Properties of Bessel Functions with Green's Function Problem Examples PhL 8.4.11

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1. Selected formulas

These are taken from AS2010 and verified against green Jackson.

The Bessel Functions:

$J_\nu(z) \equiv (z/2)^\nu \Sigma_{k=0}^{\infty}(-z^2/4)^k/[k!\,\Gamma(\nu+1+k)]$ cuts all run $(0,-\infty)$

$Y_\nu(z) \equiv [\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)]/\sin(\pi\nu)$ $\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z) = \sin(\pi\nu)\,Y_\nu(z)$

$H_\nu^{(1)}(z) \equiv J_\nu(z) + i\,Y_\nu(z)$

$H_\nu^{(2)}(z) \equiv J_\nu(z) - i\,Y_\nu(z)$

The Modified Bessel Functions:

$I_\nu(z) \equiv i^{-\nu} J_\nu(iz)$ cuts run $(0,+i\infty)$

$K_\nu(z) \equiv (i\pi/2)\, i^\nu\, H_\nu^{(1)}(iz)$ = "MacDonald function"

$I_\nu(z)\,K_\nu(z) = (i\pi/2)\,J_\nu(iz)\,H_\nu^{(1)}(iz)$

Negative order:

$J_{-\nu}(z) = \cos(\pi\nu)J_\nu(z) - \sin(\pi\nu)\,Y_\nu(z)$ // Maple confirmed

$Y_{-\nu}(z) = \sin(\pi\nu)J_\nu(z) + \cos(\pi\nu)\,Y_\nu(z)$ // Maple confirmed

$H_{-\nu}^{(1)}(z) = e^{+i\pi\nu} H_\nu^{(1)}(z)$ // AS

$H_{-\nu}^{(2)}(z) = e^{-i\pi\nu} H_\nu^{(1)}(z)$ // AS

$K_{-\nu}(z) = K_\nu(z)$

$I_{-\nu}(z) = I_\nu(z) + (2/\pi)\sin(\pi\nu)\,K_\nu(z)$

Small z:

$J_\nu(z) \approx (z/2)^\nu / \Gamma(\nu+1)$ $\nu \neq -1,-2...$

$Y_\nu(z) \approx -(\Gamma(\nu)/\pi)\,(z/2)^{-\nu}$ $\mathrm{Re}(\nu) > 0$

$H_\nu^{(1)}(z) \approx -i(\Gamma(\nu)/\pi)\,(z/2)^{-\nu}$ $\mathrm{Re}(\nu) > 0$

$H_\nu^{(2)}(z) \approx +i(\Gamma(\nu)/\pi)\,(z/2)^{-\nu}$ $\mathrm{Re}(\nu) > 0$

$I_\nu(z) \approx (z/2)^\nu / \Gamma(\nu+1)$

$K_\nu(z) \approx (z/2)^{-\nu}\,\Gamma(\nu)/2$ $\mathrm{Re}(\nu)>0$

$K_0(z) \approx -\ln(z) - 0.5772$

Large z:

$J_\nu(z) \approx \sqrt{2/(\pi z)}\ \cos(z - \nu\pi/2 - \pi/4)$

$Y_\nu(z) \approx \sqrt{2/(\pi z)}\ \sin(z - \nu\pi/2 - \pi/4)$

$H_\nu^{(1)}(z) \approx \sqrt{2/(\pi x)}\ e^{+i(z-\nu\pi/2-\pi/4)}$

$H_\nu^{(2)}(z) \approx \sqrt{2/(\pi x)}\ e^{-i(z-\nu\pi/2-\pi/4)}$

$I_\nu(z) \approx e^z / \sqrt{2\pi z}$ $\qquad \arg(z)$ in $(-\pi/2,\pi/2)$

$K_\nu(z) \approx e^{-z} \sqrt{\pi/(2z)}$ $\qquad \arg(z)$ in $(-3\pi/2,3\pi/2)$

Negative argument:

$J_\nu(-z) = e^{\mp i\pi\nu} J_\nu(z)$ $\qquad \mathrm{Im}(z) \gtrless 0$

other cases can be obtained from this case, see GR4 p 968

ODE Forms (see below) [If k^2 terms have opposite sign, then replace Z by I/K.]

