Sneddon Parallel Treatment of the two dual integral equation J_0 cases PhL 7.25.10

Things are so *slippery* here, that I want to try an *exact parallel development* so I can see *exactly* how the two cases are different. It is like handling mercury, you simply cannot track it.

Treatment of the First Pair

1. Statement of the Pair:

$$\int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) A(\xi) = f(\rho) \qquad \rho < 1 \qquad 3.2.1$$

$$\int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) = 0 \qquad \rho > 1 \qquad 3.2.2$$

2. Definition of the two functions

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) A(\xi) \qquad \text{all } \rho$$

$$G(\rho) \equiv \int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) \qquad \text{all } \rho$$

3. Ansatz for the Form of A(ξ) and its parts form

$$A(\xi) = \xi \int_0^1 dt\, \varphi(t) \cos(\xi t) \qquad 3.5.1$$

$$A(\xi) = \varphi(1)\sin\xi - \int_0^1 dt\, \varphi'(t)\sin(\xi t) \qquad 3.5.2$$

4. Proof that this Form solves the second pair member. This requires showing that G(ρ>1) = 0 if we use the ansatz form. We install the parts form of A(ξ) into G to get:

$$G(\rho) = \int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) = \int_0^\infty d\xi\, J_0(\xi\rho)\{ \varphi(1)\sin\xi - \int_0^1 dt\, \varphi'(t)\sin(\xi t)\}$$

$$= \varphi(1) \int_0^\infty d\xi\, J_0(\xi\rho) \sin\xi - \int_0^1 dt\, \varphi'(t) \int_0^\infty d\xi\, J_0(\xi\rho) \sin(\xi t) \qquad \text{p 75 A}$$

The discontinuous integral of interest here is this

$$\int_0^\infty dx\, J_0(\rho x) \sin(ax) = 0 \qquad a < \rho \qquad \text{// which are 2.1.14}$$

$$= 1/\sqrt{a^2-\rho^2} \qquad a > \rho$$

We have ρ > 1 for first integral, and we have ρ > 1 > t for second integral, so both are 0. QED.

5. Now, install the non-parts form of A(ξ) into the first of the dual pair F(ρ), for which ρ < 1

$$\int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) A(\xi) = \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) [\xi \int_0^1 dt\, \varphi(t) \cos(\xi t)]$$

$$= \int_0^1 dt\, \varphi(t) \int_0^\infty d\xi\, J_0(\xi\rho) \cos(\xi t) \qquad (*)$$

$$= [\int_0^\rho dt\, \varphi(t) + \int_\rho^1 dt\, \varphi(t)] \int_0^\infty d\xi\, J_0(\xi\rho) \cos(\xi t) \qquad \text{// since } \rho < 1, \text{ can do this}$$

Now we need the other discontinuous integral which is this:

$$\int_0^\infty dx\, J_0(\rho x) \cos(ax) = 1/\sqrt{\rho^2 - a^2} \qquad a < \rho \qquad \text{// which are 2.1.13}$$

$$= 0 \qquad a > \rho$$

In the second integral, $\rho < t$ so that term vanishes. In the first integral, $t > \rho$ and we are left with

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) A(\xi) = \int_0^\rho dt\, \varphi(t) / \sqrt{\rho^2 - t^2} \qquad \rho < 1 \qquad 3.5.6$$

In the other case that $\rho > 1$, we have $\rho > t$ in (*) and we don't split into two integrals and we have

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) A(\xi) = \int_0^1 dt\, \varphi(t) / \sqrt{\rho^2 - t^2} \qquad \rho > 1 \qquad 3.5.7$$

6. Use the Abel inversion formula. First, quote the inversion theorem of interest from Chap 2

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^{1/2} = g(x)$ interval for x,t is [a,b]

$$\Rightarrow \quad f(t) = (2/\pi)\partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1/2} \}$$

and now make edits for our current situation

S1: $\int_0^\rho dt\, \varphi(t) / \sqrt{\rho^2 - t^2} = f(\rho)$ interval for x,t is [a,b]

$$\Rightarrow \quad \varphi(t) = (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2 - \rho^2} \}$$

Applying this to 3.5.6 we get this inversion

$$\varphi(t) = (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2 - \rho^2} \} \qquad 3.5.9$$

We install this into the non-parts Form for A(ξ) and get our result

$$A(\xi) = \xi \int_0^1 dt\, \varphi(t) \cos(\xi t) = \xi \int_0^1 dt\, [(2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}] \cos(\xi t)$$

$$= (2/\pi)\, \xi \int_0^1 dt \cos(\xi t)\, \partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}$$

and this agrees with the claim of Polyanin.

