Sneddon Chap 7 Notes: Integral Rep Methods PhL 8.6.10

This chapter is a grab bag.

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Overview (n pages, written 12.17.10)

This chapter is titled "integral representation methods". But apart from "series" forms, almost all methods we have seen in this entire book are integral representation methods. The $\rho = \infty$ cylindrical Smythian form is an integral representation for the potential, for example. And Sned constantly uses helper functions whose integrals are some functions of interest, such as partner functions to prescribed driver functions. Really this chapter should be entitled "a grab bag of potential-theory problem-solving methods not included in earlier chapters".

In **Section 7.1 (Kobayashi potential)**, we solve the general Dirichlet $f(\rho,\theta)$ dual integral equation disk problem using the Kobayashi potential method as a demonstration of the method. This is the problem Copson solved in 1947 with another method, as we saw in Chap 3. We start with the familiar cylindrical Smythian atomic form. However, the two coefficients involved are assumed to be expressible for example as $a_m(u) = \Sigma_\mu A_m^\mu u^{-\lambda} J_\mu(u)$ where μ assumes certain values. That is, each original coefficient $a_m(u)$ is now a lincomb of J's and powers with some new coefficients A_m^μ. By selecting a clever set of values for μ and $\lambda = 1/2$, the Smythian form then automagically satisfies the second of the duals, a method we have seen before with J series, probably making use of the same J function property from Chap 2. A closed form solution for the A_m^μ is found. Sned's reason for having this book section is that this Kobayashi method is one of only two known methods (1966) of handling the potential problem of two coplanar disks, which he will address in Chapter 8.

In **Section 7.2** (**Dovnorovich's Solution)** Sned discusses a method which, as best I can tell, this solution is exactly Stakgold's "integral equation method" applied to a flat metal surface holding some σ. I recall how Stak shows that for a *flat* surface, you automatically get $\sigma = 0$ outside your surface when you use the $\int dA\ \sigma / R$ form for your potential, a fact appearing as p 202 A here (z=0!).

The rest of this section proposes a method that led me into many days of pain. You are supposed to consider $Ku = \mu u$ as an integral eigenvalue problem with kernel $K = 1/R$ in 2D. Once you get the

eigenfunctions of K, it is then trivial to solve a Dirichlet problem because K is then diagonal and everything diagonalizes and the solution is simple. BUT, I could not see any way to solve the integral EF problem even for a simple disk as the flat surface. See details in folder " A 2D integral EV problem". Sned does quote some other sources here: Zaremba 1946, Boussinesq 1885, and a Mikhlin book.

Section 7.3 (**Galin's theorem**) is an rather obscure theorem pertaining to a flat elliptical surface. If you happen to know that your potential is a polynomial in x and y on this surface, then σ on the surface is given by 7.3.3 which is another poly of x,y divided by a square root. An example is an ellipse with a constant potential (which is obviously a valid poly in x,y !) and you get just the square root part which is in fact the known result from ellipsoidal coordinate analysis. Sned proves Galin's theorem using Lamé functions, but I did not read the proof.

In **Section 7.4** ("A Simple Superposition Method") Sned reviews work of **Szegedin 1957** who showed that you can construct the Beltrami f(ρ) disk Dirichlet problem solution by doing my Red Flag superposition of simple Weber disk problems! The guy superposed concentric disks with a weight function K(a), and then later Sned showed that for a given f(ρ) you can invert to get K(a).

Section 7.5 (**Green**) provides a Collins 1959b solution of the Beltrami f(ρ) disk Dirichlet problem (Sned always calls this "the first basic problem") which uses the complex z+it integral representation for potential V as shown in 7.5.1 p 209. This z+it stuff was mentioned back in Chapter 3.6. Collins went on to solve a strange looking "radiation" disk problem shown in 7.5.18 where we have the mixed boundary condition stuff on the disk and V = 0 outside, and this ends up as a Fred 2 equation. There is some mention here of the alternative theorem of Stak and whether a Fred has a solution or not. I think Sned likes to associate the z+it representation idea with a certain latter-day Green, **Albert Edward Green** (1912-1999), who worked on z+it as early as 1949 and showed that the representation really is a viable one in a 1954 paper with Zerna. Green was a mechanics guy, stresses and cracks and all that stuff. Sneddon closes out this section with mention of z+it work he did with Lowengrub in 1962.