$-(xu')' + \nu^2 u/x - k^2 x u = 0$ $\qquad Z_\nu(kx)$ $\qquad (*)$

$-(z^2u')' - k^2z^2u + (\nu^2 - 1/4)\,u = 0$ $\qquad Z_\nu(kz)/\sqrt{z}$ $\qquad (**)$

Wronskians:

$W[J_\nu(z), Y_\nu(z)] = 2/(\pi z)$

$W[J_\nu(z), J_{-\nu}(z)] = -\sin(\pi\nu)\ 2/(\pi z)$

$W[J_\nu(z), H_\nu^{(1,2)}(z)] = \pm i\ 2/(\pi z)$

$W[H_\nu^{(1)}(z), H_\nu^{(2)}(z)] = -2i\ 2/(\pi z)$

$W[K_\nu(z), I_\nu(z)] = 1/z$

$W[I_\nu(z), I_{-\nu}(z)] = -\sin(\pi\nu)\ 2/(\pi z)$

Note on Wronskians : $W_z[I_\nu(z), I_{-\nu}(z)] = W_z[J_\nu(iz), J_{-\nu}(iz)] = W_{-i\xi}[J_\nu(\xi), J_{-\nu}(\xi)] = i\, W_\xi[J_\nu(\xi), J_{-\nu}(\xi)]$ $= i\ \{ -\sin(\pi\nu)\ 2/(\pi\xi)\} = i\ \{ -\sin(\pi\nu)\ 2/(\pi i z)\} = -\sin(\pi\nu)\ 2/(\pi z) = W[J_\nu(z), J_{-\nu}(z)]$. So you don't just naively replace $z \to iz$ here; you have to consider the meaning of the derivative in W. On the other hand, we can say that $W_z[u(z),v(z)] = f(z) \Rightarrow W_{kz}[u(kz),v(kz)] = f(kz)$ and here $z \to kz$ works fine. In other words, the meaning of $W[u(kz),v(kz)]$ is really $W_{kz}[u(kz),v(kz)]$.

All these functions are analytic in ν. Cuts in z as specified above.

2. Forms of the homogeneous Bessel Equation and their solutions

(a) The basic single-parameter equation and solution is this: (Schaum p 136 for example)

$x^2y'' + xy' + (x^2 - \nu^2)y = 0$ $\qquad y = J_\nu(x)$

$xy'' + y' + xy - \nu^2 y/x = 0$

$(xy')' + xy - \nu^2 y/x = 0$

$-(xy')' + \nu^2 y/x - xy = 0$

(b) Suppose y(x) solves the above ODE. Then what equation does w(x) ≡ y(kx) solve? To answer this, we can define x' = kx and then we have [we are just doing this case as an example]

w(x) = y(kx)	w'(x) = ky'(kx)	w"(x) = k^2y"(kx)	dx' = kdx
w(x) = y(x')	w'(x) = ky'(x')	w"(x) = k^2y"(x')	$\partial_{x'} = k^{-1}\partial_x$

We know from (a) that

xy" +y' + xy - ν^2y/x = 0	// quoted from part (a)
x∂_x^2y(x) +∂_xy(x) + xy(x) - ν^2y(x)/x = 0	// wrote out derivatives
x'$\partial_{x'}^2$y(x') +$\partial_{x'}$y(x') + x'y(x') - ν^2y(x')/x' = 0	// changed x to x' everywhere
(kx)$k^{-2}\partial_x^2$y(kx) +$k^{-1}\partial_x$y(kx) +kxy(kx) - ν^2y(kx)/(kx) = 0	// installed above facts
x ∂_x^2y(kx) +∂_xy(kx) +k^2xy(kx) - ν^2y(kx)/x = 0	// multiplied by k
x ∂_x^2 w(x) +∂_x w(x) +k^2x w(x) - ν^2 w(x)/x = 0	// replaced y with w
x w" +∂_x w' +k^2x w - ν^2 w/x = 0	// deleted obvious arguments
x w" +∂_x w' +k^2x w - ν^2 w/x = 0	// removed blanks from line
(xw')' +k^2x w - ν^2 w/x = 0	// combine first two terms
- (xw')' - k^2x w + ν^2 w/x = 0	// change signs.

