r < r'$$

$$\int_0^\infty J_0(r'x) J_0(rx)\, dx = (2/\pi) \int_0^{\min(r,r')} dt \,/\, [\sqrt{r'^2-t^2}\,\sqrt{r^2-t^2}\,]$$

Mandal's Google book "advances in dual integral" offers this result

$$\int_0^\infty J_\nu(xt)\, J_\nu(yt)\, dt = \frac{2}{\pi (xy)^\nu} \int_0^{\min(x,y)} \frac{s^{2\nu} ds}{\sqrt{(x^2 - s^2)(y^2 - s^2)}}, \qquad \text{(A.1.15)}$$

which is a generalization of the result just above. It seems odd that this fact does not appear in Bateman. I perused the Bessel chapter and did not see it. An equation saying "integral = integral" never appears in organized special function discussions. Perusing Watson's Bessel treatise I don't see it either, but I do see this interesting integral which seems an improvement on the GR result quoted above:

$$(2) \qquad \int_0^\infty e^{-at} J_\nu(bt)\, J_\nu(ct)\, dt = \frac{1}{\pi \sqrt{(bc)}} Q_{\nu-\frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2bc}\right). \qquad (*)$$

The above integral also appears on page 696 of GR7 6.612.3

3. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{\pi\sqrt{\gamma\beta}}\, Q_{\nu-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}\right)$$
$$[\mathrm{Re}\,(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}] \quad \text{WA 426(2), ET II 50(17)}$$

Note added: We now have independent confirmation of this triple result (see quotes above)

$$\int_0^\infty J_m(rx)J_m(r'x)\,dx = (2/\pi)\,(rr')^{-m} \int_0^{\min(r,r')} ds\, s^{2m} / [\sqrt{r^2-s^2}\,\sqrt{r'^2-s^2}\,]$$

$$= (1/\pi)\,(rr')^{-1/2}\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')]$$

According to our Bateman quote above we have

$$\int_0^\infty J_\nu(ax)J_\nu(bx)\,dx = b^\nu\, a^{-\nu-1}\,\Gamma(\nu+1/2)/[\Gamma(\nu+1)\Gamma(1/2)] * F(\nu+1/2,1/2;\nu+1;b^2/a^2)$$

Maple does not give any reduction of this to standard functions.

As for the integral on the right, GR p 280 is at least on the topic, but does not have anything with a general power like this. // I looked again 12.16.10 and still cannot find anything. Maybe I can make use of this integral p 317 of GR7

3. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}\,dx = \mathrm{B}(\lambda,\mu)\;{}_2F_1(\nu,\lambda;\lambda+\mu;\beta)$$
$$[\mathrm{Re}\,\lambda > 0, \quad \mathrm{Re}\,\mu > 0, \quad |\beta| < 1] \qquad \text{WH}$$

Let's set $\nu = 1/2$ and $\mu = 1/2$ and this says

$$\int_0^1 dx\, x^{\lambda-1} / [\sqrt{1-x}\,\sqrt{1-\beta x}\,] = B(\lambda,1/2)\, F(1/2, \lambda; \lambda+1/2; \beta)$$

Start with

$$I = (2/\pi)\,(ab)^{-\nu} \int_0^{\min(a,b)} ds\, s^{2\nu} / [\sqrt{a^2-s^2}\,\sqrt{b^2-s^2}\,]$$

First replace with $s^2 = a^2x$ so $2sds = a^2dx$, so we get $x = (s/a)^2$ so $\min(a,b) \to \min(1,(b/a)^2)$

$$= (2/\pi)\,(ab)^{-\nu} \int_0^{\min(1,(b/a)^2)} ds\, s\, s^{2\nu-1} / [\sqrt{a^2-s^2}\,\sqrt{b^2-s^2}\,]$$

$$= (1/\pi)\,(ab)^{-\nu} \int_0^{\min(1,(b/a)^2)} a^2dx\; [ax^{1/2}]^{2\nu-1} / [\sqrt{a^2-a^2x}\,\sqrt{b^2-a^2x}\,]$$

$$= (1/\pi)\,(ab)^{-\nu}\,a^{2+2\nu-1}\int_0^{\min(1,(b/a)^2)} dx\; x^{\nu-1/2} / [ab\sqrt{1-x}\sqrt{1-(a/b)^2x}\,]$$

