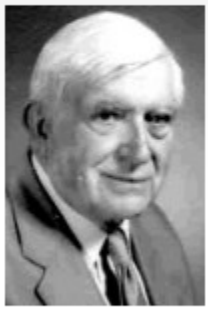
Sneddon entire book review PhL 1.7.11

Mixed BV Problems in Potential Theory, Ian Sneddon, 1966 (I call him "Sned")

FCAA is a journal: Fractional Calculus and Applied Analysis. Ian almost reached his 81st birthday.

Ian N. Sneddon † (December 12,1919 – November 4, 2000)



Name: **Ian N. Sneddon † (December 12,1919 – November 4, 2000)**

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Was a Founding Editor of the journal FCAA, since 1998

Fields of interest:

fractional calculus

Integral transforms, special functions

- integral and differential equations, potential theory

wave mechanics, theoretical physics

Books and Surveys in FCAA area:

I.N. Sneddon:
Fourier Transforms.
McGraw-Hill, New York, 1951.

I.N. Sneddon:
Special Functions of Mathematical Physics and Chemistry.
Wiley-Interscience, New York, 1956.

I.N. Sneddon:
Elements of Partial Differential Equations.
McGraw Hill 1957, Dover 2006.

I.N. Sneddon:
Mixed Boundary Value Problem in Potential Theory.
North-Holand Publ. Co., Amsterdam, 1966.

I.N. Sneddon:
The Use of Integral Transforms.
McGraw-Hill, New York, 1972.

I.N. Sneddon:
The use in mathematical analysis of EK operators and some of their applications.
In: *Fractional Calculus and Its Applications* (Proc. Int. Conf. New Haven, 1974), L.N.M. 457, 1975, 37-79.

I.N. Sneddon:
The Use of Operators of Fractional Integration in Applied Mathematics.
RWN - Polish Sci. Publishers, Warszawa-Poznan, 1979.

The book I have is the 4th one down the above list: **Mixed BV Problems in Potential Theory.** I have a complete binder hard copy of the book which I checked out from Marriott Library.

In **Chapter 1** Sneddon presents some problems which motivate the study of dual integral and dual series equations. A huge area of potential theory is the theory of crack formation in solids, and the deformation of solids under application of a punch, something I know nothing about. I am mainly interested in the electrostatics problems of course.

Chapter 2 provides most of the required mathematical results used in the rest of the book. It is simply amazing how much math information is packed into this 36 page chapter, most of which I never saw before.

(1) **integrals** involving **Bessel functions**, including different forms of the **Hankel transform**, one called "the modified Hankel transform" S which has this form

$$S_{\eta,\alpha}\{ f(r); \mu \} = 2^{\alpha} \mu^{-\alpha} \int_0^{\infty} dx\, x^{1-\alpha} J_{2\eta+\alpha}(\mu x) f(x)$$

The famous Weber 1873 discontinuous integrals are derived as well as generalizations thereof.

(2) **sums** involving **Bessel functions** (this stuff is needed for dual *series* work later)

(3) **transforms** treated as integral equations: The **Srivastav** and **generalized Abel** transforms.

(4) introduction of the **EK integral operators I and K** which are really just fancied-up generalized Abel transforms and are thus triangular matrices. I think I make these two points better than he does. The whole section on **fractional integral operators** just shows that you can *interpret* the generalized Abel transforms for general power α as continuations of multiple integrations off the integers. Sned is a "fractional calculus" fan and wanted to work this in to his discussion. So here is what I and K look like:

$$I_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{-2\alpha-2\eta} \int_0^x dt\, t^{2\eta+1} F(t) [x^2 - t^2]^{\alpha-1}$$

$$K_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{2\eta} \int_x^\infty dt\, t^{-2\eta-2\alpha+1} F(t) [t^2 - x^2]^{\alpha-1}$$

(5) Sned then derives a set of **relationships** between the **I, K and S** operators. These are so important that he lists them in his sole **appendix**, which I have placed right after the binder title page, enlarged. All of the "magic" is really contained in this page of relations.

