ODE essay applied to Bessel Functions PhL 8.5.11

Here is the Bessel equation from Schaum

$x^2y'' + xy' + (x^2-\nu^2)y = 0 \qquad y = J_\nu(x)$

Let's try to put this into some of our essay "forms".

Canonical form: $y'' + y'/x + (1-\nu^2/x^2)y = 0 \qquad L = D^2 + pD + q$

$\text{so } p = 1/x \qquad q = (1-\nu^2/x^2) = (x^2-\nu^2)/x^2$

Note that p has a simple pole at x = 0 and q has a double pole there. We expect a power branch point in the solution at x = 0. Therefore **x = 0 is a regular singular point**. In my ODE notes 1 we do a "Mobius transformation" to find that at z = ∞ we have a 4th order pole in p(1/z) and this means that our Bessel equation has an **irregular singularity at z =** ∞. Thus, the Bessel ODE is non-Fuchsian.

Frobenius form: $x^2y'' + xy' + (x^2-\nu^2)y = 0$ // about z = 0, so a = 0

$P = 1 = P_0 \quad Q = (x^2-\nu^2) = Q_0 + Q_2z^2 \qquad Q_0 = -\nu^2 \;\; Q_2 = 1$

The **indicial equation** is $[r^2 + (P_0-1)r + Q_0] = 0$ from those same notes, so we have

$r^2 + (1-1)r - \nu^2 = 0 \qquad r = \pm\nu \qquad r_1 = \nu \quad r_2 = -\nu \qquad r_1-r_2 = 2\nu$

If $2\nu \neq$ integer, the two solutions near x=0 have this form (using my ODE notes 1 conventions)

$w_1(x) = x^\nu \; \Sigma_{n=0}^{\infty} A_n x^n \qquad = J_\nu(x)$
$w_2(x) = x^{-\nu} \Sigma_{n=0}^{\infty} B_n x^n \qquad = J_{-\nu}(x)$

and these are in general the **two independent solutions.** If $r_1-r_2 = 2\nu = s$, a positive integer, meaning that ν is then either a positive integer or half integer, then we have case (6) in my same notes:

$w_1(x) = J_\nu(x)$ // the **first kind** solution
$w_2(x) = J_\nu(x) \ln(x) + x^{-\nu} \Sigma_{n=0}^{\infty} C_n x^n$

You could of course take any linear combination of these two to get an alternate second independent function, such as

$w_2'(x) = \alpha\, w_1(x) + \beta\, w_2(x)$

$= \beta\, J_\nu(x) \ln(x) + \alpha\, x^\nu \; \Sigma_{n=0}^{\infty} A_n x^n + \beta\, x^{-\nu} \; \Sigma_{n=0}^{\infty} C_n x^n$

If you look at the expression GR7 page 911 you see this expansion of the Weber function,

2. $$\pi\, Y_n(z) = 2\, J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$

$$- \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)]$$

and it appears to be exactly of the form of our $w_2'(x)$ with $\beta = (2/\pi)$ and some α.

So why was the Weber second kind function selected the way we now see it? The current definition is this

$Y_\nu(z) \equiv [\cos(\pi\nu) J_\nu(z) - J_{-\nu}(z)] / \sin(\pi\nu)$ // the **second kind** solution

[Notice by the way that in GR4 this was called $N_\nu(z)$, and in GR7 it is $Y_\nu(z)$.] When $\nu \neq$ integer, we know that the above $Y_\nu(z)$ is a viable second independent function, since we know $J_{\pm\nu}(z)$ are a viable pair. If you take the limit $\nu \to n$ of $Y_\nu(z)$ you have a 0/0 l'Hopital situation, and Bateman shows on p7,8 that the limit gives the big result expression quoted above, which we know is an independent solution. So this shows that the $Y_\nu(z)$ defined above is a viable second solution both when ν = integer and when $\nu \neq$ integer (I am not sure what happens at half integers, skip that for now). But there are likely *other* viable functions you could construct that would have this property. For example, here is candidate:

$\mathcal{Y}_\nu(z) = \alpha\, Y_\nu(z) + \beta J_\nu(z) + \gamma\, J_{-\nu}(z)$

with more or less general constants α,β,γ.