Therefore, we know that $J_\nu(kx)$ solves this last equation, and notice that k appears in only one term and it is squared. So we have established

-(xu')' + ν^2u/x - xu = 0	$Z_\nu(x)$	Z = J, Y or an H
-(xu')' + ν^2u/x - k^2xu = 0	$Z_\nu(kx)$	

Huge lists of other ODE forms appear GR4 p 971, GR7 p 931. Here is just one example

3. $$u'' + \frac{1-2\alpha}{z}u' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 + \frac{\alpha^2-\nu^2\gamma^2}{z^2}\right]u = 0 \qquad u = z^\alpha Z_\nu\left(\beta z^\gamma\right)$$ JA

If we set α = -1/2 and γ = 1 and β = k this becomes

u" +(2/z)u' + [k^2 + ((1/4) - ν^2)/z^2] u=0	u = Z_ν(kz)/$\sqrt{z}$	
z^2u" +2zu' + [k^2z^2 - (ν+1/2)(ν-1/2)] u=0	u = Z_ν(kz)/$\sqrt{z}$	
(z^2u')' + [k^2z^2 - (ν+1/2)(ν-1/2)] u=0	u = Z_ν(kz)/$\sqrt{z}$	
- (z^2u')' - k^2z^2u + (ν+1/2)(ν-1/2) u=0	u = Z_ν(kz)/$\sqrt{z}$	
- (z^2u')' - k^2z^2u + (ν^2-1/4) u=0	u = Z_ν(kz)/$\sqrt{z}$	(**)
- (z^2u')' - k^2z^2u + n(n+1) u=0	u = $Z_{n+1/2}$(kz)/$\sqrt{z}$	

So let's now restate our two principle results to this point:

-(xu')' + ν^2u/x - k^2xu = 0	$Z_\nu(kx)$	(*)
- (z^2u')' - k^2z^2u + n(n+1) u=0	$Z_{n+1/2}$(kz)/$\sqrt{z}$	(**)

3. The 3D and 2D Helmholtz problems.

In this section, I will temporarily use $L_{Helm} = -\nabla^2 - k^2$ with k>0, so in Stak terms $\lambda = +k^2$. You always have to think separately about the two possible signs of the H parameter λ, and here we assume +. In a wave situation where we have $e^{\pm i\omega t}$, we know $\partial_t^2 \rightarrow -k^2$and we arrive at our situation here for the H equation arising from the wave equation. In a heat situation where we have e^{-at}, we have $\partial_t = -a$ and $\lambda = a > 0$. But if we just think of a "Hemholtstatics problem", λ could have either sign.

Equation (**) is the homogeneous radial equation for $(-\nabla^2 - k^2)u(r,\theta,\varphi) = 0$. To see why, we write

$$(-\nabla^2 - k^2)u(r,\theta,\varphi) = (-\nabla_r^2 - (1/r^2)L^2 - k^2)Y_{nm}(\theta,\varphi)R(r) = 0$$

$$\nabla_r^2 u = (1/r^2)(r^2 u')' \qquad L^2 Y_{nm} = n(n+1)\, Y_{nm}$$

$$(-\nabla_r^2 + (1/r^2)L^2 - k^2)Y_{nm}(\theta,\varphi)R(r) = 0$$

$$(-\nabla_r^2 + (1/r^2)n(n+1) - k^2)Y_{nm}(\theta,\varphi)R(r) = 0$$

$$(-\nabla_r^2 + (1/r^2)n(n+1) - k^2)\, R(r) = 0$$

$$-(1/r^2)(r^2 R')' + (1/r^2)n(n+1)R - k^2 R = 0$$

$$-(r^2 R')' + n(n+1)R - k^2 r^2 R = 0$$

which does agree with the above. The radial functions are then $R = J_{n+1/2}(kr)/\sqrt{r}$. One often uses the functions $j_n(r) \equiv \sqrt{\pi/2r}\ J_{n+1/2}(r)$ in this case, which are called the **spherical Bessel functions**, and more generally $j_n(kr) \equiv \sqrt{\pi/2kr}\ J_{n+1/2}(kr)$. See green Jackson p 539.