(6) Finally, Sned gives various properties of the **Jacobi polynomials** in both their regular $P_n^{(\alpha,\beta)}(\cos\theta)$ form and also in their "shifted form" $\mathcal{F}_n(\alpha,\beta;x)$ used by certain sources. When $\alpha = \beta = m$, the $P_n^{(\alpha,\beta)}(z)$ are proportional to the associated Legendre $P_n^m(z)$ as shown on p 56, which has ramifications for problems like the charged bowl.

In **Chapter 3** Sned discusses **the classical disk Dirichlet problem** (radius 1). When you use a cylindrical Smythian form for the potential, you get these dual integral equations (he uses ξ where I usually use k)

$\int_0^\infty dk\, k^{-1} A(k) J_0(k\rho) = f(\rho,\theta)$	$\rho<1$	potential specified for $\rho < 1$
$\int_0^\infty dk\, A(k) J_0(k\rho) = 0$	$\rho>1$	charge $\sigma \sim \partial_z$(potential) = 0 for $\rho > 1$

The simple charged disk problem has f = 1, and can be solved by just inspecting the Weber discontinuous integrals (1873). The case f = f(ρ) I call the Beltrami problem 1881 and this gets solved in closed form using the Abel transform and parts integration (easier to follow Sned's own version of this). In 1872 Kelvin published his oblates solution to the charged disk problem (but he solved it in 1847). Then 70 years goes by from Beltrami until Copson in 1947 solves the general case f = f(ρ,θ) using two sequential Abel transforms and some tricky integral representation work, and the solution can be found in Polyanin. Sned also does a certain "second problem" which has ξ^{+1} in place of ξ^{-1} in the first equation, and this relates to crack theory and this is stated I think incorrectly in Polyanin (got no response to my email).

In **Chapter 4** (54 pages), Sned looks more generally at **dual integral equations** of two different forms. In both cases, the kernel of the integral equation has the form $k^a\, g(xk)$ and so is a function of the product of the two variables times a power. Certainly this is a very special case of a kernel: $\int dk\, k^a g(kx)A(k)=F(x)$.

(1, Bessel) First, Sned considers this **Bessel form** which encompasses our disk problem of the previous chapter,

$$\int_0^\infty dk\, k^{-2\alpha}\, A(k)\, J_\mu(xk) = F(x) \qquad x \text{ in } I_1 \qquad \text{// Titchmarsh type duals}$$

$$\int_0^\infty dk\, k^{-2\beta}\, A(k)\, J_\nu(xk) = G(x) \qquad x \text{ in } I_2 \qquad (0,\infty) = I_1 \cup I_2$$

A special case of these dual integral equations was solved by Titchmarsh in 1948 and that names sticks. Sned shows that after you redefine the functions a bit,

$$A(k) = k\,\Psi(k) \qquad f_1(x) = 2^{2\alpha}\, x^{-2\alpha}\, F(x) \qquad g_2(x) = 2^{2\beta}\, x^{-2\beta}\, G(x)$$

these dual equations become

$$2^{2\alpha}\, x^{-2\alpha} \int_0^\infty dk\, k^{-2\alpha+1}\, \psi(k)\, J_\mu(xk) = f_1(x) \qquad x < 1$$

$$2^{2\beta}\, x^{-2\beta} \int_0^\infty dk\, k^{-2\beta+1}\, \psi(k)\, J_\nu(xk) = g_2(x) \qquad x > 1$$

which are just a pair of modified Hankel transforms, but having different parameters

$$S_1\Psi = f_1 \qquad x \text{ in } I_1 \qquad S_1 = \mathcal{S}_{\mu/2-\alpha,2\alpha}$$
$$S_2\Psi = g_2 \qquad x \text{ in } I_2 \qquad S_2 = \mathcal{S}_{\nu/2-\beta,2\beta}$$

