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Overview (written 1.7.11, 2.5 pages)

In **Section 8.1** (18 p) Sned deals with the most famous capacitor problem in electrostatics: the **parallel disk capacitor** (radius 1, separation κ). It is to me a fascinating subject.

He first reviews Love's 1949 work. The potential in 8.1.2 is modeled as a sum of two sums, where each sum is of oblate atoms for its disk. Somewhere a limit is taken so each plate is a disk and not an oblate spheroid. The end result is a very simple looking Fred 2 integral equation 8.1.6 for a function f(t), called **Love's Integral Equation** (and there are current papers concerning this equation!). If you solve this for f(t), then 8.1.7 tells you the capacitance C, and equation 8.1.4 tells you the potential everywhere, and from this you could get σ. So although this problem apparently has no closed-form solution, the Love solution is exact to the extent you can solve Love's equation. Love was extending earlier 1924 work by **Nicholson**. Sned does not explain all the work, but it uses a z+it form and involves tricky properties of associated Legendre functions.

Sned then rolls his own alternate derivation of the Love solution. He starts by making a three-region Smythian form, and ends up somehow with dual integral equations 8.1.13 where ± refers to the two disks having the same or opposite potentials. He shows how this leads to the same Love solution. Sned's dual integral equation has the "weighted form" he treated many times earlier in the book, so here we finally see an application of the weighted dual integral equation of Bessel type.

Next, Sned shows how you can do a Newton iterative kernel type solution to the Love equation, and he ends up with some numerical results in a table on page 238 for capacitance. The numbers are relative to the capacitance of far-separated disks (which is the sum of the two known disk caps). The integral equation gets harder to solve as the plates get closer. In 1966 someone had done $\kappa = 0.1$, but I just found a

numerical paper date 2009 that takes this down to $\kappa = .0001$! [By the way, this paper uses a combination of numerical *and* analytic work! It is not just a simple 3D mesh thing.] Of course for very small κ, you approach the known classical parallel plate capacitor limit where edge effects can be ignored.

Sned shows how you arrive from his model at the Kirchhoff 1877 approximate formula for capacitance C (8.1.28) which is valid for $\kappa << 1.5$. A table on page 241 shows how extremely close this formula gets to the exact numerical solution for κ in the range 0.1 to 0.8.

Sned goes on to discuss several other approaches to this classic problem.

In **Section 8.2** (6p) Sned addresses the Beltrami $f(\rho)$ **Dirichlet disk** problem, but the disk lies halfway between two **parallel infinite plates** which have V = 0. Sned gives 3 different approaches, all of which lead the same Fred 2 equations, so this is not a closed form problem. His first approach uses a two-region Smythian form 8.2.5 which is of cylindrical atoms in a special way. This leads to a weighed dual integral pair 8.2.7 with simple weigh k(u) as shown. His approach is similar to the "Sned's way" Beltrami solution, but because of the weight, he ends up with Fred 2 equation 8.2.10 instead of a closed form solution. His second approach (Collins) uses a triple fit with z+it integrands as in 8.2.16, and this ends up with the same Fred 2. His third approach (Williams) makes use of the Green's function for the parallel plates. Once again he ends up with the same old Fred 2.

In **Section 8.3** (7p) we now put our radius-1 Beltrami Dirichlet **disk** $f(\rho)$ inside an infinite grounded **cylinder** of radius $a > 1$. Again Sned presents several different approaches, all of which lead to a Fred 2 equation. His first approach uses the unusual cylindrical Smythian form 8.3.4 which contains a *continuous* k spectrum and both the J_0 and I_0 Bessel functions with coefficients A and B. I guess the J_0 alone cannot handle this problem. This gives three integral equations, but by assuming form 8.3.8 this is reduced to dual integral equations. This all leads to Fred 2 8.3.12, so no closed form solution (even if f = 1). His second approach is the Williams Green's Function method. Collins has a third method mentioned briefly. Then on page 256 Sned gives a fourth approach which uses a series Smythian form with the expected $J_0(ak_n) = 0$ spectrum. This gives a dual series situation of the type already studied in this book, and I guess this also gives a Fred 2. Even for the simplest Beltrami case $f(\rho) = 1$, there is no closed form solution, but he does perturbation theory where 1/a is the smallness parameter, and this leads to some numeric data for capacitance shown page 259, but I have not examined that data. I don't think any approximation formulas of the Kirchhoff type are given, there might be something like that. (I keep thinking 3D conformal mapping of some sort would be useful.)