Here is the corresponding 2D Helmholtz problem

$$(-\nabla^2 - k^2)u(r,\varphi) = (-\nabla_r^2 - (1/r^2)\partial_\varphi^2 - k^2)e^{in\varphi}\ R(r) = 0$$

$$\nabla_r^2 u = (1/r)(ru')' \qquad \partial_\varphi^2\, e^{in\varphi} = -n^2\, e^{in\varphi}$$

$$(-\nabla_r^2 - (1/r^2)\partial_\varphi^2 - k^2)e^{in\varphi}\ R(r) = 0$$

$$(-\nabla_r^2 + (1/r^2)n^2 - k^2)e^{in\varphi}\ R(r) = 0$$

$$-(1/r)(rR')' + (1/r^2)n^2 R - k^2 R = 0$$

$$-(rR')' + n^2 R/r - k^2 rR = 0$$

and this matches our $J_n(kr)$ form of part (b) above, so in this 2D problem the radial functions are just $J_n(kr)$. All these functions of course have 2nd and 3rd kind analogs, I am just using J here to represent the class.

4. How are the ODE equations different for the I and K functions?

For example, consider these two:

$-(xu')' + \nu^2 u/x - k^2 xu = 0$	$Z_\nu(kx)$
$-(z^2u')' - k^2z^2u + n(n+1)\,u=0$	$Z_{n+1/2}(kz)/\sqrt{z}$

All that happens is the k^2 term changes sign, so we have

$-(xu')' + \nu^2 u/x + k^2 xu = 0$	$K_\nu(kx)$	or I_ν
$-(z^2u')' + k^2z^2u + n(n+1)\,u=0$	$K_{n+1/2}(kz)/\sqrt{z}$	

5. Examples of Problems involving Bessel Functions

Example 1: The 3D free-space Helmholtz radial Green's Function problem.

If we define the Helmholtz operator as $-\nabla^2-\lambda$, and do separation for a φ azisym problem like the free-space situation, and if we assume $w = \Sigma_{n=0}^{\infty}\, u_n(r)\, P_n(\cos\theta)$, and if we put our unit point charge on the +z axis at $z = r_0$, we arrive at this radial Green's Function equation (as I show in Stak Ex 7.31) (I suppress the n index on u_n) :

$$-(r^2u')' - [\lambda r^2 - n(n+1)]\,u = \delta(r-r_0)\,(2n+1)/4\pi \qquad u(\infty) = 0 \quad u(0) = \text{finite}$$

If we then set $\lambda = -k^2$ as Stak always does, we get

$$-(r^2u')' + k^2r^2u + n(n+1)\,u = \delta(r-r_0)\,(2n+1)/4\pi$$

Rather than find u directly for this equation, let's instead treat this more generic equation ($\nu = n+1/2$)

$$-(r^2g')' + k^2r^2g + (\nu^2-1/4)\,g = \delta(r-r_0) \qquad \Rightarrow \dim(g) = 1/L$$

We recognize this as (**) but with $k^2 \to -k^2$. Thus, we expect our free-space Green's Function to have the form

$$g = C\,[\, I_\nu(kr_<)/\sqrt{r_<}\,]\,[\, K_\nu(kr_>)/\sqrt{r_>}\,] = (C/\sqrt{rr_0}\,)\, I_\nu(kr_<)\, K_\nu(kr_>) \qquad \Rightarrow \dim(C) = 1$$

We like this form on $(0,\infty)$ due to the small-r limit of I, and the large r limit of K:

$$I_{n+1/2}(kr)/\sqrt{r} \to r^{-1/2}\,(kr/2)^\nu / \Gamma(\nu+1) = r^{-1/2}\,(kr/2)^{n+1/2}/\Gamma(n+3/2) \sim r^n = \text{finite} \quad r\to 0$$
$$K_{n+1/2}(kr)/\sqrt{r} \to r^{-1/2} e^{-kr}\sqrt{\pi/(2kr)} \sim e^{-kr}/r \to 0 \quad r\to\infty$$

We want to compute C. We know staring at our ODE above that jump = $-1/r_0^2$. But we know

jump = $C[I_\nu(kr_0)/\sqrt{r_0}] [K_\nu(kr)/\sqrt{r}]' - C[I_\nu(kr)/\sqrt{r}]' [K_\nu(kr_0)/\sqrt{r_0}]$