In **Section 7.6** (Some Bowl Problems) Sned reviews some Collins 1959 work which uses an exceedingly strange integral representation for the potential V see 7.6.1. It involves a complex R distance like thing with a sec(x/2) factor stuck in the integrand. Collins is able to do the azisym Dirichlet bowl problem with an f(θ) prescribed potential, using in the end a simple Abel inversion. Collins goes on to do some similar bowl problems including a Neumann bowl. Remember the title of this chapter, and this is just a weird complex integral representation method which does not have a very intuitive discussion. I of course did not spend much time reading this stuff.

In **Section 7.7** Sned reviews some Collins 1961a work on the Dirichlet problem for **two collinear bowls** whose sphere's don't intersect. The same complex potential with sec(x/2)/R is used as in Section 6, but we have one such integral rep for each bowl. This is a monster long section I did not study and you end up with some Fred 2 equations. I think WD Collins (Prof Derek Collins) is still going at Sheffield as an "honorary academic staff member", he has his name on papers in the 2000's on new topics.

In **Section 7.8** Sned discusses a z+it method attack on the **flat annular ring Dirichlet** problem. This time he has to set up at least three regions in space with Smythian form type potentials, and this all ends up in a Fred 2 situation. Collins is here as well, but so are others. I think at the very end he quotes the result when

you have a constant potential on the ring. I did not study this section, but it is the first appearance of the annular ring solved in this book.

7.1 Kobayashi Potentials

We have this "general form" shown in 7.1.1 which is just the usual atomic form in cylindricals for no ρ boundary, but the form is written in a special way such that the coefficients have slightly strange form. The normal coefficients might be $a_m(u)$ and $b_m(u)$ for the cos and sin terms, but here they are written

$$a_m(u) = \Sigma_\mu A_m^\mu u^{-\lambda} J_\mu(u)$$
$$b_m(u) = \Sigma_\mu B_m^\mu u^{-\lambda} J_\mu(u)$$

which just says that the u-dependence of the coefficients is *assumed* to be a sum of Bessel functions with power, where μ is the summation index, but it could be any such sum, perhaps even an integral on μ. Although fairly general, this "form" obviously does not cover all possible potentials, it is just a class.

So this general form is called (by Sned) a Kobayashi potential form (1931) and we then consider our usual mixed situation where we have some specified potential on a disk of unit radius (in the z = 0 plane), and outside that disk we have 0 charge density. If we make an assumption about the sum on μ, we get 7.1.2 which is a particular example of a Kobayashi potential. Using some fancy property from Ch 2, we find that this Kobayashi potential automatically satisfies the Neumann equation of our dual pair 7.1.3 and 7.1.4. [I think we have seen this idea in previous chapters where a series was assumed. Surely this is the main *motivation* for writing the coefficients above as sums of J functions. If you select a certain μ set, then you have solved your second dual and you hope your form has enough flex to solve the first.]

Sned then proceeds to find a solution for the first of the pair. He gives the A_m^μ in 7.1.6 as a double integral over the Dirichlet prescribed potential f(ρ,θ) on the disk, and B_m^μ is a similar integral. Notice that the shifted Jacobi polynomials are involved here -- that script F function (and their orthogonality)

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Since the form assumed for the potential is an integral over u, this method can be put in this grab bag chapter on "integral representation methods". That is to say, the Smythian Form is an integral rep.
(Actually it is part integral and part sum).

7.2 Dovnorovich's Solution of The First Basic Problem

This is Dirichlet for a localized, arbitrarily shaped but simply-connected flat "cookie".

Review of Stakgold Vol I page 200. We know we can configure the same boundary value problem in either ODE or integral equation terms. Maybe L has to be self-adjoint for this to work. One idea is that the kernel of the integral equation is the Green's Function of the ODE. The eigenfunctions of the ODE eigenvalue problem $L\varphi_n = \lambda_n\varphi_n$ will be solutions of the Fred 2 homo integral equation eigenvalue problem usually written as $K\varphi_n = \mu_n\varphi_n$ where $\mu_n = 1/\lambda_n$. If you want to solve a Fred 2 inhomo integral

equation of the form Ku = μu + f, one method is to expand u and f on the φ_n of the homo problem, perhaps with coefficients a_n and b_n. Then you can solve your inhomo integral equation if you can get the a_n expressed in terms of the b_n. Stak does this for his Example 5 where $L = -\partial_x^2$ for various values of μ (EV or non EV as shown on page 201).

