The Sneddon Matrix Problem PhL 12.12.10

In his Chapter 4, Sneddon presents a method of solving dual integral equations which I have interpreted in terms of matrix linear algebra. As outlined in the meta notes for that chapter, the key idea is this claim:

1. The Problem

Given two invertible matrices S_1 and S_2 , it is possible to find two triangular matrices I and K such that

$$IS_1 = KS_2 \equiv S$$

and matrix S will be invertible. Matrix K is upper triangular and I is lower triangular.

The problem is to show that the claim is true, and to come up with the matrices I and K to make it work.

2. Matrix Decomposition Forms

After some initial attempts to solve this little problem, I decided it better to review matrix theorems which might be relevant. I have some notes on triangular matrices on page 3 of section 7 of my matrix binder, but the notes don't clearly state the theorems they are about, so here is a web shot:

(1) First, here is the LU decomposition idea:

Let A be a square matrix. An **LU decomposition** is a decomposition of the form

$$A = LU,$$

where L and U are lower and upper triangular matrices (of the same size), respectively. This means that L has only zeros above the diagonal and U has only zeros below the diagonal. For a 3×3 matrix, this becomes:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

There are slightly obscure conditions on matrix A such that LU exists (quoted later in this doc).

(2) The next idea is this:

An **LDU decomposition** is a decomposition of the form

$$A = LDU,$$

where D is a diagonal matrix and L and U are *unit* triangular matrices, meaning that all the entries on the diagonals of L and U are one.

(3) A third idea is this:

An **LUP decomposition** (also called a **LU decomposition** with partial pivoting) is a decomposition of the form

$$PA = LU,$$

where L and U are again lower and upper triangular matrices and P is a permutation matrix, i.e., a matrix of zeros and ones that has exactly one entry 1 in each row and column.

Wiki on permutation matrices notes that $P^{-1} = P^T$ which is just some other permutation matrix, so you could write

A = PLU

But it turns out you can also say (as a general form)

A = LUP

Here are comments from another wiki location:

LU decomposition [edit]

- Applicable to: square matrix A
- Decomposition: $A = LU$, where L is lower triangular and U is upper triangular
- Related: the LDU decomposition is $A = LDU$, where L is lower triangular with ones on the diagonal, U is upper triangular with ones on the diagonal, and D is a diagonal matrix.
- Related: the LUP decomposition is $A = LUP$, where L is lower triangular, U is upper triangular, and P is a permutation matrix.
- Existence: An LUP decomposition exists for any square matrix A. When P is an identity matrix, the LUP decomposition reduces to the LU decomposition. If the LU decomposition exists, the LDU decomposition does too.[1]
- Comments: The LUP and LU decompositions are useful in solving an n-by-n system of linear equations $Ax = b$. These decompositions summarize the process of Gaussian elimination in matrix form. Matrix P represents any row interchanges carried out in the process of Gaussian elimination. If Gaussian elimination produces the row echelon form without requiring any row interchanges, then $P=I$, so an LU decomposition exists.

This claims that every square matrix A has an LUP (but may not have an LU). I think I could prove these claims using my notes.

3. Do these decompositions help me solve my problem?

$IS_1 = KS_2 \equiv S$

Let's reframe the problem with different symbols:

$LA = UB \equiv C$

where L and U of course mean upper and lower triangular, and A and B are invertible. Given A and B, we are supposed to find L and U such that LA = UB.

So we could write

$A = L_a U_a P_a$
$B = L_b U_b P_b$
$C = L_c U_c P_c$

We know the X_a matrices, we don't know the X_c matrices or L and U. But let's make the ansatz that matrix C exists such that LA = UB ≡ C, and see where that leads us. We have three basic unknowns in our problem: L, U, C. And we have already listed all usable information except A and B are invertible.

Fact A: If we knew U, we could compute L as follows (and vice versa on the right)

$L = UBA^{-1}$ $U = LAB^{-1}$

All we know about BA^{-1} is that it is some invertible matrix, maybe call that M. Then we have

$L = UM$ $U = LN$

The claim is that, given any invertible M, you can find a U that converts M to an L. That seems rather amazing. And similarly for U = LN. If we wrote out N we can then say

$U = L\ L_n U_n P_n = L_{n'} U_n P_n$

Using the LUP decomposition does not seem very useful here. So consider U = LN by brute force

$U = LN$

$U_{ij} = \Sigma_{k=1}^{n} L_{ik} N_{kj}$ (*)

This is n^2 equations which involve the $2n^2$ unknowns U_{ij} and L_{ik}. But we have more information:

$U_{ij} = 0$ if $i > j$

$L_{ij} = 0$ if $i < j$

So from (*) we may write

$0 = \Sigma_{k=1}^{n} L_{ik} N_{kj}$ for $i > j$

But since L is triangular as well, the sum is only partial and we can write

$$0 = \Sigma_{k=1}^{i} L_{ik} N_{kj} \qquad \text{for } i > j \qquad (**)$$

We are trying to find L_{ik} to make this work.