The two terms involving $\partial_r(1/\sqrt{r})$ cancel when $r=r_0$, so we can simplify the above to

jump = $(C/r_0) \{[I_\nu(kr_0)] [K_\nu(kr)]' - [I_\nu(kr)]' [K_\nu(kr_0)]\} = (Ck/r_0)W[I_\nu(kr_0), K_\nu(kr_0)]$

$= (Ck/r_0)(-1/kr_0) = -C/r_0^2$

Comparison tells us that C = 1. Thus we have obtained our generic Green's function

$- (r^2g')' + kr^2g + (\nu^2-1/4)\, g = \delta(r-r_0) \quad \Rightarrow \quad g = (1/\sqrt{rr_0})\, I_\nu(kr_<)\, K_\nu(kr_>)$ ****

If we now set $\nu = n+1/2$ and add factor $(2n+1)/4\pi$ to the RHS, we get

$- (r^2g')' + kr^2g + n(n+1)\, g = \delta(r-r_0)\, (2n+1)/4\pi$

$\Rightarrow \quad g = [(2n+1)/4\pi]\, (1/\sqrt{rr_0})\, I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>)$

Using $I_\nu(z)\, K_\nu(z) = (i\pi/2)\, J_\nu(iz)\, H_\nu^{(1)}(iz)$ we could write this as

$g = (i\pi/2)\, [(2n+1)/4\pi]\, (1/\sqrt{rr_0})\, J_{n+1/2}(ikr_<)\, H_\nu^{(1)}{}_{n+1/2}(ikr_>)$

$= (i/8\sqrt{rr_0})\, (2n+1)\, J_{n+1/2}(ikr_<)\, H_\nu^{(1)}{}_{n+1/2}(ikr_>)$

and this is the form I obtained in Exercise 7.31.

We can then insert this into an expansion for w we find

$w = \Sigma_{n=0}^{\infty}\, u_n(r)\, P_n(\cos\theta) =$

$= \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, [(2n+1)/4\pi]\, (1/\sqrt{rr_0})\, I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>) = E_3(\lambda = -k^2)$

But we know that (see Stak p 55 5.119 and 5.121)

$E_3(\lambda=-k^2) = (1/2\pi)\, (k/2\pi R)^{1/2} K_{1/2}(kR) = e^{-kR}/(4\pi R)$

so we then have this little addition theorem

$e^{-kR}/(4\pi R) = (1/4\pi)(1/\sqrt{rr_0})\, \Sigma_{n=0}^{\infty}\, P_n(\cos\theta)\, (2n+1)\, I_{n+1/2}(kr_<)\, K_{n+1/2}(kr_>)$ ****

$$R^2 = r^2 + r_0^2 - 2rr_0\cos\theta$$

which we see in 7.225. I cannot find this in Bateman. But Bateman does have p 43 (3) which we can apply with $\nu = 1/2$ to get

$$R^{-1/2}K_{1/2}(R) = (rr_0/2)^{-1/2}\Gamma(1/2) \times \Sigma_{n=0}^{\infty} (1/2 + n)C^{(1/2)}_n(\cos\theta)\, I_{n+1/2}(r_<)\, K_{n+1/2}(r_>)$$

But we know from Bateman p 179 (3) that $C^{(1/2)}_n(\cos\theta) = P_n(\cos\theta)$, so we then have

$$R^{-1/2}K_{1/2}(R) = (rr_0/2)^{-1/2}\sqrt{\pi}\ (1/2)\ \Sigma_{n=0}^{\infty}\ (2n+1)\ P_n(\cos\theta)\ I_{n+1/2}(r_<)\ K_{n+1/2}(r_>)$$

$$= \sqrt{2\pi}\ (1/2)\ 1/\sqrt{rr_0}\ \ \Sigma_{n=0}^{\infty}\ (2n+1)\ P_n(\cos\theta)\ I_{n+1/2}(r_<)\ K_{n+1/2}(r_>)$$

So Bateman's result gives

$$K_{1/2}(R)/\sqrt{2\pi R} = (1/2)\ 1/\sqrt{rr_0}\ \ \Sigma_{n=0}^{\infty}\ (2n+1)\ P_n(\cos\theta)\ I_{n+1/2}(r_<)\ K_{n+1/2}(r_>)$$