In **Section 8.4** (5 p) does the **coplanar disk Dirichlet problem** where each disk has a distinct Copson $f(\rho,\varphi)$. As usual, a Smythian form is concocted as a sum of two objects, one for each disk with separate cylindrical coordinates for each. You might have expected an integral Smythian form, and it is in a way. Each object is a sum is of cylindrical atoms, but in a very special way that falls into the Kobayashi potential category see 8.4.1 with a continuous μ spectrum. We are not azisym, $m \neq 0$ is present, and we have four sets of coefficients A,B,C,D, very messy! Many equations later, we arrive at two pairs of rather ugly-looking dual series equations p 264 A and B. The kernels in these things (kernels are K and H) are variations of the triple-J integral G object 8.4.9. Each sum is in fact a double sum, something totally new to this book for dual series (perhaps one comes from the $m \neq 0$ effect). We are quietly referred to K's 1939 paper on how you might solve such series! I think Sned regards this as an exceedingly difficult problem. No numerical data is presented.

In **Section 8.5** (2p) Sned considers a **capacitor** consisting of **two identical coplanar metal strips** extruding in the z direction. This is of course the 2D problem of two equal wire segments on the x axis, I think the first and perhaps only 2D problem addressed in this entire Sneddon book. For the right side segment, we have three regions which are $x<a$, $a\le x\le b$, $x>b$. Happily, a simple *2D* Smythian form is used, which of course involves *trig* functions, ie, $\sin(kx)e^{-k|y|}$ as atoms. This leads to a triple *trig* type integral equation p 265A. Using a method mentioned earlier in the book, this can be converted to a dual series equation p 266B, and this just happens to be one of those trig dual series we studied earlier and which has a closed form solution! The helper function A(u) is as in 8.5.6, σ(x) in 8.5.7, V(x,y) in 8.5.3, we have everything. σ(x) is computed and it very simply stated as 8.5.8 -- I could have guessed the form of the answer! So to Sned's credit, in this section he uses two of the math methods we worked pretty hard to learn earlier in the book.

In **Section 8.6** (4 p) Sned treats the **annular metal disk with V = 1**, radii are b > a. This problem trivially is seen to be of the triple J_0 integral equation form. This corresponds to the problem Sned studied in his section 6.3 but with the weight set to h = 0. Even in this simple case, as I noted there, this problem boils down to a Fred 2. In this application, that Fred 2 is 8.6.7 for H(φ) with kernel in 8.6.8. Once you know H, the charge σ and total Q hence capacitance are easy to compute. Numeric data is presented which shows if the hole has half the outer radius, you still get 98% of the solid disk capacitance (save metal!) One could take a limit of this problem to get data for a flat wire ring, something I wonder about from time to time.

In **Section 8.7** Sned reminds us of the solution to the **Dirichlet f(θ) = 1 bowl** he already got in Section 5.6, resulting from treatment of the simplified Jacobi polynomial dual series equations. Then he solves the problem again using the Collins Section 7.6 method which uses that strange complex 1/R object in an integral representation for the potential. The solution comes out unbelievably simple, see 8.7.6 for V(r,θ) in terms of certain simple γ and γ' objects.

Sned's final problem is grounding this same bowl and putting it in a uniform E field. This leads to dual series equations now with F(θ) = cosθ instead of F = 1, and the same Section 5.6 method gives a simple closed-form result for a_n as shown in 8.7.10. Sned then redoes this using the Collins Section 7.6 method and gets a relatively simple closed-form no-sum no-integral result for the potential. He credits Ferrers in 1882 for first solving this problem, and several later people.

8.1 The Circular Plate "Condenser"

1926 Gloss. Terms Electr. Engin. (B.S.I.) 43 Condenser..It is suggested that the new term **capacitor** shall be used for this device in order to avoid confusion with a steam ‘Condenser’. -OED

This is the second "classical problem", first was the charged disk. It has a long history given on page 230.