He shows how you can find EK operators I and K such that $IS_1 = KS_2 = S$ so you end up with

$$S\,\Psi = I\,f \qquad I = I_{\mu/2+\alpha,\lambda-\mu} \qquad \lambda = (\mu+\nu)/2 + \beta - \alpha$$
$$S\,\Psi = Kg \qquad K = K_{\lambda-\nu/2-\beta,\nu-\lambda} \qquad S = S_{\mu/2-\alpha,\lambda-\mu+2\alpha}$$

and then you solve for Ψ by inverting either equation using an inverse modified Hankel. Once you find Ψ, you can get the partner functions from $S_1\Psi = f_1$ and $S_2\Psi = g_2$. This general approach I call the **Peters Solution** 1961. Sned goes on to do the Titchmarsh special case which has $\beta = 0$ and $\mu = \nu$ and is on interval (0,1). He then goes through some related but alternative solution methods: the **Noble** Solution of 1958, and the **Gordon-Copson** solution of 1954 and 1961. He then specializes to J_0 which is the azisym case and states a lot of detailed results.

(2, Trig) Second, Sned considers a certain **trig form** of dual integral equations, for example,

$$\int_0^\infty dk\, k^{-1} A(k) \cos(k\rho) = \varphi_1(\rho) = F(x) \qquad \rho < 1$$

$$\int_0^\infty dk \quad A(k) \cos(k\rho) = \chi_2(\rho) = 0 \qquad \rho > 1$$

Sned assumes a series solution form which automatically satisfies the second equation and he gets a solution which is a bit ugly, but at least it is a solution. He does four cases here, k^{-1} and k^{+1} each with cos and sin. These equations seem harder to solve than having the Bessel J functions as kernels, and you can see that he has limited his interest to one simple power situation.

Sned then considers adding an arbitrary **weight function** to the first equation of the Bessel type dual integral equations and of course this forces you into the Fred 2 situation. All through the book this is the last resort. If you cannot find a closed form solution, at least you want to cast things in terms of a Fred 2 single variable integral equation which you can solve numerically. (p 106)

After all this, Sned backs off and considers "**the general problem**" of dual integral equations where you have two arbitrary kernels L_1 and L_2, not just the simple Bessel and trig forms considered so far. You are lucky to get to a Fred 2 situation in the general case. I suspect there are many pairs of duals that nobody knows how to even cast into a numerical Fred 2 problem.

His last gasp is then "**systems" of** n pairs of dual integral equations with n potentials Ψ_i. Recall that multiple potentials are used in crack problems and in certain obscure applications. In each dual equation pair, the second equation has no factors other than $J_{\mu i}$, but the first has $c_{ij}(x)$, see p 129.

In **Chapter 5** (44 pages) , Sned looks more generally at **dual integral *series*** of several forms:

(1,Bessel) First, Sned considers this dual series of the **Bessel form**, where $J_\nu(k_n a) = 0$,

$$\Sigma_{n=1}^\infty k_n^{-2p} a_n J_\nu(\rho k_n) = f_1(\rho) \qquad f_1 = 0 \text{ for "problem (b) "} \qquad I_1 = (0,1)$$

$$\Sigma_{n=1}^\infty a_n J_\nu(\rho k_n) = g_2(\rho) \qquad g_2 = 0 \text{ for "problem (a) "} \qquad I_2 = (1,a)$$

$$2^{2\alpha} x^{-2\alpha} \int_0^\infty dk\, k^{-2\alpha+1} \psi(k) J_\mu(xk) = f_1(x) \qquad x < 1$$

$$2^{2\beta} x^{-2\beta} \int_0^\infty dk\, k^{-2\beta+1} \psi(k) J_\nu(xk) = g_2(x) \qquad x > 1$$

where I have shown the Peters dual integral pair below for comparison. As usual, series are "harder to deal with" so he considers only a simple power case as shown. In the analogy we have $\psi(k_n) = a_n$. Sned discusses two methods of solving the dual series equations above, and both are very ugly. The first method (Cooke and Tranter 1959) assumes a series form for a_n whose coefficients b_m must be determined from an ∞ x ∞ Cramer's problem $\Sigma_{m=0}^\infty b_m B_m = E$. The second method (Sneddon and Srivastav 1964) instead uses cascaded integral representations for a_n which leads to a Fred 2 problem. So basically there is no closed analytic form for even the simple Bessel series case shown above.