But from above we have

$$(1/2\pi)\ (k/2\pi r)^{1/2}K_{1/2}(kr) = e^{-kr}/(4\pi r)$$
$$(1/2\pi)\ (1/2\pi R)^{1/2}K_{1/2}(R) = e^{-kR}/(4\pi R)$$
$$K_{1/2}(R)/\sqrt{2\pi R} = 2\pi\ e^{-kr}/(4\pi R) = e^{-kR}/2R$$

So Bateman's result becomes

$$e^{-kR}/(4\pi R) = (1/4\pi)\ 1/\sqrt{rr_0}\ \ \Sigma_{n=0}^{\infty}\ (2n+1)\ P_n(\cos\theta)\ I_{n+1/2}(r_<)\ K_{n+1/2}(r_>)$$

which agrees with the Stak result.

Example 2: The 2D free-space Helmholtz radial Green's Function problem.

If we define the Helmholtz operator as $-\nabla^2-\lambda$, and if we assume $w = \Sigma_n\ u_n(r)\ e^{in\theta}$, and if we put our unit point charge on the +x axis ($\theta=0$) at $x = r_0$, we arrive at this radial Green's Function equation (as I show in Stak Ex 7.34 part (b), although there I was doing range (a,∞), whereas here I am doing $(0,\infty)$)(I suppress the n index on u_n) :

$$-(ru')' - \lambda ru + n^2u/r = \delta(r-r_0)/2\pi \qquad u(\infty) = 0 \quad u(0) = \text{finite}$$

If we then set $\lambda = -k^2$ as Stak always does, we get

$$-(ru')' + k^2ru + n^2u/r = \delta(r-r_0)/2\pi$$

Rather than find u directly for this equation, let's instead treat this more generic equation

$$-(ru')' + k^2 ru + \nu^2 u/r = \delta(r-r_0) \qquad \dim(u) = 1$$

We recognize this as (*) but with $k^2 \rightarrow -k^2$. Thus, we expect our free-space Green's Function to have this form,

$$g = C\,[\,I_\nu(kr_<)]\,[\,K_\nu(kr_>)] \qquad \Rightarrow \dim(C) = 1$$

We like this form on $(0,\infty)$ due to the small-r limit of I, and the large r limit of K:

$$I_\nu(kr) \rightarrow (kr/2)^\nu / \Gamma(\nu+1) \quad \sim r^\nu = \text{finite} \quad r\rightarrow 0 \qquad \nu = 0,1,2...$$
$$K_\nu(kr) \rightarrow e^{-kr}\sqrt{\pi/(2kr)} \sim e^{-kr}/r \rightarrow 0 \quad r\rightarrow\infty$$

We want to compute C. We know staring at our ODE above that jump = $-1/r_0$. But we know

$$\text{jump} = C[\,I_\nu(kr_0)]\,[\,K_\nu(kr)]' - C[\,I_\nu(kr)]'\,[\,K_\nu(kr_0)] = Ck\,W[I_\nu(kr_0), K_\nu(kr_0)]$$

$$= Ck(-1/kr_0) = -C/r_0$$

Comparison tells us that C = 1. Thus we have obtained our generic Green's function

$$-(ru')' + k^2 ru + \nu^2 u/r = \delta(r-r_0) \qquad \Rightarrow \qquad g = I_\nu(kr_<)\,K_\nu(kr_>) \qquad ****$$

If we now set $\nu = n$ and add factor $1/2\pi$ to the RHS, we get

$$-(r^2 g')' + kr^2 g + n(n+1)\,g = \delta(r-r_0)\,(2n+1)/4\pi \qquad \Rightarrow \qquad g = (1/2\pi)\,I_n(kr_<)\,K_n(kr_>)$$

We can then insert this into an expansion for w we find

$$w = (1/2\pi)\,\Sigma_n\; I_n(kr_<)\,K_n(kr_>)\,e^{in\theta} = E_2(\lambda=-k^2)$$

But we know that (see Stak p 266 76.154)

$$E_2(\lambda=-k^2) = (i/4)H_0^{(1)}(ikR) = (i/4)(2/i\pi)K_0(kR) = (1/2\pi)K_0(kR) \qquad \text{// as in 7.155}$$

so we then have this little addition theorem

$$K_0(kR) = \Sigma_n\; I_n(kr_<)\,K_n(kr_>)\,e^{in\theta} \qquad R^2 = r^2 + r_0^2 - 2rr_0\cos\theta \qquad ****$$

which we see sort of in 7.166, and in Bateman p 44 top,

Example 1A: The 3D Helmholtz radial exterior-to-sphere Green's Function problem.