8.1.1 Love's Integral Equation

Separation is κ, radii are 1, z axis obvious. The setup includes two cylindrical and two oblate coordinate systems. Sned's oblates are called u,v and in terms of MF we have

ξ = shv
η = sinu

$x = a\sqrt{\xi^2+1}\sqrt{1-\eta^2}\cos\varphi = a\,\text{chv}\cos u\cos\varphi$
$y = a\sqrt{\xi^2+1}\sqrt{1-\eta^2}\sin\varphi = a\,\text{chv}\cos u\sin\varphi$
$z = a\,\xi\,\eta = a\,\text{shv}\sin u$

So u is the complement of the polar angle I would say offhand. a is the spheroid focal distance, and when you flatten a spheroid to a disk, a is the radius, and here a = 1.

Having introduced v and u as above, Sned at once drops back to ξ and η, but naturally he has them *exactly reversed* from MF:

MF:	$P_n^m(\eta)\,Q_n^m(i\xi)\,e^{im\varphi}$	ξ = spheroid label
Sned:	$P_n^m(\xi)\,Q_n^m(i\eta)\,e^{im\varphi}$	η = spheroid label

OK, all fine. P 231 A shows a pretty useful integral rep for the product $P_n(\xi)Q_n(i\eta)$ in which our (z+it) denominator appears. This looks just like 7.5.15 in which we had j(t) being an odd function, but we have it here where $j(t) = P_n(t)$ which can be odd or even! Notice $z = (1-\xi^2)(1+\eta^2)$..

But we need to back up. The ansatz for the day is 8.1.2 which gives V as a sum of oblate atoms of the two systems, all being azisym of course. The ± here indicates which of two problems we are doing -- equally charged disks, or oppositely charged ones. Perhaps this form gives rise to the correct symmetry properties in z, but Sned does not give details (eg, only even $P_N\,Q_N$ appear).

So plug A into 8.1.2 to get B where we have one set of coefficients called a_{2n} (2n = even). Define f(t) as lincomb of the a_{2n} on P_{2n} and 8.1.4 results with newly defined G. Love 1949 quickly arrives at a relatively simple Fred 2 for f(t) as in 8.1.6. [I think somewhere along the line Love will take the limit $\eta\to 0$ so the each disk, starting as a spheroid, becomes a disk.] Sned says this integral equation arises when you enforce the BC's on the two disks, but he does not show how, and it is not obvious to me, but I have seen lots of Fred 2's arise in other examples so not a surprise. Solve this Fred 2 for f(t), shoot that into 8.1.4 and you have your answer as an integral.

Sned regards Love's derivation of this integral equation as somewhat tricky involving Legendre sub details, so he will now do an alternate derivation. He writes my usual Smythian integral form with J_0 and sets things up as a PhL 3-region problem. Potentials are called V_+, V_- and V in between the plates. In true Smythian form, these forms pre-match at the two planes. He writes the $\partial_z V$ continuity in each plane outside its disk and this gives p 233C as an integral constraint on A(u). Sned then gives us a dual integral equation pair 8.1.9 for both plates at $+V_0$, but the pair 8.1.11 if plates are at $\pm V_0$. He then combines these into a single dual pair 8.1.13 where upper sign means the equal V_0 on both. What we see is one of our earlier dual study cases -- where we threw in a **weight function** -- Section 4.6. When we apply that work already done, we get the same Love integral equation. So I could track all this if I wanted.

We now turn to solving this integral equation.

8.1.2 Solving Love's Integral Equation

Sned writes the solution as a set of iterated kernels, which was one of Stak's methods. See 8.1.17. Sned then runs off four "lemmas" which are meant to justify this iterated method as being reasonable, and we are then on page 237 at the pencil line. We now look at some numerical work, but we note that

$C_\kappa = (1/\pi) \int_0^1 d\rho\, f(\rho;\kappa)$ // the capacitance of the two disks, κ separation, unit radius

where recall κ is the disk separation (1 is the radius). The table on page 238 gives the beautiful numerical results for various κ values. As κ gets very large, I expect the result to be some kind of reciprocity theorem result related to the capacitance of just a single disk which we know is $2/\pi$. Certainly for the equally charged problem you should get twice the cap of one disk, which would then be $1/\pi$ so the table would in that case go to 1. This seems to be the case for both problems, I am not surprised. The great sphere plays a role in one solution and not the other. Notice his last entry is κ = 20 which is a large separation. As κ gets smaller, the numeric work gets harder, iterated series convergence is slower.