$$= (1/\pi)\,(ab)^{-\nu-1}\,a^{2+2\nu-1}\int_0^{\min(1,(b/a)^2)} dx\; x^{\nu-1/2} / [\sqrt{1-x}\sqrt{1-(a/b)^2x}\,]$$

$$= (1/\pi)\,b^{-\nu-1}\,a^{\nu}\int_0^{\min(1,(b/a)^2)} dx\; x^{\nu-1/2} / [\sqrt{1-x}\sqrt{1-(a/b)^2x}\,]$$

Now define $\beta = (a/b)^2$ and ASSUME now that $b/a > 1$ so that $\min(1,(b/a)^2) = 1$ and $\beta < 1$. Then we have

$$= (1/\pi)\,b^{-\nu-1}\,a^{\nu}\int_0^{1} dx\; x^{\nu-1/2} / [\sqrt{1-x}\sqrt{1-\beta x}\,]$$

We can then use our quoted integral above with $\lambda-1 = \nu-1/2$ so $\lambda = \nu+1/2$ and we get

$$= (1/\pi)\,b^{-\nu-1}\,a^{\nu}\;B(\nu+1/2,1/2)\;F(1/2,\,\nu+1/2;\,\nu+1;\,\beta)$$

So at this point we have shown that

$$(2/\pi)\,(ab)^{-\nu}\int_0^{\mathbf{min(a,b)}} ds\; s^{2\nu} / [\sqrt{a^2-s^2}\sqrt{b^2-s^2}\,]$$

$$= (1/\pi)\,b^{-\nu-1}\,a^{\nu}\;B(\nu+1/2,1/2)\;F(1/2,\,\nu+1/2;\,\nu+1;\,(a/b)^2) \qquad \text{when } \beta < 1,\quad \beta = (a/b)^2$$

In this case the F function converges, by the way. Now hold on this for a moment, and taking a hint from (*) above, let us consider

$$Q_{\nu-1/2}[\,(a^2+b^2)/(2ab)] \;\equiv\; Q_{\nu-1/2}[z]$$

Now here is an interesting Maple fact. If we define z as shown above, then we find that

$$[\,z - \sqrt{z^2-1}\,] / [\,z + \sqrt{z^2-1}\,] = (a/b)^2 = \beta$$
$$\sqrt{z^2-1} = (b^2-a^2)/(2ab)$$
$$z + \sqrt{z^2-1} = b/a$$
$$z - \sqrt{z^2-1} = 1/b$$

With this as a further hint, we now look at Bateman page 137 (45) which sys, with $\mu = 0$,

$$Q_\nu(z) = \sqrt{\pi}\,\Gamma(1+\nu)\,(b/a)^{-1-\nu} / \Gamma(\nu+3/2) * F(1/2,\,1+\nu;\,\nu+3/2;\,[\,z - \sqrt{z^2-1}\,] / [\,z + \sqrt{z^2-1}\,])$$

=>

$$Q_{\nu-1/2}(z) = \sqrt{\pi}\,\Gamma(1+\nu-1/2)\,(b/a)^{-1-(\nu-1/2)} / \Gamma(\nu-1/2+3/2) * F(1/2,\,1+\nu-1/2;\,\nu-1/2+3/2;\,\beta)$$

$$= \sqrt{\pi}\,\Gamma(\nu+1/2)\,(b/a)^{-1/2-\nu} / \Gamma(\nu+1) * F(1/2,\,1/2+\nu;\,\nu+1;\,\beta)$$

Solving for the F we get

$$F(1/2,\ 1/2+\nu;\ \nu+1;\ \beta) \ = \ \Gamma(\nu+1) \,/\, [\sqrt{\pi}\ \Gamma(\nu+1/2)\ (b/a)^{-1/2-\nu}]\ Q_{\nu-1/2}(z)$$