We follow Example 1 down to the generic equation (a is radius of sphere, r_0 radius of point source)

$$- (r^2 g')' + k^2 r^2 g + (\nu^2 - 1/4)\, g = \delta(r - r_0) \qquad \Rightarrow \dim(g) = 1/L$$

but now we have to take g to have the form

$$g = C \left[(I_\nu(kr_<)/\sqrt{r_<}\,)\,(K_\nu(ka)/\sqrt{a}\,) - (K_\nu(kr_<)/\sqrt{r_<}\,)\,(I_\nu(ka)/\sqrt{a}\,) \right] \left[K_\nu(kr_>)/\sqrt{r_>} \right]$$

which is a Smythian form making exterior g(r=a) = 0, our sphere BC. As before, we know jump = $-1/r_0^2$. When we compute the jump from the above form for g, the second term will make no contribution to the jump since W[K,K] = 0. The remaining term is just what we had in Example 1 except we have an extra factor of $(K_\nu(ka)/\sqrt{a}\,)$. Thus, we shall obtain $C\,(K_\nu(ka)/\sqrt{a}\,) = 1$ instead of C = 1, and g will be

$$g = \left[(I_\nu(kr_<)/\sqrt{r_<}\,)\,(K_\nu(ka)/\sqrt{a}\,) - (K_\nu(kr_<)/\sqrt{r_<}\,)\,(I_\nu(ka)/\sqrt{a}\,) \right] \left[K_\nu(kr_>)/\sqrt{r_>} \,/\, K_\nu(ka)/\sqrt{a} \right]$$

Each term here has $\sqrt{r_<}\,\sqrt{a}\,\sqrt{r_>}\,/\sqrt{a} = \sqrt{r_<}\,\sqrt{r_>} = \sqrt{rr_0}$ in the denominator, so simplify to

$$g = (1/\sqrt{rr_0}\,)\left[I_\nu(kr_<)\, K_\nu(ka)) - K_\nu(kr_<)\, I_\nu(ka) \right] \left[K_\nu(kr_>)/\, K_\nu(ka) \right]$$

Thus we have this result

$$- (r^2 g')' + k^2 r^2 g + (\nu^2 - 1/4)\, g = \delta(r - r_0) \qquad g(a) = g(\infty) = 0$$

$$g = (1/\sqrt{rr_0}\,)\left[I_\nu(kr_<)\, K_\nu(ka)) - K_\nu(kr_<)\, I_\nu(ka) \right] \left[K_\nu(kr_>)/\, K_\nu(ka) \right] \qquad ****$$

Now moving back from the generic to the Helmholtz problem, we have ($\nu = n+1/2$)

$$- (r^2 u')' + k^2 r^2 u + n(n+1)\, u = \delta(r - r_0)\,(2n+1)/4\pi$$

$$g = (2n+1)\,(1/4\pi\sqrt{rr_0}\,)\left[I_{n+1/2}(kr_<)\, K_{n+1/2}(ka)) - K_{n+1/2}\,(kr_<)\, I_{n+1/2}(ka) \right] \left[K_{n+1/2}(kr_>)/\, K_{n+1/2}(ka) \right]$$

and then the solution to our exterior-to-grounded-sphere Helmholtstatics 3D Green's Function problem is ($\lambda = -k^2$)

$$w(r,\theta) = (1/4\pi\sqrt{rr_0}\,)\,\Sigma_{n=0}^{\infty}\,(2n+1) P_n(\cos\theta) \,*$$
$$\left[I_{n+1/2}(kr_<)\, K_{n+1/2}(ka)) - K_{n+1/2}\,(kr_<)\, I_{n+1/2}(ka) \right] \left[K_{n+1/2}(kr_>)/\, K_{n+1/2}(ka) \right]$$