Of course if the plates are very close, you should get C = Area/(4πκ) in cgs, which is $\pi\, 1^2/4\pi\kappa = 1/4\kappa$ in which case inf(t) = $\pi/4\kappa$ as he says in 8.121. Since this is a cgs result, we have r = 1 cm for our disks.

[Note added 1.7.11: consider κ = 0.1 which is pretty close spacing. The parallel plate model says C = $1/4\kappa = 2.5$. The Love thing gives $C = (1/\pi) \int_0^1 d\rho\, f(\rho;\kappa) = (1/\pi) * 9.233$ according to p 238 A = 2.94. This might be reasonable since charge runs to edges and therefore even in this case the edges make significant contributions to the capacitance.]

Sned comments on "desktop computers" of this time. That might be the HP-2115 with 16 KB RAM, 16-bit, 8 MHz. This would have had software floating point, perhaps on the order of a 1 MIP machine, maybe even 0.1 MIP. Alta is maybe 1000 MIPs.

8.1.3 Approximate Solutions

Before computers, this was your only hope, and this subject was reviewed by **Hutson** in 1963. He shows that 8.1.28 is a reasonable approximation I guess as long as $\pi/2\kappa >> 1$ so $\kappa << \pi/2 = 1.5$. This is then an approx when you are getting closer to the parallel plate limit. This was correctly done by **Kirchhoff** in 1877, pretty impressive. Notice how close this approx comes in the table on page 241 last two columns! Even today, I think this is the main approximation one uses within its range.

Sned discusses several other approx methods, one of Cooke (called "**Maxwell's**") which differs from Kirchhoff's by a constant small amount. Then **Noble** did some kind of variational approach. **Reich** somehow related the problem to a 1D random walk and was able to do Monte Carlo numerical work.

Comments: So OK, this is one of the two most basic "classical" problems in electrostatics. At least in 1966 there was no closed form answer, only the Love integral equation with its iterative solution. Probably if there *were* a closed form solution, someone would have found it in the last 150 years! By closed form, I mean an expression for say C which is perhaps an integral or series in special functions. Even multiple integrals and/or multiple sums. [Closed form has other definitions!] But all we have for this problem is an iterative algorithm. In contrast, the single disk has a closed form solution for everything.

You could try some kind of dual oblate coordinates where a surface is somehow two spheroids.

8.2 Disk between two grounded parallel plates [p 247]

This problem doesn't *look* very hard at first blush. He says the general case is doable (disk not in center plane, plates have prescribed potentials instead of V = 0), but he is going to do the simple case of grounded plates and distance f from disk to each plane, and disk has prescribed azisym f(ρ). The BC's are all stated on page 247, no problem. Then he takes 8.2.5 for his Smythian Form. [This came up somewhere else I think in this book, but I have no way to search for it.] If f = z, $V_+ = 0$. And $\partial_z V_+ = -\int du\, A(u) J_0(\rho u)$ which I think you can show has to vanish outside the disk at z = 0 which gives 8.7.2b, meaning no σ in the z = 0 plane outside ρ = 1. You can see in general from 8.2.5 that we have a linear combination of sh(zu) and ch(zu) and is thus composed of cylindrical atoms. Now the first of the dual pair arises from BC 8.2.3 where we get tanh inside the integral. But where is the "1" coming from? Well, he has written th(fu) = 1 - [1-th(fu)] = 1 - k(u), fine.

So pause: we now have a dual pair 8.2.7 which, if we can solve for A(u), solves our problem. This dual pair basically says we have Dirichlet on the disk with prescribed f(ρ), and Neumann=0 outside the disk, all in the z = 0 plane. So we have the weight function form, but this time he shuns that and goes with "the Sned method" which was demonstrated in Section 3.5, just a simplification of the Beltrami method for the charged disk. First up at bat then is the helper function φ(t) in 8.2.9. Recall that this "form" causes the second dual to be automatically satisfied because it is one of those Weber discontinuous integrals. [Again, notice the idea of selecting a multiplicative function inside an integral representation (here for A(u), and the multiplicative function is cos(ut), such that, if φ(t) is smooth and boring, gets a result you want, namely, satisfies the second dual.)] Walking through steps of Section 3.5, Sned ends up quickly with 8.2.10 which is a Fred 2 for φ(t) (the kernel of which is an integral 8.2.12 in which weight k(u) appears). In Section 3.5 he got a closed form solution, but the difference here is that we have that k(u) thing in our first dual, so we get a Fred 2 instead. The inhomo term is h(t) top page 249, the kernel is 8.2.12 which he then shows is symmetric.

