

Zeros of the Macdonald function of complex order

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Abstract

The z -zeros of the modified Bessel function of the third kind $K_\nu(z)$, also known as modified Hankel function or Macdonald function, are considered for arbitrary complex values of the order ν . Approximate expressions for the zeros, applicable in the cases of very small or very large $|\nu|$, are given. The behaviour of the zeros for varying $|\nu|$ or $\arg \nu$, obtained numerically, is illustrated by means of some graphics.

Key words: Macdonald function, modified Bessel function of the third kind, Hankel function, zeros

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1 Introduction.

The relevance of Bessel functions to the solution of a great variety of problems in Physics and Engineering is widely known and does not need to be stressed.

In particular, zeros of the different kinds of those functions are closely related to energies of bound or resonant physical systems. For instance, the ν -zeros of the Hankel function $H_\nu(z)$ for positive values of z determine the poles of the amplitude of scattering of various kinds of waves by spheres and cylinders [20]. Almost four decades ago we considered [10] the zeros of the modified Bessel function of the third kind, $K_\nu(z)$, also known as modified Hankel function or Macdonald function. Information about the z -zeros of $K_\nu(z)$ for positive order ν was found in the book by Watson [33]. However, our interest, motivated by a quantum mechanical problem, was on z -zeros when ν assumes purely imaginary values. For that case it was already known [12,18] that $K_\nu(z)$ presents an infinity of zeros, all of them located on the positive real semiaxis of the complex z -plane. We showed that these positive zeros, labelled according to decreasing values, form a sequence tending to be a geometric progression with ratio $\exp(-\pi/|\nu|)$. We gave, also, asymptotic approximations to the position of the largest zeros for large values of $|\nu|$. Further discussions of the properties of the zeros of $K_\nu(z)$, ν pure imaginary, have been carried out by Laforgia [21] and by Dunster [7]. Zeros of $K_\nu(z)$ with complex ν have been less studied. In view of their application in the analysis of electromagnetic wave propagation in anisotropic inhomogeneous conducting media, Nalesso [26] has considered the ν -zeros of $K_\nu(t\nu)$ for fixed positive values of t . But, to our knowledge, a discussion of the z -zeros of $K_\nu(z)$ for arbitrary complex ν is still lacking.

In recent years, stimulated by review papers by Lozier and Olver [23,24] and by the DLMF project, a considerable number of new algorithms for the computation of special functions have been published. Concerning the Macdonald function, Refs. [9,13,14,15,16,17,29] are to be noticed. As a general rule, algorithms lose accuracy in the vicinity of the zeros. Hence the interest of having a knowledge, at least approximate, of the location of those zeros. On the other hand, $K_\nu(z)$ presents only a finite number of z -zeros in the Riemann sheet $-\pi < \arg z \leq \pi$ if ν is real, whereas an infinity of zeros occur if ν is pure imaginary. It has therefore seemed to us interesting to study the evolution of the z -zeros of $K_\nu(z)$ as the modulus and/or argument of ν are continuously changed.

In what follows, we consider only values of ν in the quadrant

$$0 \leq \arg \nu \leq \pi/2,$$

and z is assumed to lie in the principal Riemann sheet, $-\pi < \arg z \leq \pi$, for large values of ν . The symmetry relations

$$K_{-\nu}(z) = K_\nu(z), \quad K_{\overline{\nu}}(\overline{z}) = \overline{K_\nu(z)}, \quad (1)$$

allow to extend our results to values of ν in all quadrants. On the other hand,

the relation [1, Eq. 9.6.4]

$$K_\nu(z) = \frac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}\left(ze^{\frac{1}{2}\pi i}\right), \quad (2)$$

valid when K and $H^{(1)}$ are analytically continued to every Riemann sheet, proves that the pattern of z -zeros of the Hankel function $H_\nu^{(1)}(z)$ results from that of $K_\nu(z)$ by rotation by an angle of $+\pi/2$ around the origin.

We discuss in Section 2 the behaviour of the zeros for decreasing values of $|\nu|$. The case of large values of $|\nu|$ is considered in Section 3, where asymptotic approximations to the zeros are given. Numerical results for the zeros of $K_\nu(z)$ with moderate $|\nu|$ are presented graphically in Section 4. Finally, the particular case of real ν is commented in Section 5.