Since the k_n come from $J_\nu(k_n a) = 0$, the above form fits into the Fourier-Bessel universe which we can regard perhaps as going to the dual integral equations in a Hankel limit.

(1, Bessel Dini) Sned then considers a different definition of k_n which is $(k_n a)J_\nu'(k_n a) + H\, J_\nu(k_n a) = 0$. This corresponds to what we always called the "inhomo boundary condition" in 1D Stakgold work, but instead of being at the endpoints of (a,b), this is on a cylindrical boundary in cylindrical coordinates. This is a locally mixed boundary condition that Sned calls the "radiation condition" for an obscure reason. With this mixed boundary condition, the dual series shown above *look* the same, but are called **Dini Series** since they have a different k_n spectrum. The transform instead of being the Fourier-Bessel then becomes the **Hankel-Schwartz transform**. Needless to say, finding a solution is at least as complicated as in the non-Dini case, and Sned shows how you can grind things down to a Fred 2 equation.

(2, Trig) Second, Sned considers the **trig form** of a dual series where $\sin(k_n a) = 0$ and $k_n = n\,\pi/a$. But to simplify, he sets $a = \pi$, so then $k_n = n$ and our BC is then $\sin(n\pi) = 0$, so our spectrum is then $k_n = n = 1,2,3...$ (and -2p of the above Bessel series form is now replaced by just +p) (this is 5.4.1)

$$\Sigma_{n=1}^{\infty}\, n^{+p}\, a_n \sin(nx) = f_1(x) \qquad f_1 = 0 \text{ for "problem (b) "} \qquad I_1 = (0,d)$$
$$\Sigma_{n=1}^{\infty}\, a_n \sin(nx) = g_2(x) \qquad g_2 = 0 \text{ for "problem (a) "} \qquad I_2 = (d,a)$$

$$\int_0^{\infty} dk\, k^{-1} A(k) \sin(k\rho) = \varphi_1(\rho) = F(x) \qquad \rho < 1$$
$$\int_0^{\infty} dk\, A(k) \sin(k\rho) = \chi_2(\rho) = 0 \qquad \rho > 1$$

and again on the last two lines I show the corresponding dual integral trig equations, and now $a_n = A(k_n)$. Trig was harder than Bessel for dual *integral* equations, but somehow in the dual *series* equations the trig is "easy", at least for the simple powers $p = \pm 1$. Sned provides closed form cascaded integral style solutions for $p = \pm 1$ and problems a and b.

The corresponding cosine dual series requires $\cos(k_n a) = 0$ and in this case the spectrum is $k_n = (n\text{-}1/2)$ with $n = 1,2...$ (so $k_n \geq 0$). So in this case $\cos(k_n x) = \cos[(n\text{-}1/2)x]$ and the dual series becomes (this is 5.4.2)

$$\Sigma_{n=1}^{\infty}\, (n\text{-}1/2)^{+p}\, a_n \cos[(n\text{-}1/2)x] = f_1(x) \qquad f_1 = 0 \text{ for "problem (b) "} \qquad I_1 = (0,d)$$
$$\Sigma_{n=1}^{\infty}\, a_n \cos[(n\text{-}1/2)x] = g_2(x) \qquad g_2 = 0 \text{ for "problem (a) "} \qquad I_2 = (d,a)$$

and Sned gives a solution to this dual series problem as well (problem a and problem b).