At this point I went off and did Problem of the Day and the wagon crashed into a ditch and I then was off to the Cod. Go fix that wreck up first, then you are allowed to continue here trying to get an overview.

Suddenly things are more general and we are talking of a potential prescribed on a simply connected region S of the z = 0 plane (with σ = 0 outside). Here we represent that potential as the usual Stak integral of σ/R 7.2.4, and I am on familiar ground. This is a "simple layer" problem. In the usual way, we set z = 0 and this gives 7.2.6 which we would like to use to solve for σ(x,y) [here called p(u,v)] The claim is that the kernel here (which is just 1/R) is H-S so it has a set of eigenfunctions $\varphi_i(x,y)$ on which we can expand p. When the dust settles, we get 7.2.11 as the solution for σ. This is one of Stak's standard solutions I recall somewhere. We need to know the φ_i, then the projections f_i onto them, and the EV's λ_i. Notice that the second dual equation is automatically satisfied since we have z = 0 killing the second thing off outside region S (see p 202 A). [It is interesting to think of $\int \sigma/R$ as just one of many possible integral representation forms. This chapter certainly has many.]

So, could you use this method to solve the disk or iris problem? I have no idea what the 2D EF's are of this K operator, but we are referred to an Integral Equations book by **Mikhlin** of 1957, something I have never heard of. This is a 340 page book costing maybe $12 at Abe books! Marriott has it right now

	Location	Collection	Call No.	Description	Check Out Period	Due date	Due hour	Barcode	Note	SFX
Request	Marriott Library	LVL 1: General Collection	QA431 .M466 1957		60 Days	On Shelf		31114004174696		

so I could "check it out" without buying it. I see Sneddon has coauthored an integral equations book with Mikhlin!

So OK, the eigenfunctions of K would be specific to the region S, so there won't be any one-size-fits-all solution to this problem. Dovnorovich is a 1957 reference, so recent work.

7.3 Galin's Theorem

This is certainly an oddball theorem. Suppose you have a problem where you have Dirichlet on an ellipse in the z=0 plane, and Neumann=0 outside it. Then if V is some Laplace solution for the $z \geq 0$ half space, and if V is a poly p_n of degree n in x and y at z = 0, then Galin tells us the "general form" of σ on the ellipse. This general form is shown in 7.3.3 and you must determine there an unknown poly P_n of degree n. So in this kind of problem, you can reduce things to a Cramer's rule problem for those poly coefficients. Notice that for that ellipse being a circle and the potential p_n is a constant, then P_n = constant and we get the usual formula for σ on a charged disk! Of course I happen to know the solution for the charged ellipse problem from the old Kelvin form and from ellipsoidal coordinates, but I cannot find the result for σ right now, but later I may come back and verify that the Galin form is correct!

Sned in fact proves the Galin theorem using ellipsoidals, the first time such have appeared in this book. I skip this proof.

7.4 A simple superposition method

We learned about the "Weber solution" for the charged disk problem back in Ch 3. Suppose you were to superpose the Weber solution $U(\rho,z,a_1)$ for a disk of radius a_1 with that $U(\rho,z,a_2)$ of a disk of radius a_2 . Then on the inner disk you have V = 2 but in the annular region you have V = 1. According to my Red Flag Superposition theorem, which talks about $S_{int} = S_1 \cap S_2$, we find that the intersection of the two pieces of metal in problems 1 and 2 is the inner disk S_1, so we have $S_{int} = S_1$ and superposition is valid! So you could consider $V(\rho,z,a) = \int_0^a da' K(a') U(\rho,z,a')$ as a continuous superposition of Weber solution with radii a' < a, and this is 7.4.2 (I guess u is Sned's generic dummy integration variable, but for me using the same u is confusing, since u usually means k). Now

$$V(\rho,0,a) = \int_0^a da' K(a') U(\rho,0,a') = \int_0^a da' K(a') [H(\rho<a') 1 + H(\rho>a') (2/\pi)\sin^{-1}(a'/\rho)]$$