We can now insert this into our starting position above

$$(2/\pi)\ (ab)^{-\nu}\ \ \int_{\mathbf{0}}^{\mathbf{min(a,b)}} ds\ s^{2\nu} \,/\, [\ \sqrt{a^2-s^2}\ \sqrt{b^2-s^2}\ ]$$

$$= (1/\pi)\ b^{-\nu-1}\ a^{\nu}\ B(\nu+1/2,1/2)\ F(1/2,\ \nu+1/2;\ \nu+1;\ (a/b)^2) \qquad \text{when } \beta < 1, \quad \beta = (a/b)^2 \quad \text{line A}$$

$$= \ (1/\pi)\ b^{-\nu-1}\ a^{\nu}\ B(\nu+1/2,1/2)\ \Gamma(\nu+1) \,/\, [\sqrt{\pi}\ \Gamma(\nu+1/2)\ (b/a)^{-1/2-\nu}]\ Q_{\nu-1/2}(z)$$

$$= \ (1/\pi)\ b^{-\nu-1}\ a^{\nu}\ (b/a)^{1/2+\nu} B(\nu+1/2,1/2)\ \Gamma(\nu+1) \,/\, [\sqrt{\pi}\ \Gamma(\nu+1/2)]\ Q_{\nu-1/2}(z)$$

$$= \ (1/\pi)\ b^{-1/2}\ a^{-1/2}\ \{\Gamma(\nu+1/2)\Gamma(1/2)/\ \Gamma(\nu+1)\}\ \ \Gamma(\nu+1) \,/\, [\sqrt{\pi}\ \Gamma(\nu+1/2)]\ Q_{\nu-1/2}(z)$$

$$= \ (1/\pi)\ b^{-1/2}\ a^{-1/2}\ \{\sqrt{\pi}\ \}\ /\ [\sqrt{\pi}\ ]\ Q_{\nu-1/2}(z)$$

$$= \ (1/\pi)\ (ab)^{-1/2}\ Q_{\nu-1/2}[\ (a^2+b^2)/(2ab)]$$

So finally we have this part of the pie:

$$(2/\pi)\ (ab)^{-\nu}\ \ \int_{\mathbf{0}}^{\mathbf{min(a,b)}} ds\ s^{2\nu} \,/\, [\ \sqrt{a^2-s^2}\ \sqrt{b^2-s^2}\ ] \ = \ \ (1/\pi)\ (ab)^{-1/2}\ Q_{\nu-1/2}[\ (a^2+b^2)/(2ab)]$$

This was for $\beta < 1$, but then we argue sym $a \leftrightarrow b$ to get same result for $\beta > 1$ etc etc. The other piece of th pie is to use Bateman as quoted above

$$\int_{\mathbf{0}}^{\infty} J_\nu(ax)J_\nu(bx)\ dx \ = b^{\nu}\ a^{-\nu-1}\ \Gamma(\nu+1/2)/[\Gamma(\nu+1)\Gamma(1/2)]\ *\ F(\nu+1/2,1/2;\nu+1;b^2/a^2)$$

But let's swap $a \leftrightarrow b$ and write this as

$$\int_{\mathbf{0}}^{\infty} J_\nu(ax)J_\nu(bx)\ dx \ = a^{\nu}\ b^{-\nu-1}\ \Gamma(\nu+1/2)/[\Gamma(\nu+1)\Gamma(1/2)]\ *\ F(\nu+1/2,1/2;\nu+1;a^2/b^2)$$

$$= \ b^{-\nu-1}\ a^{\nu}\ \ \Gamma(\nu+1/2)\ \Gamma(1/2)/[\Gamma(\nu+1)\Gamma(1/2)^2]\ *\ F(\nu+1/2,1/2;\nu+1;a^2/b^2)$$

$$= \ (1/\pi)b^{-\nu-1}\ a^{\nu}\ B(\nu+1/2,1/2)\ F(\nu+1/2,1/2;\nu+1;a^2/b^2)$$

$$= (2/\pi)\ (ab)^{-\nu}\ \ \int_{\mathbf{0}}^{\mathbf{min(a,b)}} ds\ s^{2\nu} \,/\, [\ \sqrt{a^2-s^2}\ \sqrt{b^2-s^2}\ ] \qquad \text{// by line A above}$$