(2, Trig Dini). Sned then considers the Dini Series situation where now the boundary condition is still $(k_n a)J_\nu'(k_n a) + H\, J_\nu(k_n a) = 0$. There are two trig cases of interest here, based on the fact that

$$J_{1/2}(x) = \sqrt{2/\pi}\, \sin(x)/\sqrt{x} \qquad \text{and} \qquad J_{-1/2}(x) = \sqrt{2/\pi}\, \cos(x)/\sqrt{x}$$

As I show in my notes, $\nu = 1/2$ and $\nu = -1/2$ combined with $H = 1/2$ lead to these two pairs of dual series

$$\Sigma_{n=1}^{\infty} (n-1/2)^p a_n \sin(x(n-1/2)) = F(x) \qquad \text{// Sned 5.4.3}$$
$$\Sigma_{n=1}^{\infty} \quad a_n \sin(x(n-1/2)) = G(x)$$

$$(\alpha/2)a_0 + \Sigma_{n=1}^{\infty} n^p a_n \cos(nx) = F(x) \qquad \text{// Sned 5.4.4}$$
$$\Sigma_{n=1}^{\infty} \quad a_n \cos(nx) = G(x)$$

where there is a special issue with the a_0 coefficient in the second case where a divide by 0 situation I think allows a free parameter ($\alpha/2$) to exist here. Sned gives solutions to both these "Dini" trig series for the two problems and for $p = \pm 1$.

In Section 5.5 Sned treats dual series equations with full **Jacobi Polynomials** (K an K' are gamma functions, see p 166 which shows two versions of this dual series pair)

$$\Sigma_{n=0}^{\infty} K_n(\alpha,\beta) A_n P_n^{(\alpha,\beta)}(\cos\theta) = F(\theta) \qquad \theta \text{ in } (0,\varphi)$$
$$\Sigma_{n=0}^{\infty} K'_n(\alpha,\beta) A_n P_n^{(\alpha,\beta)}(\cos\theta) = G(\theta) \qquad \theta \text{ in } (\varphi,\pi)$$

Srivastav 1964c was able to obtain a closed-form solution even to this general case, as Sned shows in great detail. That is pretty amazing.,

Finally, in Section 5.6 Sned specializes the above Jacobi thing to the case $\alpha = \beta = m = 0$ and this yields a closed form solution to the **charged bowl problem**! The closed form solution is more general that that, allowing for any Dirichlet $F(\theta)$ on the bowl. The famous $(2n+1)$ factor arises somehow from those K and K' factors.

In **Chapter 6** (44 pages), Sned opens with two triple relation examples. The first is an annular temperature reservoir applied to a half-space "plate" which gives a triple *integral* equation set of this form

$$\int_0^{\infty} dk \quad A(k) J_\nu(\rho k) = E(\rho) \qquad x \le a \qquad\qquad E = 0$$

$$\int_0^{\infty} dk\, k^{-2\alpha} A(k) J_\nu(\rho k)) = F(\rho) \qquad a \le x \le b \qquad\qquad F = f(\rho)\ ,\ \alpha = 1/2$$

$$\int_0^{\infty} dk \quad A(k) J_\nu(\rho k) = G(\rho) \qquad x \ge b \qquad\qquad G = 0$$

where I show the general case Sned is going to treat, and on the right is the special case of the heat example. The second example is a triple Legendre series which I will describe below. Cooke in 1963 shows how (Section 6.2) you can take this "triple Titchmarsh form" and think of it as two pairs of dual integral equations, and this leads to a pair of cross-coupled Fred 2 equations which can then be decoupled into just a pair of Fred 2's.

In Section 6.3 Sned then sets $\nu = 0$ for the azisym case, and also $\alpha = 1/2$ and $E = G = 0$, but also throws in a weight function h. With or without this weight function, you end up with a Fred 2 (6.3.15 p 187)! This suggests to me then that the Beltrami annular ring problem has no closed form solution! (we shall see in Chapter 8). In this section Sned encounters my "Q object" angular integral and its two alternate integral forms, but only for $\nu = 0$, and I think this is when I got all that fully derived and verified.

In Section 6.4 Sned reviews 1960 work of Tranter on the above triple integral set with G = 0. By assuming a series form for the potential with coefficients a_n, Tranter is able to write a dual series for a_n so he has in effect converted a triple integral problem into a dual series problem 6.4.7. But the series contain fully general Jacobi polynomials and no solution is known in general. But he then specializes this thing to the case $\nu = 1/2$ and $\alpha = 1/2$ and the Jacobi's simplify to trig, and you end up with one of our four "trig dual series" treated in Chapter 5, which has a closed form solution. I think he gets solutions for all four cases where $\alpha = \pm 1/2$ and $\nu = \pm 1/2$. This is fine, but I don't know any problems that result in a triple integral set with $\nu = \pm 1/2$. This is the end of Sned's comments on the triple integral equations.