The reason $Y_\nu(z)$ was selected as the form above is not always mentioned in books, and I will mention it here. Suppose you DO define it as shown above. We already know the large-z limits of $J_{\pm\nu}(z)$ which we obtain somehow before we have ever heard of our second kind function

$J_\nu(z) \approx \sqrt{2/(\pi z)}\ \cos(z - \nu\pi/2 - \pi/4)$
$J_{-\nu}(z) \approx \sqrt{2/(\pi z)}\ \cos(z + \nu\pi/2 - \pi/4)$

Now consider what this implies for the large z behavior of $Y_\nu(z)$:

$Y_\nu(z) \equiv [\cos(\pi\nu) J_\nu(z) - J_{-\nu}(z)] / \sin(\pi\nu)$

$\to [\cos(\pi\nu)\sqrt{2/(\pi z)}\ \cos(z - \nu\pi/2 - \pi/4) - \sqrt{2/(\pi z)}\ \cos(z + \nu\pi/2 - \pi/4)] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ [\cos(\pi\nu) \cos(z - \nu\pi/2 - \pi/4) - \cos(z + \nu\pi/2 - \pi/4)\ ] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ [(1/2)\cos(z - \nu 3\pi/2 - \pi/4) + (1/2)\cos(z + \nu\pi/2 - \pi/4) - \cos(z + \nu\pi/2 - \pi/4)\ ] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ \ [(1/2)\cos(z - \nu 3\pi/2 - \pi/4) - (1/2)\cos(z + \nu\pi/2 - \pi/4)\ ] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ \ (1/2)\ [\cos(z - \nu 3\pi/2 - \pi/4) - \cos(z + \nu\pi/2 - \pi/4)\ ] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ \ (1/2)\ [2\ \sin(z - \pi\nu/2 - \pi/4)\ \sin(\pi\nu)\ ] / \sin(\pi\nu)$

$= \sqrt{2/(\pi z)}\ \ \sin(z - \pi\nu/2 - \pi/4)$

Thus, $Y_\nu(z)$ was selected (by Hankel in 1869, see WW p 363 footnote and p 371) so that we get these large z behaviors

$$J_\nu(z) \quad \to \sqrt{2/(\pi z)}\ \cos(z - \nu\pi/2 - \pi/4)$$
$$Y_\nu(z) \quad \to \sqrt{2/(\pi z)}\ \sin(z - \nu\pi/2 - \pi/4)$$

In other words, Y is the same as J but cos is replaced with sin. So in a sense, J and Y are partner functions in the sense that cos and sin are partner functions. They are in phase quadrature for example. One thing we can then do is this:

$$H_\nu^{(1)}(z) \equiv J_\nu(z) + i\, Y_\nu(z) \ \to \sqrt{2/(\pi z)}\ e^{+i(z - \nu\pi/2 - \pi/4)} \qquad \text{// } \textbf{third kind} \text{ solutions}$$
$$H_\nu^{(2)}(z) \equiv J_\nu(z) - i\, Y_\nu(z) \ \to \sqrt{2/(\pi z)}\ e^{-i(z - \nu\pi/2 - \pi/4)}$$

These forms have much significance (one moment please) and then for real ν and z we see that Y is elected to be the imaginary part of these forms, with J being the real part. When we do 2D PDE evolution equations, we find that $e^{\pm ir}/\sqrt{r}$ represent outgoing and ingoing cylindrical waves, with similar results in other dimensions. Another perhaps more important significance is that if we set $z = i\xi$, then we find that H $H_\nu^{(1)}(i\xi) \ \sim e^{-\xi z}$ and we have a function that decays exponentially which can be used in the general Bessel world to solve problems with regions reaching out to infinity.

To summarize, here is the "batting order" for Bessel equation solutions. First we do Frobenius and we obtain the general independent solutions $J_{\pm\nu}(z)$ expressed in the traditional Frobenius form, and these both have the obvious power branch cut going to the left at z = 0. Nothing at all mysterious here, we just turned the Frobenius crank.

The function $Y_\nu(z)$ is then formed as a linear combination of $J_\nu(z)$ and $J_{-\nu}(z)$ which maintains its independence from $J_\nu(z)$ even when ν is an integer, AND which has an asymptotic form making it a partner of $J_\nu(z)$ in the sense of sine partner to cosine. Then finally we define the Hankel functions in the way shown above, and we obtain from them desirable large z exponential behaviors.

Except for special values of ν, all four Bessel functions are **cut in the z plane** on (-∞,0). As a function of ν, they are all analytic in ν. For the modified Bessel functions, since $K_\nu(z) \equiv (i\pi/2)\ i^\nu\ H_\nu^{(1)}(iz)$ for example, the cut runs iz on (0,-∞) which means z on (0,+i∞) and this gives rise to strange phase restrictions on the validity of equations involving these functions.