Starting below the pencil line on page 249, Sned then **rederives his Fred 2 equation** for φ(t) using a different starting integral rep for V. Instead of doing V in terms of A(u) and then A(u) in terms of φ, here he directly puts V in terms of φ with one of those z+it deals, and Collins gets a credit. But he has to add together three of the z+it forms to make it work (I don't think this is a 3-region problem, we just have three functions as shown). About 20 equations later at the top of p 251 he has rederived that same Fred 2.

Now Sned will **derive the same integral equation a 3rd time** using the Williams method of Section 2.3. I just reviewed those notes and it was pretty messy. So he is off. He utilizes the Green's function G, (he does not do that very often!) , he has a distance R in cylindricals between a point in the z=0 plane (the Green's point charge location) and a general point, he has a simple layer σ = g/ρ on the disk. Then he claims to know the Green's function .

Aside: This is a problem I recently worked on, see Cartesians section "Atoms and Problems...". My result was this (my plates at z = 0 and z = S)

$$u(z,\rho,\varphi) = (4q/S) \Sigma_{nz} \sin(k_z z_>) \sin(k_z z_<) \Sigma_m \varepsilon_m K_m(k_z \rho_>) I_m(k_z \rho_<) \cos(m\varphi)$$

where $k_z = n_z(\pi/S)$ sums are: $\Sigma_{nz=1}^{\infty}$ and $\Sigma_{m=0}^{\infty}$

but of course we know there is also a J J form of this thing. My notes there quote blue Jackson on that

(b) Show that an alternative form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\ e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\ \sinh[k(L - z_>)]}{\sinh(kL)}$$

Jackson's plates are at z = 0 and z = L. I presume if we shift the plate location and write the usual real form instead of complex form, the above Jackson result would become Sned 8.2.23

Now back up a bit. R is our Green's distance. Equation 8.2.21 is just $V(x) = \int dS\ g(x,S)\ \sigma(s)$, where we then integrate over azimuth and this becomes $V(x) = \int \rho' d\rho'\ K_1(\rho,\rho')\ \sigma(\rho') \equiv \int d\rho'\ K_1(\rho,\rho')\ g(\rho')$. So the object K_1 is just Green's integrated over azimuth.

He now integrates his Green's expansion over azimuth to get an expansion for K_1 as in 8.2.24 where now K_0 has the meaning in 8.2.25. He then introduces a new helper function S(t) as the fractional transform of g(ρ') in 8.2.27 (inverted in the next line). Many lines later he arrives at 8.2.30 which is a Fred 2 for S(t) with f(ρ) involved in the driver term. He then claims that S(t) = φ(t)/2 from our previous work, and the same integral equation is thus obtained (many details).

Conclusion: So the simple f(ρ) Dirichlet disk between two grounded planes turns out to be another *non* closed form problem! We end up with a 1D Fred 2 that has to be solved, then we have the solution.

8.3 Disk inside a grounded cylinder [p 253]

We have our lonely radius-1 disk sitting in the cylinder of radius a > 1, he draws a picture (Smythe shuns such pictures). The three BC's are shown at page bottom, all very clear. Now a very contrived Smythian form is proposed in 8.3.4 which consists of two terms. This form is truly strange. Yes, it is made of valid cylindrical atoms. It is a linear combination of the two usual atomic forms, which are these (Curvilinear..)

expo osc osc

(1) $[\ e^{+kz}, e^{-kz}\ ]\ [\ J_m(k\rho), N_m(k\rho)]\ [\ \sin(m\varphi),\cos(m\varphi)]$

osc expo osc

(2) $[\ \sin(\kappa z), \cos(\kappa z)\ ]\ [\ I_m(\kappa\rho), K_m(\kappa\rho)]\ [\ \sin(m\varphi),\cos(m\varphi)]$

We are azisym so m = 0 everywhere. I have never seen such a linear combination used before! Certainly it is a valid ansatz and is OK at ρ = 0 so no N or K functions. By then writing out the BC's, Sned gets three conditions on his two coefficient functions here called A(u) and B(u). The middle one says no σ outside the disk in the z = 0 plane, and applies only to A(u) (derivative kills off the second term, so that is the motivation for this form!). We can elevate A(u) in terms of the helper φ(t) as we just did in the last section (the Sned Method) and automatically satisfy the second BC, so only two remain.