Now I am going to copy and paste the entire section above, and just edit in changes as needed.

Treatment of the Second Pair

1. Statement of the Pair:

$$\int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho) A(\xi) = f(\rho) \qquad \rho < 1 \qquad 3.5.15$$

$$\int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) = 0 \qquad \rho > 1 \qquad 3.5.16$$

2. Definition of the two functions

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho) A(\xi) \qquad \text{all } \rho$$

$$G(\rho) \equiv \int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) \qquad \text{all } \rho$$

3. Ansatz for the Form of A(ξ) and its parts form

$$A(\xi) = 1 \int_0^1 dt\, \varphi(t) \sin(\xi t) \qquad \varphi(0) = 0 \qquad 3.5.17$$

$$A(\xi) = \xi^{-1}\{- \varphi(1)\cos\xi + \int_0^1 dt\, \varphi'(t) \cos(\xi t)\} \qquad \text{p 77 A; we used } \varphi(0) = 0 \text{ !}$$

4. Proof that this Form solves the second pair member. This requires showing that G(ρ>1) = 0 if we use the ansatz form. We install *the parts form* of A(ξ) into G to get:

$$G(\rho) = \int_0^\infty d\xi\, J_0(\xi\rho) A(\xi) = \int_0^\infty d\xi\, J_0(\xi\rho)\, \xi^{-1}\{- \varphi(1)\cos\xi + \int_0^1 dt\, \varphi'(t) \cos(\xi t)\}$$

$$= - \varphi(1) \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) \cos\xi - \int_0^1 dt\, \varphi'(t) \int_0^\infty d\xi\, \xi^{-1} J_0(\xi\rho) \cos(\xi t)$$

But this does not work now because the two integrals diverge at the *low* end! So we have to change the thread right at this point. So let's try instead installing the *non-parts form* of A(ξ) into G:

$$G(\rho) = \int_0^\infty d\xi\, J_0(\xi\rho)A(\xi) = \int_0^\infty d\xi\, J_0(\xi\rho)\, 1 \int_0^1 dt\, \varphi(t)\, \sin(\xi t)$$

$$= \int_0^1 dt\, \varphi(t) \int_0^\infty d\xi\, J_0(\xi\rho)\, \sin(\xi t)$$

Comments: (1) since we used the non-parts form, we have the sin trig function here, which is then the same as we had for the first dual pair; (2) since we used the non-parts form, there is only one of these J_0 sin integrals, not two as we had before; (3) since we used the non-parts form, we have φ(t) appearing instead of φ'(t) as last time !!

The discontinuous integral of interest here is this (same as for first pair)

$$\int_0^\infty dx\, J_0(\rho x)\, \sin(ax) = 0 \qquad a < \rho \qquad \text{// which are 2.1.14}$$

$$= 1/\sqrt{a^2-\rho^2} \qquad a > \rho$$

Since our second dual equation is for ρ > 1, we have ρ > 1 > t for our integral above, so the integral is 0. Thus, we find the same fact we found last time, which is that G(ρ>1) = 0. I think this is WHY we used a sin in the ansatz form instead of a cos for this case!

5. Now, install the *non-parts form* of A(ξ) into the first of the dual pair F(ρ), for which ρ < 1

$$\int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)A(\xi) = \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)\, [1 \int_0^1 dt\, \varphi(t)\, \sin(\xi t) \quad]$$

$$= \int_0^1 dt\, \varphi(t) \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)\, \cos(\xi t) \qquad (*)$$

But again we have a problem: this integral now diverges at the *high* end. So we make our second change in the thread at this point, and we instead install *the parts form* into the first of the dual pair:

$$\int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)A(\xi) = \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)[\ \xi^{-1}\{-\varphi(1)\cos\xi + \int_0^1 dt\, \varphi'(t)\, \cos(\xi t)\}\ ] \qquad \text{p 77A}$$

$$= \int_0^\infty d\xi\, J_0(\xi\rho)\ \{-\varphi(1)\cos\xi + \int_0^1 dt\, \varphi'(t)\, \cos(\xi t)\}$$

$$= -\varphi(1) \int_0^\infty d\xi\, J_0(\xi\rho)\, \cos\xi + \int_0^1 dt\, \varphi'(t) \int_0^\infty d\xi\, J_0(\xi\rho)\, \cos(\xi t) \qquad \text{p 77 B}$$