2 Zeros of $K_\nu(z)$ for $|\nu| \ll 1$.

The particular case of pure imaginary ν revealed [10] that, as $|\nu|$ goes to zero, all zeros of $K_\nu(z)$ approach the origin. Therefore we start by assuming $|z| \ll 1$ in the expression of $K_\nu(z)$ as a sum of two convergent ascending series [1, Eqs. 9.6.2 and 9.6.10]

$$K_\nu(z) = \frac{(\pi/2)}{\sin(\nu\pi)} \left(\frac{(z/2)^{-\nu}}{\Gamma(1-\nu)} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1-\nu)_k} - \frac{(z/2)^\nu}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1+\nu)_k} \right), \quad (3)$$

valid for ν different from an integer. Obviously, the zeros of $K_\nu(z)$ should satisfy the relation

$$(z/2)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \frac{\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1-\nu)_k}}{\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1+\nu)_k}}, \quad (4)$$

that, in the case of being $|z| \ll 1$, can be approximated by

$$(z/2)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \left(1 + \frac{\nu}{2(1-\nu^2)} z^2 + O(z^4) \right), \quad |z| \ll 1. \quad (5)$$

By taking logarithms, one obtains, as a first approximation,

$$\log |z| + i \arg z \simeq -\frac{n\pi i}{\nu} + \log 2 + \frac{1}{2\nu} \log(\Gamma(1+\nu)/\Gamma(1-\nu)) + \frac{z^2}{4(1-\nu^2)}, \quad (6)$$

where n could in principle take the values $n = 0, \pm 1, \pm 2, \dots$, although, according to that obtained in the case of ν being pure imaginary, actually existing zeros correspond to

$$n = 1, 2, 3, \dots$$

These values of n will be used as a label to characterize each zero in the form z_n . Ignoring, in the crudest approximation, the last term in the right hand side of Eq. (6), one has

$$\log |z_n| \simeq \Re \left(\frac{1}{\nu} (-n\pi i + \frac{1}{2} \log(\Gamma(1+\nu)/\Gamma(1-\nu))) \right) + \log 2, \quad (7)$$

$$\arg z_n \simeq \Im \left(\frac{1}{\nu} (-n\pi i + \frac{1}{2} \log(\Gamma(1+\nu)/\Gamma(1-\nu))) \right). \quad (8)$$

The behavior of the zeros z_n as $|\nu|$ decreases can be immediately obtained from these expressions by using the series expansion [1, Eq. 6.1.34]

$$\frac{1}{\Gamma(1+\nu)} = 1 + \gamma\nu + c_3\nu^2 + \dots \quad (9)$$

to obtain

$$\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} = 1 - 2\gamma\nu + 2\gamma^2\nu^2 + \dots, \quad (10)$$

where γ represents the Euler constant, $\gamma = 0.5772156649\dots$. By retaining terms up to $O(|\nu|^{-1})$, it turns out

$$\log |z_n| \simeq -n\pi \frac{\Im \nu}{|\nu|^2} + \log 2 - \gamma, \quad (11)$$

$$\arg z_n \simeq -n\pi \frac{\Re \nu}{|\nu|^2}. \quad (12)$$

These two equations allow us to obtain a picture of the behaviour of the zeros of $K_\nu(z)$ as $|\nu|$ tends to zero. For instance, let us assume that $|\nu|$ decreases with $\arg \nu$ fixed. If $\arg \nu = \pi/2$, the zeros z_n approach the origin along the positive real semiaxis, as described in Ref. [10]. If $0 < \arg \nu < \pi/2$, $|z_n|$ tends to zero whereas $\arg z_n$ decreases without limit. In other words, the zeros approach the origin spiraling clockwise infinitely. According to Eq. (12), most of zeros (or even all of them, for sufficiently small $|\nu|$) occur outside the principal Riemann sheet. For $\arg \nu = 0$, Eqs. (11) and (12) are no more valid because, as we will see later, the possibility exists of z_n going to infinity for certain values of ν .

The behaviour of the zeros in this case is much more complicated and will be described in Section 5.

3 Zeros of $K_\nu(z)$ for $|\nu| \gg 1$.

We have already mentioned that Eq. (2) allows one to obtain the zeros of the Macdonald function from those of the Hankel function, and viceversa. Previous research, by other authors [3,11,19,20,25,27], on the ν -zeros of the Hankel function shows that, for large values of $|z|$, they occur at $\nu \simeq z$. Asymptotic expansions of $H^{(1)}$ valid in the transition region should then be used to obtain approximate expressions of its z -zeros for $|\nu| \gg 1$. From Eqs. 9.3.23 and 9.3.24 of Ref. [1], by using Eqs. 9.1.3 and 10.4.9, a convenient expansion can be written, namely

$$H_\nu^{(1)}(\nu + \theta\nu^{1/3}) \sim \frac{2^{4/3}}{\nu^{1/3}} e^{-i\pi/3} \left(\text{Ai}(-2^{1/3}e^{i2\pi/3}\theta) \sum_{k=0}^{\infty} f_k(\theta)\nu^{-2k/3} + \frac{2^{1/3}}{\nu^{2/3}} e^{i2\pi/3} \text{Ai}'(-2^{1/3}e^{i2\pi/3}\theta) \sum_{k=0}^{\infty} g_k(\theta)\nu^{-2k/3} \right), \quad (13)$$