OK, enough of the details. He arrives at a Fred 2 for this φ(t) in 8.3.12 with kernel 8.3.13 and driver function 8.3.10.

One more comment on his Smythian form: although we have V = 0 on the cylinder, the form still has a continuum of k values, it does not use k_n where $J_0(ak_n) = 0$. Somehow the coefficients have to make V = 0 on the cylinder, which condition is 8.3.7.

He then rederives this Fred 2 using the Williams method, as we did in the last section. The helper function is again called S(t).

On page 256 (between two pencil lines) he just comments that Collins has another solution, but he is not reviewing that here.

Then he sort of starts over with a new Smythian form which does involve the λ_n zeros of the J's. This is how I might have started into this problem in the first place, since that makes V = 0 on the cylinder be automatic. So now 8.3.14 is our Smythian form which is a series, not an integral. The problem is now cast into a dual series equations problem, rather than a dual integral equations problem, as stated in 8.3.15. This is something Sned did back in Section 5.2.1. This leads to numerical work with some constants called M_n which are certain Bessel integrals you have to compute. End of that section.

Top page 257: What about the case V = 1 for the disk? Even for this case we have no closed form result. So if smallness parameter is 1/a (cylinder radius large relative to disk radius), we can do the usual iteration method, and this turns out to involve those same M_n constants just mentioned in the last sub section. He writes $\varphi(t) = \Sigma_n\, g_n(t)$ and writes out g_1 through g_6 explicitly, each is a polynomial in t! As before, the capacitance C of our cylinder-contained disk is the integral of $\varphi(t)$ and when we put in these six terms, we get 8.3.19 or alternatively 8.3.20. This is certainly a down-to-earth result!

He lists some of the M_n on page 258. I just had Maple compute these things:

```
> M0 := evalf(int(BesselK(0,w)/BesselI(0,w), w=0..infinity));
                                M0 := 1.367676862
> M2 := evalf(int(w^2*BesselK(0,w)/BesselI(0,w), w=0..infinity));
                                M2 := .6468892585
> M3 := evalf(int(w^4*BesselK(0,w)/BesselI(0,w), w=0..infinity));
                                M3 := 2.069859770
```

Sned in 1966 would have been pretty impressed that I could compute these things to 10 decimal places in about 3 seconds.

Sned wraps up this section by quoting some work by Mathur (unpublished at the time) who also solved the V = 1 integral equation and tables of his results are presented on page 259.

Obviously we know the solution to this problem as $a \to \infty$. For $a = a+\varepsilon$ we could probably also come up with some exact limit, though that is not quite so obvious. A capacitor made from two closely spaced thin ribbons in a circular shape. Maybe this could just be a parallel plate capacitor.

Conclusion: Once again, we have a problem which has no closed-form solution, only a Fred 2 integral equation situation. This is true even of V = 1 on the disk.

8.4 Coplanar Disks [p 259]

Sned regards this as a "difficult problem" and the only way he knows to do it is with the Kobayashi potential method. The picture is page 260 where z comes out, and there are the two disks. As he has done before, he takes for his Smythian form an appropriate form for each disk and adds them together. I commented above how this bypasses the red flag metals theorem since the coefficients in each sub form

are not meant to solve problems 1 and 2. Now for each disk, the form used is that specialized Kobayashi form 7.1.2 which, due to an obscure fact, satisfies the Neumann=0 outside the disk. In other words, the A(u) we usually use has the special form as a sum over J functions with certain weights. In any event, we can just regard 8.4.1 as our "ansatz" and see if it flies. [coordinates are ρ,φ for one disk, $\bar{\rho}$ and $\bar{\varphi}$ for the other]. Notice that this is NOT an azisym problem so we have Σ_m now floating around. And there are now *four* sets of coefficients, the largest number he has dealt with in this book I think. He carries on and on, things a bit messy since m is around. Finally, several pages later on page 264, we end up with four coupled "integral equations" as shown in p 264 A and B.