Now we need the other discontinuous integral which is this: (same as for first pair)

$$\int_0^\infty dx\, J_0(\rho x)\cos(ax) = 1/\sqrt{\rho^2-a^2} \qquad a<\rho \qquad \text{// which are 2.1.13}$$

$$= 0 \qquad a>\rho$$

We are doing $\rho < 1$ now for first of the dual pair, so the first integral vanishes and we then have

$$\int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)A(\xi) = \int_0^1 dt\, \varphi'(t) \int_0^\infty d\xi\, J_0(\xi\rho)\cos(\xi t)$$

$$= \left[\int_0^\rho dt\, \varphi'(t) + \int_\rho^1 dt\, \varphi'(t) \right] \int_0^\infty d\xi\, J_0(\xi\rho)\cos(\xi t) \qquad \text{// since } \rho<1\text{, can do this}$$

This is exactly the situation we had for the first pair, except here we have $\varphi'(t)$ instead of $\varphi(t)$, so the same conclusions will occur: In the second integral, $\rho < t$ so that term vanishes. In the first integral, $t > \rho$ and we are left with

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)A(\xi) = \int_0^\rho dt\, \varphi'(t)/\sqrt{\rho^2-t^2} \qquad \rho<1 \qquad \text{p 77 C}$$

In the other case that $\rho > 1$, we have $\rho > t$ in (*) and we don't split into two integrals and we have

$$F(\rho) \equiv \int_0^\infty d\xi\, \xi^{+1} J_0(\xi\rho)A(\xi) = \int_0^1 dt\, \varphi'(t)/\sqrt{\rho^2-t^2} \qquad \rho>1 \qquad \text{not shown}$$

6. Use the Abel inversion formula. First, quote the inversion theorem of interest from Chap 2

S1: $\int_a^x dt\, f(t) / [x^2 - t^2]^{1/2} = g(x)$ interval for x,t is [a,b]

=> $f(t) = (2/\pi)\partial_t \{ \int_a^t du\, u\, g(u) / [t^2 - u^2]^{1/2} \}$

and now make edits for our current situation

S1: $\int_0^\rho dt\, \varphi'(t) / \sqrt{\rho^2-t^2} = f(\rho)$ interval for x,t is [a,b]

=> $\varphi'(t) = (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}$

Applying this to p77C above and we get this inversion

$$\varphi'(t) = (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \} \qquad \text{p 77 D}$$

We install this into the non-parts Form for $A(\xi)$ and get our result -- but no can do! The non-parts form does not have $\varphi'(t)$ in it. So instead let's try installing this into the parts form:

$$A(\xi) = \xi^{-1}\{- \varphi(1)\cos\xi + \int_0^1 dt\, \varphi'(t)\, \cos(\xi t)\}$$

$$= - \varphi(1)\, \xi^{-1}\, \cos\xi + \xi^{-1} \int_0^1 dt\, \cos(\xi t)\, \varphi'(t)$$

$$= - \varphi(1)\, \xi^{-1}\, \cos\xi + \xi^{-1} \int_0^1 dt\, \cos(\xi t)\, \{ (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}\}$$

But this is not a form anybody claims, and it has that $\varphi(1)$, a non-starter.

So instead, let's go back to

$$\varphi'(t) = (2/\pi)\partial_t \{ \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}$$ p 77 D

and integrate this from 0 to t. The LHS gives just $\varphi(t)$ since $\varphi(0) = 0$. The RHS is the RHS without the ∂_t,

$$\varphi(t) = (2/\pi) \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2}$$ 3.5.18

Now install this into the non-parts Form to get

$$A(\xi) = 1 \int_0^1 dt\, \varphi(t)\, \sin(\xi t) = \int_0^1 dt\, \sin(\xi t)\, \{(2/\pi) \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2} \}$$

$$= (2/\pi) \int_0^1 dt\, \sin(\xi t) \int_0^t d\rho\, \rho\, f(\rho) / \sqrt{t^2-\rho^2}$$ 3.5.19

and this agrees with Sneddon but disagrees with the claim of Polyanin! I now think Polyanin has an error here! Amazing, since this is one of the simples canonical cases!