$$= \int_\rho^a da' K(a') + \int_0^\rho da' K(a') (2/\pi)\sin^{-1}(a'/\rho) \qquad \text{if } \rho \text{ is in the integration range } \rho<a$$

$$= \int_0^a da' K(a') (2/\pi)\sin^{-1}(a'/\rho) \qquad \text{if } \rho > a$$

and I have found a small typo. We are doing **Szegedin 1957** here. So for each weight function K(a') which you specify, you have created a Laplace solution. For some odd reason Szegedin did not realize a simple fact which Sneddon pointed out in a letter I guess that you can invert the above to determine K(a') rather than assign it. So prescribe $V(\rho,0,a) = f(\rho)$ on the disk of radius a, then K(a') is determined. In order to invert the above, you have to do "backwards parts" by first defining K_1 as the integral of K to get things into one of our standard inversion forms, and you find that $K_1(u) = \partial_u$ [integral of f] and then you end up with $K(u) = \partial_{u^2}$ [...] as shown in 7.4.6. Somehow this has to agree with the Beltrami solutions we found earlier in the book.

Now as an example Szegedin took $K(u) = u^{n-1}$ and found some Laplace solutions. OK, fine.

7.5 Green's solution of the First Basic Problem (and related things)

We must first recall from Section 3.6 that we could write the solution to the "Beltrami problem" (that is, potential on disk is some $f(\rho)$) in the form 7.5.1 [the z + it thing] where g (formerly called φ in "Sneddon's Beltrami simplified solution") is the "helper function". In this section, Sned presents some work of **Green and Zerna** (1954) (obviously not George Green!) which is just a rigorous proof that the form 7.5.1 really is a solution of this disk problem, called the First Basic Problem in Chapter 1. There is lots of mention of things being "continuous" and the proof concludes on the bottom of page 210, and I did not study it.

[Comment: I glossed over this "continuous" stuff, but it is a key point in the following sense: We often represent our potential as an integral of some coefficient "helper" function h(x) times some other function o(x) which is highly singular at certain locations. We say that the coefficient function h(x) is "smooth", so we can therefore do various manipulations. The singular nature of the integrand is "exposed" in the "other function" o(x) which in itself allows certain manipulations, such as a transform. I think that this idea of exposing the singular nature of an integrand is really the name of the game here. So my point is that there is "more to it" than just "renaming" your integrand I(x) to be a product of functions h(x)o(x) where we "just" replace I(x) as an unknown with h(x) as the unknown. Perhaps this idea is the basis of what is now called "regularization". This chapter, then, is really a study of various selections of o(x) for various situations.]

"Methods based on integral representations of harmonic functions" is our Chapter 7 really. Harmonic means solving Laplace in 3D.

At top of page 211, he just differentiates 7.5.1 to have a partner expansion for $\partial_z V$.

The mid section on page 211 considers the same f(ρ) Beltrami problem but with f(ρ) restricted to being an even function of ρ. In this case, our z + it form simplifies to 7.5.15 where j(t) is an odd helper function [Note: this is an integral representation for the potential -- V is an integral of j(t) -- and remember this whole chapter is entitled "integral representation methods".] Things now fit one of our standard invertible forms when we state the BC condition, so we then know j(t) as an integral of f(ρ), and the problem is solved. So this can be considered another solution of the Beltrami problem, it is pretty compact, and is Collins 1959b.

Page 212-213 sees **Collins** attacking a "radiation" BC problem with this same method. I just use that quoted term to refer to the idea that you have $\partial_z V + kV = f(\rho)$ as a very "mixed" BC situation for $\rho < 1$. This is a BC that is "mixed" at every point on the boundary. In this problem, we have a 3D half space defined by $z \geq 0$, and our "boundary" is the entire disk $\rho < 1$ at z=0, and on this boundary we have this fancy mixed BC just stated. Sned shows how for this situation [also, V = 0 outside the disk] you end up with a Fred 2 integral equation for the helper function j(t) as shown 7.5.24 with quite a simple kernel. The driving term is called $\ell(t)$ which is related to f(ρ) by two integrals as shown p 212.