So we are finally all done and we have shown that

$$\int_0^\infty J_\nu(ax)J_\nu(bx)\,dx = (2/\pi)\,(ab)^{-\nu} \int_0^{\min(a,b)} ds\, s^{2\nu} / [\sqrt{a^2-s^2}\,\sqrt{b^2-s^2}\,]$$

$$= (1/\pi)\,(ab)^{-1/2}\, Q_{\nu-1/2}[\,(a^2+b^2)/(2ab)]$$

which also agrees with the Watson result I quoted above. If is then the ν = 0 application that appears on page 185 in Sneddon

$$\int_0^\infty J_0(ax)J_0(bx)\,dx = (2/\pi) \int_0^{\min(a,b)} ds / [\sqrt{a^2-s^2}\,\sqrt{b^2-s^2}\,]$$

$$= (1/\pi)\,(ab)^{-1/2}\, Q_{-1/2}[\,(a^2+b^2)/(2ab)]$$

end of digression

6.4 Reduction of Triple Integral to Dual Series

Here Sned quotes some Tranter work of 1960 where he takes our Titchmarsh 6.2.1 triple integral situation with the added assumption that G = 0. He uses no h weight function here. For α = 1/2 I show his result in brackets on page 188. He ends up with a dual series pair for a_n which you then jam into 6.4.10 to get your result A(u). Now the dual series happens to be that sin(n-1/2)θ "Dini thing" he studied in Chapter 5 (I wondered why we studied it). This is treated on page 158 p = +1 (α = 1/2) part (ii) with F=0. In his work there, he seems to get h_2 as in 5.4.45,and then the a_n are given in 5.4.44 and we do indeed seem to have a closed form solution! Now back to Chapter 6, if we have α = 1/2, we get the result just noted. But if α = -1/2, we get my pencil bracket on page 189 top. Both these results require that ν = 1/2 in the Titchmarsh's, so this does not apply to the azisym case! He has similar solutions for ν = -1/2. For general ν, you have to solve the dual series 6.4.7 which he claims are too hard to solve in general.

So here is the conclusion: For general ν, even if G = 0, you do get a dual series, but it is way to hard to solve. If we take four special cases ν = ±1/2 and α = ±1/2, we can get closed form solutions, and these don't apply to the azisym case which is ν = 0.

6.5 Triple Integral Series involving Legendre Polynomials

Sned's opening comment is that there has not been much "interest" in triple *series* situations, but he is rolling out a piece of work done by Collins 1962b. We have our two "forms" as shown 6.5.1 and 2, with the usual (2n+1) in one of them (this is the derivative one) and a little weight function added to the other. These are series, remember. Our three ranges will be based on $0 < \alpha < \beta < \pi$.

Sned then proposes that we look at two triple options. In the first, we have the weighted form prescribed in the middle region which is like having a potential prescribed on a spherical barrel. In the second we have the other form prescribed in the middle region, which is like having a charge distribution

σ prescribed on the barrel. In either case, the other two regions are disjoint and have 0 driver functions, so each case really is a triple series case. First kind and second kind.

The first action is to elevate the $P_n(\cos\theta)$ which appears in each form φ and χ into the Mehler integral representation 2.6.20, so our forms become 6.5.7 and 8. The second action is to assume you can write the prescribed function f(θ) as a Taylor series in tan(θ/2) as shown in 6.5.9. Actually it is a double sided series so right name is Laurent (Taylor has positive powers only). We break our f(θ) series into two positive series as shown bottom p 191, fine, $f_1 + f_2$

Then the coefficients C_n are broken into $A_n + B_n$ where the A_n are associated with f_1 and so on. Doing this causes the triple to becomes a quadruple thing shown in 6.5.12-15. The original middle range for our original triple is really the intersection of the two middle lines here 13 and 14, and each f_i has been quietly extended beyond its intended range (after all, we have known series) as shown in my pencil graphic. Roughly, Sned takes our quadruple thing and treats it as two sets of dual integral equations, but they are cross linked because each has A_n and B_n. He then applies that Mehler thing to everything and away we go with 9 pages of details to get the result for our two "triple options" assumed at the start. For the first case, we end up with a set of four Fred 2 equations! The second case gives a set of only two Fred 2's. For each case he considers special subcases.

So ends this chapter. I wonder if he will be calling on any of these triple results in his grand finale electrostatics chapter 8 ?