He then turns his attention to a certain triple *series* equation set:

	first problem	second problem
$\Sigma_{n=0}^{\infty}\ (2n+1)\ C_n P_n(\cos\theta) = E(\theta)$	0	$f(\theta)$
$\Sigma_{n=0}^{\infty}\ (1 + H_n)\ C_n P_n(\cos\theta) = F(\theta)$	$f(\theta)$	0
$\Sigma_{n=0}^{\infty}\ (2n+1)\ C_n P_n(\cos\theta) = G(\theta)$	0	$f(\theta)$

where H_n is a weight function. If $H_n = 0$, the first problem would be a Beltrami barrel problem which is in fact his opening section example ($\sigma = 0$ in both off-barrel regions), while the second problem is two facing spherical bowls. For the first problem, he ends up with four Fred 2's to solve! If $H_n = 0$ this reduces to two Fred 2's. For the second problem he ends up with two Fred 2's, and if $H_n = 0$ this becomes one Fred 2. So this suggests that maybe the barrel problem is like the annulus problem and has no closed form solution! That would be consistent with the fact that they are related by inversion.

Chapter 7 is titled "integral representation methods". Really this chapter should be entitled "a **grab bag** of potential-theory problem-solving methods not included in earlier chapters".

First up (Section 7.1) in this grab bag is **the Kobayashi potential** method of 1931. The potential is expressed as the usual cylindrical Smythian form, but the coefficients are expanded as a sum of Bessel J functions $a_m(k) = \Sigma_\mu\ A_m^\mu\ k^{-\lambda} J_\mu(k)$ where you get to select whatever summation set of μ you want. Sned's motivation for mentioning this method is that it is one of only two known methods (1966) of handling the potential problem of two coplanar disks, which he will address in Chapter 8. By picking a certain set $\{\mu\}$ of discrete values, Sned proceeds to solve the Copson 1947 disk Dirichlet problem for $f(r,\theta)$. The solution A_m^μ are as shown in 7.1.6 p 200 (but our old shifted Jacobi $\mathcal{F}$ friend is involved). This result is a little messy since it requires integrating $f(r,\theta)$ against $\mathcal{F}$, but at least it is a closed form result. He does not compare this to the Copson result, but he does then consider the azisym m=0 case and for f = 1 he obtains the classic Weber result for the charged metal disk.

Next up (Section 7.2) is what Sned calls **Dovnorovich's Solution**, a 1957 thesis about punches. As best I can tell, this solution is exactly Stakgold's "integral equation method" applied to a flat metal surface holding some σ. I recall how Stak shows that for a *flat* surface, you automatically get $\sigma = 0$ outside your surface when you use the $\int dA\ \sigma / R$ form for your potential, a fact appearing as p 202 A here (z=0!).

The rest of this section proposes a method that led me into many days of pain. You are supposed to consider $Ku = \mu u$ as an integral eigenvalue problem with kernel $K = 1/R$ in 2D. Once you get the

eigenfunctions of K, it is then trivial to solve a Dirichlet problem because K is then diagonal and everything diagonalizes and the solution is simple. BUT, I could not see any way to solve the integral EF problem even for a simple disk as the flat surface, nor could I even determine what the boundary conditions should be. See details in folder " A 2D integral EV problem". Sned does quote some other sources here: Zaremba 1946, Boussinesq 1885, and a 1957 Mikhlin book.

Section 7.3 **Galin's theorem** is an rather obscure theorem pertaining to a flat elliptical surface. If you happen to know that your potential is a polynomial in x and y on this surface, then σ on the surface is given by 7.3.3 which is another poly of x,y divided by a square root. An example is an ellipse with a constant potential (which is obviously a valid poly in x,y !) and you get just the square root part which is in fact the known result from ellipsoidal coordinate analysis. Sned proves Galin's theorem using Lamé functions, but I did not read the proof.