Sned then makes some Stak-familiar comments. We have a Fred 2 inhomo here where the eigenvalue quantity is our mixed BC constant k (Stak's λ). We know that there is an associated EV equation $Ku = (1/\lambda)u$ I think. We know that there is an "alternative theorem" which says that the "range" shrinks as the "nullspace" increases. This means that if λ is an eigenvalue, then you have "some nullspace" and that means there will be restrictions on the functions $\ell(t)$ (Stak f(x)) for which Fred 2 inhomo solutions will exist! Sned points out that we know there is at least one EV, so yes, for this value of k, there will be some nullspace, and that means our solution is not unique (for that EV). I think Sned's main point here is that as long as we avoid this one "bad" value of k (and I suppose other possible EV's), our Collins BV problem 7.5.18 will have a *unique* solution. So here is a nice tie-in Sned to Stak. Again Sned quotes the Mikhlin book which specializes in integral equations (Stak is much more general).

The next little shot is a **Sneddon and Lowengrub** effort (1962) where they put V = 0 on the disk, but then let f(ρ) specify the charge density *outside* the disk. They use an integral representation like 7.5.1 but the integral ranges (1,∞) instead of (0,1) I guess because we assume our external potential decays at ∞ (he in fact specifies this fact). He says you can show this thing is harmonic for any g(t). Several times he makes the point: you can differentiate through the integral sign provided the integrand is continuous. Doing a little trick with a helper function Φ, they arrive at the conclusion 7.5.30 which gives g(t) of our integral rep as an integral of f(ρ), problem solved.

7.6 Spherical Cap Problems [p 215]

In Section 7.5 we used the integral rep 7.5.1 (the z+it thing) for doing a few problems with the half space $z \geq 0$ where we had some kind of BC's at $z = 0$ with respect to a disk centered at the cylindrical coordinates origin. We are now going to move to a different geometry, and the variable ρ of the previous world will become polar angle θ going down the cap I am sure.

The ansatz here is the integral rep shown in 7.6.1 which strikes me as unusual in that R does not seem to be any distance, and is in fact complex. [This is a "selection" of $o(x) = \sec(x/2)/R$ commented on above!] However, the integration range is symmetric $(-\alpha,\alpha)$ so I guess that makes things work, and $g(x)$ is assumed even (x is an angle). Soon α will be the angle of a bowl at $r = 1$ in sphericals. At r=1 this R thing becomes 7.6.4 and he worries about the phase as you move past the edge of the bowl, all familiar to me. [The function $o(x)$ is "handling" the bowl geometry.] So our first problem is the **Dirichlet bowl** with $f(\theta)$ specified. As 7.6.9 shows, $f(\theta) = F(\tan\theta/2)$, a reparam, and then we get G from F in 7.6.12, and G is the same reparam of g so we have found g(t) hence our solution !

[Sned won't say it, but I am pretty sure g_1 is the charge density σ and R really is the distance to a point where we then observe the potential V, so this really is just the usual σ/R integral, but with R somehow being complex. This is our Stak thing of solving first for the simple layer σ (his I) and then use that to get the potential. I know that $\sec(\theta/2)$ thing comes in somehow from a scale factor or something, maybe it is part of the true distance R.... just winging this comment.]

So what just happened (we are at mid page 217)? Collins just cranked out a Dirichlet bowl solution starting with this strange integral representation 7.6.1. I know there is something going on here that is not being stated, maybe a toroidal coordinates thing where the bowl is a surface of constant potential, or maybe oblates of some kind. I recall all these half-angle things floating around in my past "bowl work" based on inversion.

Recall that the bowl analysis usually uses series in sphericals, not an integral, so here we have a rare treatment I think where the bowl problem is handled in a dual integral equation. This section is unusual in that Sned seems not to state our problem clearly. He does not say what the other BC is (no charge on the rest of the bowl I presume). I guess when he says V is continuous everywhere except on the bowl, that implies we have $\sigma = 0$ on the complementary sphere surface.

Page 217 bottom we start into a **Neumann bowl** version of the same problem. He just turns the crank and out comes a similar solution. Again, I see all the motions, but I am not "understanding" the underlying group theory here or whatever it is. Somehow 7.6.1 is a Smythian form in some set of coordinates which has m=0 atoms of the form shown. But again, the complex R is strange to me.