Now I call them integral equations because I understand that language better. Then these can be thought of as Fred-2 things with inhomo drivers C and D and barred versions of the same. In fact, however, these are coupled Fred 2 "sum" equations, not "integral" equations. I am sure there is some Fred 2 theory that applies to such things, and I imagine we have seen this before in this very book, but Stak did not mention them (well, a mesh approach to a real integral (0,∞) equation looks like one of these things.) There must be some technical term for a "summation" equation. As I have noted earlier, it is just an ∞ dimensional Cramer's Rule problem. Usually in studying integral equations I think of the kernel as a matrix anyway, and here it *really is* a matrix. The vector components are labeled by an infinite range discrete index instead of a real variable as index. I am sure the theory is all the same. Stak did not mention such things, Mikhlin might. I don't have a search handle.

Sned refers us to Kobayashi 1939 if we want to see how to solve these equations. Probably he does some kind of iterative solution where the separation distance is large. Even my canonical pdf does not go after coplanar disks!

8.5 Coplanar charged strips [p 264]

This was addressed in the canonical PDF (which I did browse a bit). The strips are infinite in z, so this is a 2D problem, a first for this book. Strips are metal with potential V = ±1. It is a 2D capacitor between two wire segments lying on the same line. In 2D, an atomic form will have expo in one variable and oscillatory in the other, and that is why we have the Smythian Form 8.5.3 where we do sine in the x direction which is where the strips live. He really means to have $e^{-u|y|}$ I think. We end up with a classic *trig triple integral* equation as in p 265 A: Neumann=0 for x<a , then Dirichlet = 1 on (a,b), then back to Neumann=0 for x > b.

Naturally Sned now refers to one of the few earlier sections of the book I completely skipped. He showed there how he could reduce the triple integral equation situation to a dual series situation. I won't go study that section now, but let's just see what he does here. His ansatz form for A(u) is 8.5.4 where he expands on to some J's with some a_n. Then the "dual series situation" is p 266 A where now the series kernel functions are not J's, but are sines and cosines of (n-1/2). He does an angle inversion to get the 0 to go with the second equation instead of the first.

But earlier in the book (Chap 5) we dealt with dual series equation with trig kernels, and we found a closed form solution, so boom, he quotes the result for a_n in 8.5.5. Our problem is now solved, A(u) is given by 8.5.6. He then computes the σ(x) charge density on the strips, result is 8.5.8 which is rather interesting. We have "the usual" peaking square root behavior along each side of the strip!

So this was a closed-form solution, we were not left with a Fred 2 to go off and solve.

8.6 Charged Annular Disk [p 267]

The first solution here follows the plan for the certain class of triple integral equations Cooke treated back in section 6.3, so it behooves us to review that section now, p 184. The triple set is 6.3.1,2,3 where we have an h(u) weight function. The solution is embedded in the Fred 2 integral equation 6.3.15 which has two added kernels and a driving function F*. This F* is computed from the central region driving function f_2. The kernel term K is a huge but closed-form mess of logs and square roots 6.3.13, while the M term an integral of h(u). Fine, Now in our current application, we have h(u) = 0, no weight function, so the M kernel term is 0. Also, we have $f_2 = 1$ on the central region. So we can directly apply Cooke using just the K kernel term which appears here as 8.6.6 (which is somewhat transformed from 6.3.13 but looks similar). Redefining various functions and variables, the Fred 2 is recast as 8.6.7 and then you see the same kernel recast into angles. [By the way, Maple V does not include a ready-to-go Fred 2 integral equation solver, nor does Maxima. I think that would be a little much to expect, but of course you could implement a selected solution method in either of these programs.] So to solve this problem, even with no weight function and even with a constant V on the annulus, you still have to solve this Fred 2 equation with its very messy kernel. Soon I am going to have to try this myself! I have never attempted such a thing. Sned goes on to write an expression for $\sigma(\rho)$ on the annulus and the total charge Q which is a simple integral of Fred 2 solution H (pencil page bottom). Some data is then presented. The question is this: given V = 1 on the annulus, what is its total charge, which is to say, what is its capacitance? The numerical results are given in a table on page 268 as a ratio over a full disk of radius b. The data is compared to some 1951 Smythe work of another paper and another method, and of course agreement is excellent. Amazingly, even if b/a = 2 (which means you have a pretty large hole), the capacitance is still 98% of that of the full disk! The reason of course is that for a simple disk, we know that "most of the charge" lies near the edges and not much is in the center, so knocking out the center does not do much! You could take a limit of the solution here when $a = b-\varepsilon$ to get the capacitance of a circular flat "wire", Sned does not do that. He has never tried problems which have the surface two dimensions down from the space in some limit. It would be interesting to see $\sigma(\rho)$ plotted versus radius. Is there a quadratic buildup on the inner edge one wonders?