Aside: The number of off diagonal elements in a square matrix is n^2-n, so each side has n(n-1)/2 elements. Thus, a triangular matrix has (n^2-n)/2 + n = n^2/2 + n/2 = n(n+1)/2 elements.

In (**), matrix L_{ik} has n(n+1)/2 elements (all unknowns). How many equations do we have in (**)? We have one equation for each pair i,j such that i>j. That number must be the same as the number of elements on a side, which is n(n-1)/2. Thus, we have n(n-1)/2 equations in n(n+1)/2 unknowns, and the difference is n, so we should be able to arbitrarily set n elements of L_{ik} and then find the other elements from these equations. That is to say, we are underdetermined by n degrees of freedom. So why not set all the diagonal elements of L_{ik} to 1, that would soak up our degrees of freedom.

OK, suppose this works and we make L have ones on the diagonal. Then, given N, we have found a matrix L such that LN is upper triangular:

$$U = LN$$

Since we have 1's on the diagonal, we know that L^{-1} exists, so we can write

$$L^{-1}U = N$$

I am pretty sure that the inverse of a triangular matrix is a triangular matrix of the same type, but might not have one's on the diagonal, so let's call L^{-1} = L'. Then we have

$$L'U = N$$

This seems to claim that ANY matrix N can be written as an LU decomposition, so we have somewhere lost our condition! Maybe our set of simultaneous equations cannot be solved in certain situations.

Go back to these equations. After we set 1's on the diagonals, we have n(n-1)/2 equations in the same number of unknowns. But in a Cramer's Rule sense, every one of these equations has 0 for the RHS.

$$0 = \Sigma_{k=1}^{i} L_{ik} N_{kj} \qquad \text{for } i > j \qquad (**)$$

What do these equations look like ? Suppose n = 3 to be concrete. Then we have

i > j

2 > 1 $\qquad 0 = L_{21}N_{11} + L_{22}N_{21}$

3 > 1 $\qquad 0 = L_{31}N_{11} + L_{32}N_{21} + L_{33}N_{31}$

$3 > 2 \qquad 0 = L_{31}N_{12} + L_{32}N_{22} + L_{33}N_{32}$

Our L matrix has $3^2 = 9$ elements in general, but since triangular, only has 6 non-zero elements. Of these, only 5 appear in our equations, L_{11} does not appear. If we set the diagonals to 1 we get

$2 > 1 \qquad 0 = L_{21}N_{11} + N_{21}$
$3 > 1 \qquad 0 = L_{31}N_{11} + L_{32}N_{21} + N_{31}$
$3 > 2 \qquad 0 = L_{31}N_{12} + L_{32}N_{22} + N_{32}$

So here we have 3 equations in 3 unknowns which are L_{21} , L_{31} and L_{32}. Rewrite as

$$\begin{aligned} N_{11}\,L_{21} &= -N_{21} \\ L_{31}N_{11} + L_{32}N_{21} &= -N_{31} \\ L_{31}N_{12} + L_{32}N_{22} &= -N_{32} \end{aligned}$$

$$\begin{pmatrix} N_{11} & 0 & 0 \\ 0 & N_{11} & N_{21} \\ 0 & N_{12} & N_{22} \end{pmatrix} \begin{pmatrix} L_{21} \\ L_{31} \\ L_{32} \end{pmatrix} = \begin{pmatrix} -N_{21} \\ -N_{31} \\ -N_{32} \end{pmatrix}$$

The matrix on the left we can call S. We need $\det S \neq 0$ to have our solution. But this says

$$N_{11} \det \begin{pmatrix} N_{11} & N_{21} \\ N_{12} & N_{22} \end{pmatrix} \neq 0$$

If $N_{11} = 0$ we cannot solve except: if N_{21} happens to be 0, then any L_{21} is OK.

4. Restate what we think we have shown.

Matrices A and B are invertible, and we want to find L and U such that

$$LA = UB$$

Rewrite this as

$$U = L(AB^{-1})$$

$$U = LN$$

where $N = AB^{-1}$ is an invertible matrix. Assume for the moment that L is invertible, so we can write

$$L'U = N \qquad \text{where } L' = L^{-1}$$

Now assume further that we set the diagonal elements of L' all to one. This L' will be invertible for sure, and its inverse is our L.

Now, as long as N is a "reasonable" matrix, it has a unique L'U decomposition where L' has unit diagonal elements,

> An invertible matrix admits an *LU* factorization if and only if all its leading principal minors are non-zero. The factorization is unique if we require that the diagonal of *L* (or *U*) consist of ones. The matrix has a unique *LDU* factorization under the same conditions.

You see here stated what I mean by "reasonable": principle minors are all non-zero. This is not necessarily true for an invertible matrix, where all we know is that a certain sum of these minors is not zero, so this condition is more stringent than the condition of just being invertible.

So here is how we solve our problem:

(1) write out AB^{-1} in the unique decomposition $AB^{-1} = L'U$ where L' has ones on the diagonal. Just assume that AB^{-1} is a "reasonable" matrix and therefore has such a decomposition.