7.7 A Dirichlet Problem with Two Spherical Caps

The picture on page 219 shows them. The z axis goes to the right, the problem is at least azisym! The potential is prescribed on both caps! How do you write a form for V that is harmonic and does the right stuff on these caps? An obvious start is the sum of the two single-cap terms shown in 7.7.2 with helper functions now g_1 and g_2. This superposition seems to violate my red flag metal superposition theorem, but we are not really claiming that each term is a solution of problem 1 or 2.

This section, no surprise, is 7 long pages of details. We end up with a system of two Fred 2's for some functions G_1 and G_2. Sned spends a whole page showing how this can be reduced to a single Fred 2 for a function called L, as in 7.7.25.

When each cap is held at a constant potential, then you can be talking two metal caps and then 7.7.34 confirms my feeling that g_1 and g_2 really are the charge densities on these caps.

Question: in the more general case -- let's just go back to the f(ρ) Beltrami disk -- when potential varies on the BC surface, we know it is not metal. But when you jam some Dirichlet potential on it, do you create a sticky charge density on it? You might. I hate to admit it, but even after doing this for a year or so, I don't have a solid answer to that question. What about f(θ) on the unit circle in 2D? The interpretation of $\partial_n g$ as a σ is always in the Green's Function problem, not the Dirichlet problem where it is the Poisson kernel. Recall from Stak notes how you think of these two problems together.

7.8 A Dirichlet Problem with an Annulus BC Surface [p 225]

OK, after a visit to Bowl World, we are back to our $z \geq 0$ half space where we specify a Dirichlet potential on some piece S of the plane $z = 0$. We have done a disk in the past, we talked about an ellipse with the Galin's Theorem. Now we attack the case where this surface S is an *annulus* between $\rho = a$ and $\rho = b$, and we have $\sigma = 0$ both inside and outside. We saw something vaguely like this mentioned in Heat Flow World at the start of Chapter 6 as an example of a triple integral equation.

In the discussion here, things are not explicitly treated as a triple deal. BUT, he is doing a superposition of four potentials $V_{0,1,2}$ and U as in 7.8.3 and somehow this does look like one of my painful "multiple region" attacks on problems like this. I think his phrase "a plane screen" is what I call an "iris". You see "region looking stuff" in all those BC conditions on page 226.

He makes use of our famous (by now) $z \pm it$ complex integral representations for each of the three V_i so each has its own g_i, top page 227. Eventually this all boils down to a Fred 2. Finally at the very end, I think he quotes the result which is "simple" where the Dirichlet condition is V = constant on the annulus, but I don't see the solution chain. I have of course just scanned this section as with the previous ones.

But if I someday want to know about a "charged metal annulus", I will come back here and study things a little more. [Boom, this comes up in Section 8.5]

Then of course what about two such annuli? And what about thin ones which make thin rings? And then what about multiple rings? There are lots of problems! I wonder if there is a directory somewhere that tells you where to look for a given problem? The Polyanin Handbook of Electrostatic Problems? Lebedev's book is a bit this way, but mixes in those other worlds.

And so ends Chapter 7. Somewhere I have to comment that there seems to have been a lot of activity in the dual integral equations world in the 1940-1964 time frame, say, with names that come up again and again, such as Collins, Kober, etc. I wonder if this activity continued unabated after the 1966 publication date of this Sneddon book? Probably yes, so this book is just a survey at one point in time. That was 44 years ago! Still, everyone who is anyone always references Sneddon for something or other! I think Sneddon realized that he was part of a burst of activity, he was a good writer and presenter, so he decided to do this documentation task for the group.

So who are these people like Collins? They are not particle physics people, nor electrostatics people per se. Perhaps not even physics people. I should look up some of the names and see where the existed on the math-physics-engineering spectra.

There are of course older names associated with the charged disk problem. Weber is 1873 and Beltrami is 1881. Gallop was 1886. But then we seem to jump to Copson 1947.

When we come to the subject of dual integral equations, Peters is 1961, Titchmarsh 1948, Noble 1958, Williams 1961, Busbridge 1938

Tranter C J : Professor of Mathematical Physics at the Royal Military College in Shrivenham

Titchmarsh E C: Savilian Professor of Geometry at the University of Oxford from 1932 to 1963.

Sneddon was Professor of Mathematics at U of Glasgow.

Collins, Derek (WD) Applied Mathematics at Sheffield, still alive.

Physics, Mathematical Physics, Theoretical Physics, Applied Mathematics, there are many names.