In Section 7.4 ("A Simple Superposition Method") Sned reviews work of **Szegedin 1957** who showed that you can construct the **Beltrami f(ρ)** disk Dirichlet problem solution by doing my Red Flag **superposition** of simple Weber disk problems! The guy superposed concentric disks with a weight function K(a), and then later Sned showed that for a given f(ρ) you can invert to get K(a).

Section 7.5 is yet another treatment of the **Beltrami** f(ρ) disk, this time with the **z+it type potential** promoted by Albert Edward **Green** (1912-1999).

In Section 7.6 (Some Bowl Problems) Sned reviews some Collins 1959 work which uses an exceedingly strange integral representation for the potential V see 7.6.1. It involves a **complex R** distance like thing with a sec(x/2) factor stuck in the integrand. Collins is able to do the **azisym Dirichlet bowl** problem with an f(θ) prescribed potential, using in the end a simple Abel inversion.

Section 7.7 extends the "complex R" model of Section 7.6 to treat **two collinear bowls**, more Derek Collins 1961 work. (now "honorary academic staff member" at Sheffield)

In Section 7.8 Sned discusses a z+it method attack on the **flat annular ring Dirichlet** problem. This time he has to set up at least three regions in space with Smythian form type potentials, and this all ends up in a Fred 2 situation.

Chapter 8 gets my title "some classical problems in electrostatics" I will just say a little bit about each problem he deals with. See the notes overview for somewhat more detail. In all cases, Sned calls upon preparatory work he did earlier in the book, so this is the Big Payoff chapter for me.

8.1 Parallel Disk Capacitor (same or opposite potentials, r=1, $\Delta z = \kappa$). This has no closed-form solution, and you get the famous Love 1949 Fred 2 integral equation. The setup involves oblates on each disk. Sned gets the same Love equation several other ways, one just a three-region Smythian form, and this lets Sned use one of his "weighted" dual integral equation methods. Some 1953 numeric data is quoted showing results down to $\kappa = 0.1$ for capacitance relative $\kappa = \infty$ separation. The integral equation gets worse as κ gets smaller, but in this same regime the 1877 Kirchhoff result gets better. I found a 2009 paper on this and sent it to Jim, it takes κ down to 0.0001. Maybe Jim can get the same results with 50 points instead of 10,000 points.

8.2 Beltrami Dirichlet Disk f(ρ) centered between two grounded parallel plates. He does this one a few different ways, all of which lead to a Fred 2, so no closed-form solution even when f = 1.

8.3 Beltrami Dirichlet Disk f(ρ) centered in a grounded cylinder. Conclusions are the same as the previous problem. This time he gets the Fred 2 four different ways!

8.4 Two coplanar disks each with its own f(r,θ) Dirichlet potential. This is a tough problem and Sned has to use the Kobayashi potential method. The final resulting series equations are so messy that Sned does not even comment on the results, we are sent off to the 1939 Kobayashi paper. I doubt things simplify much even with constant potentials on the plates. This is a $m \neq 0$ full bore problem even then.

8.5 Coplanar metal strip capacitor (like two PCB runs). This is a 2D problem which becomes a trig triple integral set. He converts this to a dual trig series using an earlier section's method, then he gets from that a closed form solution, and the resulting σ is very simple, so much so that you could just guess it.

8.6 The charged annular disk, (V = 1, radii b > a) . This becomes a triple J_0 integral equation set with no weight, treated earlier in the book, and it has no closed-form solution, just a Fred 2. Some interesting numeric work is presented which shows creating a large hole makes little difference in capacitance, which of course follows since there is not much charge away from the edge in the solid disk case.

8.7A The charged bowl. Sned restates the closed-form solution from his Jacobi polynomial dual series work (special case thereof), and gives another derivation as well.

8.7B. Grounded bowl in uniform E field. The same methods are used to get the closed-form result to this problem.