If we change to J and H functions, we will pick up a factor of ($i\pi/2$) and arguments gain factor i. The leading factor is then $(1/4\pi\sqrt{rr_0}\,)(i\pi/2) = (i/8\sqrt{rr_0}\,)$. This makes the above result for w agree with the result I got for U(r,s) in the Ex 7.33 doc, which I now quote (and in which we have n = 0)

U(r,s) =

$$= (i/8\sqrt{rr_0}\,)\left[H_{1/2}^{(1)}(\sqrt{\lambda}\, a) J_{1/2}(\sqrt{\lambda}\, r_<) - J_{1/2}(\sqrt{\lambda}\, a)\, H_{1/2}^{(1)}(\sqrt{\lambda}\, r_<) \right] \left[H_{1/2}^{(1)}(\sqrt{\lambda}\, r_>)/\, H_{1/2}^{(1)}(\sqrt{\lambda}\, a) \right]$$

where the connection is $\sqrt{\lambda} = ik = i\sqrt{s}$. The reason we only got n=0 there was due to the spherical symmetry of the problem posed in Ex 7.33. In contrast, our Helmholtstatics problem is only azisym.

Example 2A: The 2D Helmholtz radial exterior-to-circle Green's Function problem.

We follow Example 2 down to the generic equation

$$-(ru')' + k^2 ru + \nu^2 u/r = \delta(r-r_0) \qquad \dim(u) = 1$$

but now we have to take g to have the form

$$g = C\,[\, I_\nu(kr_<)K_\nu(ka) - K_\nu(kr_<)\, I_\nu(ka)]\,[\, K_\nu(kr_>)]$$

which is a Smythian form making exterior g(r=a) = 0, our circle BC. As before, we know jump = $-1/r_0$. When we compute the jump from the above form for g, the second term will make no contribution to the jump since W[K,K] = 0. The remaining term is just what we had in Example 2 except we have an extra factor of $K_\nu(ka)$. Thus, we shall obtain $C\,K_\nu(ka) = 1$ instead of C = 1, and g will be

$$g = [\, I_\nu(kr_<)K_\nu(ka) - K_\nu(kr_<)\, I_\nu(ka)]\,[\, K_\nu(kr_>)/\, K_\nu(ka)\,]$$

Thus we have this result

$$-(ru')' + k^2 ru + \nu^2 u/r = \delta(r-r_0) \qquad g(a) = g(\infty) = 0$$

$$g = [\, I_\nu(kr_<)K_\nu(ka) - K_\nu(kr_<)\, I_\nu(ka)]\,[\, K_\nu(kr_>)/\, K_\nu(ka)\,] \qquad ****$$

For the actual Helmholtz problem instead of the generic, we add factor $1/2\pi$ and set $\nu = n$. Then

$$-(ru')' + k^2 ru + n^2 u/r = \delta(r-r_0)/2\pi$$

$$g = (1/2\pi)\,[\, I_n(kr_<)K_n(ka) - K_n(kr_<)\, I_n(ka)]\,[\, K_n(kr_>)/\, K_n(ka)\,]$$

and then we have our full solution,

$$w = \Sigma_n\, u_n(r)\, e^{in\theta} = (1/2\pi)\,\Sigma_n\, e^{in\theta}[\, I_n(kr_<)K_n(ka) - K_n(kr_<)\, I_n(ka)]\,[\, K_n(kr_>)/\, K_n(ka)\,]$$

Then using $I_\nu(z)\, K_\nu(z) = (i\pi/2)\, J_\nu(iz)\, H_\nu^{(1)}(iz)$ and $K_n(x)/\, K_n(y) = H_n^{(1)}(ix)/\, H_n^{(1)}(iy)$, we can rewrite this, noting that $(1/2\pi)\,(i\pi/2) = (i/4)$,

$$w(r,\theta) = (i/4)\,\Sigma_n\, e^{in\theta}[J_n(ikr_<)\, H_n^{(1)}(ika) - H_n^{(1)}(ikr_<)\, J_\nu(ika)]\,[H_n^{(1)}(ikr_>)/\, H_n^{(1)}(ika)\,]$$

which agrees with the end of part (b) of Exercise 7.34.

Comment: Notice that, in our 1A and 2A examples above, if we take $a \to 0$, we do NOT obtain the solutions to the free-space problems 1 and 2. The reason of course is that the free-space problems do not have $w=0$ at the origin, whereas the limit of the a problems does.