Sned comments that you could do a second solution to this problem using the **Section 7.8** method, but he does not do it.

Sned then dives into a third solution method using the Copson method of **Section 3.4** (but it was Williams who applied that solution here). We are top of page 269. In 8.6.10 we have our Stak equation which says that $\int \sigma/R = 1$ since V = 1 on the iris -- again, this was my starting point for such problems! He then uses that same fascinating integral equality 3.4.6 (but in the simple case n=0) and we get 8.6.11 and this is shuffled into 8.6.12 which is still an integral equation for $\sigma(\rho)$. But then he "elevates" σ onto another function G as shown in 8.6.14, and he ends up with a Fred 2 for G in 8.6.17 with a fairly messy kernel L as shown, but with a simple driving function. Sned then shows that this integral equation is equivalent to the first method we did above.

8.7 Spherical Cap Problems [p 270]

8.7.1 The charged spherical cap (270)

It's interesting that this very last section of Sned's book addresses the subject that got my interested in this whole subject! Back in **Section 5.6** page 174 we in fact solved the bowl problem using a dual series attack on the problem. Go look there: the dual series is 5.6.5, the coefficients a_n are given by 5.6.6 in terms of a function F* which in turn is given by 5.6.4 in the general case of some F(θ) potential on the bowl! This is the entire answer. You can then specialize it to F = 1 on the bowl, and then the a_n come out as shown in 5.6.7. This result is then quoted in 8.7.1.

As a second solution, we review **Section 7.6**. This was a bowl attack based not on a dual series, but on a strange *integral* rep for V which includes sec(θ/2)/R as the "other" function. So what we do here is now the special case that f = F = V = 1 on the cap and we get a simple form then for g(x) which appeared in our integral rep of V and the solution for V is now the very simple form 8.7.3, where we still have that strange complex R thing in p 271 B. Then magically this is rewritten as 8.7.6 where suddenly all integrals are gone and we just have these γ and γ' arcsin things left, a simple form for the potential everywhere! He does not interpret the two distances r_1 and r_2 which appear in this result. This reminds me of a form for V I have seen in the canonical PDF, but I think that form was only at r = 1, but here we have it for all r right there in 8.7.6. Notice that γ and γ' are functions of r and θ. He concludes bye rederiving the expression for total charge on the bowl (which he calls Σ) and this agrees with that we get from the dual series solution.

8.7.2 Grounded spherical cap in uniform E field (272)

In 8.7.7 Sned writes V as a sum of the V of the external field (which is V = -Ez = -Ercosθ) and the V_1 due to the induced charge on the bowl. We get our dual series form for V_1 as shown p 272 B, and V_1 must cancel the external V on the bowl itself, hence 8.7.8. This all fits into the mold quoted above of Section 5.6 where now we have F(θ) = cosθ instead of F(θ) = 1. So this is basically a bowl Dirichlet problem with this prescribed F(θ). We install this F(θ) into our F* integral noted two paragraphs above and we get an incredibly simples result for F* $= \sqrt{2}\cos(3u/2)$. This leads to the a_n quoted in 8.7.10.

As a second solution, we again try the Section 7.6 method. Now our g(x) is 8.7.13 which is still very simple [it was 8.7.2 for the charged bowl] . The end result here is 8.7.14 which is a closed form solution for V where γ and γ' are the same as in the previous section above. This formula was derived earlier by four authors ranging from 1882 to 1945.

DONE!!!!