(2) We know that L' is invertible into another L-type matrix, so write $L'^{-1} = L$ which defines L. We then have U and L such that

$$LAB^{-1} = U$$

But this then says

$$LA = UB$$

and that is the problem we were trying to solve. U is the unique U that arises in the decomposition $AB^{-1} = L'U$, and L is L'^{-1}.

(3) Question: is the resulting matrix C invertible?

$$C = LA$$

$$C^{-1} = (LA)^{-1} = A^{-1}L^{-1} = A^{-1}L'.$$

Yes it is!

(4) Now, to complete our little picture, suppose you had

$$[A\Psi]_1 = f_1$$

$$[B\Psi]_2 = g_2$$

where 1 and 2 now refer to some simple partition of the index range i = 1..n. These are our "dual integral equations". We have $f = (f_1, f_2)$ and similarly for g, so we know that in the full matrix sense

$$A\Psi = f$$
$$B\Psi = g$$

but we don't know Ψ or f_2 or g_1. The trick then is to make use of our solution LA = UB. Apply L to the first and U to the second:

$$LA\Psi = Lf$$
$$UB\Psi = Ug$$

Which we can write as

$$C\Psi = Lf$$
$$C\Psi = Ug$$

Now we know that

$$Lf = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} L_{11}\, f_1 \\ * \end{pmatrix} \qquad \Rightarrow \qquad [C\,\Psi]_1 = L_{11}\, f_1$$

$$Ug = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} * \\ U_{22} g_2 \end{pmatrix} \qquad \Rightarrow \qquad [C\,\Psi]_2 = U_{22}\, g_2$$

Putting there two pieces together, we get

$$C\Psi = \begin{pmatrix} L_{11}\, f_1 \\ U_{22}\, g_2 \end{pmatrix}$$

But we know f_1 and g_2 and we know L and U, so the RHS is completely known. Then

$$\Psi = C^{-1}\begin{pmatrix} L_{11}\, f_1 \\ U_{22}\, g_2 \end{pmatrix}$$

and so Ψ is then known. But then

$$A\Psi = f$$
$$B\Psi = g$$

and we know A,B and Ψ, so we know f and g which means we know f_2 and g_1. All done!

5. What happens when we take the matrix dimension to be infinite?

Somehow, when we go to the continuum, our matrix $AB^{-1} = N$ must come out being "reasonable" all the time. Matrices A and B are modified Hankel transforms,

$$\mathcal{S}_{\eta,\alpha}\{ f(r); \mu \} = 2^{\alpha} \mu^{-\alpha} \int_0^{\infty} dx\, x^{1-\alpha} J_{2\eta+\alpha}(\mu x) f(x)$$

and matrix U=K and L=I are certain Erdelyi-Kober transforms

$$I_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{-2\alpha-2\eta} \int_0^{x} dt\, t^{2\eta+1} F(t) [x^2 - t^2]^{\alpha-1}$$

$$K_{\eta,\alpha} F(x) = 2 / \Gamma(\alpha) * x^{2\eta} \int_x^{\infty} dt\, t^{-2\eta-2\alpha+1} F(t) [t^2 - x^2]^{\alpha-1}$$

which are diagonal due to their Volterra endpoints.

To be a little more specific, we have the following infinite dimensional matrices:

$$A_{\mu x} = [\mathcal{S}_{\eta,\alpha}]_{\mu x} = 2^{\alpha} \mu^{-\alpha} x^{1-\alpha} J_{2\eta+\alpha}(\mu x) = [S_1]_{\mu x} \qquad \alpha,\eta = \alpha_1, \eta_1$$

$$B_{\mu x} = [\mathcal{S}_{\eta,\alpha}]_{\mu x} = 2^{\alpha} \mu^{-\alpha} x^{1-\alpha} J_{2\eta+\alpha}(\mu x) = [S_2]_{\mu x} \qquad \alpha,\eta = \alpha_2, \eta_2$$

$$U_{xt} = [K_{\eta,\alpha}]_{xt} = 2 / \Gamma(\alpha) * x^{2\eta} t^{-2\eta-2\alpha+1} [t^2 - x^2]^{\alpha-1} \theta(t-x) \qquad \alpha,\eta = \alpha_K, \eta_K$$

$$L_{xt} = [I_{\eta,\alpha}]_{xt} = 2 / \Gamma(\alpha) * x^{2\eta} t^{-2\eta-2\alpha+1} [t^2 - x^2]^{\alpha-1} \theta(t-x) \qquad \alpha,\eta = \alpha_I, \eta_I$$

We are given the first two A and B, and we have shown in our finite stuff above that U and L can be found and are not really unique. Sneddon produces certain K and I which are candidates for U and L and he shows that these candidates have the proper triangularity and in fact work in the sense $IS_1 = KS_2$. Notice that in general all four sets of α,η labels will be different, and that these things are NOT the matrix indices, so don't confuse that!