where $f_0(\theta) = 1$, the other functions $f_k(\theta)$ and $g_k(\theta)$ being given in Eqs. 9.3.25 and 9.3.26 of the same reference. A more powerful asymptotic expansion of $H^{(1)}$ exists [1, Eq. 9.3.37], but its coefficients are much more complicated and, although reversion of that expansion could be done by means of a procedure due to Fabijonas and Olver [8], we find preferable, for the sake of simplicity and transparency, to work with Eq. (13). A simple inspection of that expansion shows that, for large $|\nu|$, it vanishes for values of θ verifying

$$\theta_s = -2^{-1/3} e^{-i2\pi/3} (a_s + \varepsilon_s(\nu)), \quad s = 1, 2, 3, \dots, \quad (14)$$

where a_s represents each one of the zeros of the Airy function, given in Table 10.13 of Ref. [1], and $\varepsilon_s(\nu) = O(\nu^{-2/3})$ is to be determined. Trying an expansion of the form

$$\varepsilon_s(\nu) = \sum_{j=1}^{\infty} \alpha_{s,j} \nu^{-2j/3}, \quad (15)$$

and using Taylor expansions of the Airy function and of its derivative around a_s , it is straightforward, with the aid of a computer algebra software package, to check that the right hand side of Eq. (13) cancels out for

$$\begin{aligned}
\alpha_{s,1} &= -\frac{3}{10} 2^{-1/3} e^{-i2\pi/3} a_s^2, \\
\alpha_{s,2} &= -\frac{1}{700} 2^{1/3} e^{i2\pi/3} (a_s^3 + 10), \\
\alpha_{s,3} &= \frac{1}{126000} a_s (479 a_s^3 - 40), \\
\alpha_{s,4} &= \frac{1}{16170000} 2^{-1/3} e^{-i2\pi/3} a_s^2 (20231 a_s^3 + 55100), \quad \text{etc.}
\end{aligned}$$

Approximate values of the z -zeros of $H_\nu^{(1)}(z)$ are then given by

$$z_s \sim (\nu + \theta_s \nu^{1/3}), \quad |\nu| \gg 1, \quad s = 1, 2, 3, \dots \quad (16)$$

Reversion of this relation between z and ν would provide an approximation for the ν -zeros of $H_\nu^{(1)}(z)$ when $|z|$ is large,

$$\begin{aligned}
\nu_s \sim & z + 2^{-1/3} e^{-i2\pi/3} a_s z^{1/3} + \frac{1}{60} 2^{1/3} e^{i2\pi/3} a_s^2 z^{-1/3} - \frac{1}{700} (a_s^3 + 10) z^{-1} \\
& + \frac{1}{1134000} 2^{-1/3} e^{-i2\pi/3} a_s (281 a_s^3 + 10440) z^{-5/3} \\
& - \frac{1}{2619540000} 2^{1/3} e^{i2\pi/3} a_s^2 (73769 a_s^3 + 6624900) z^{-7/3}, \quad |z| \gg 1, \quad (17)
\end{aligned}$$

to be compared with those given in Refs. [3,11,19,20,27]. According to Eq. (2), the z -zeros of $K_\nu^{(1)}(z)$ are given approximately by

$$z_s \sim e^{-i\pi/2} (\nu + \theta_s \nu^{1/3}), \quad |\nu| \gg 1, \quad s = 1, 2, 3, \dots \quad (18)$$

The usefulness of Eqs. (17) and (18) is restricted to the first values of $s = 1, 2, 3, \dots$, due to the fact that $|a_s|$ increases with s . This makes the omitted terms in the expansion (17) to increase and invalidates their suppression. Besides this, $|\theta_s|$, given by (14), increases with s and, for a given ν , may reach a value such that expansion (13) ceases from being useful. If zeros corresponding to a large s are to be considered, approximate values can be obtained following a procedure, already used by Magnus and Kotin [25] and by Cochran [3], consisting in taking logarithms in Eq. (4),

$$\begin{aligned}
2\nu \log(z/2) + i2n\pi &= \log(\Gamma(\nu)) - \log(-\Gamma(-\nu)) \\
&+ \log\left(\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1-\nu)_k}\right) - \log\left(\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1+\nu)_k}\right), \quad n = 0, \pm 1, \pm 2, \dots, \quad (19)
\end{aligned}$$

and approximating the logarithms of the gamma functions by their asymptotic expansions [1, Eq. 6.1.40]. One obtains in this way

$$\begin{aligned} \log(z/2) \sim \log \nu - 1 - i\pi/2 - i(n - \tfrac{1}{4})\pi/\nu + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)\nu^{2m}} \\ + \frac{1}{2\nu} \left(\log \left(\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1-\nu)_k} \right) - \log \left(\sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1+\nu)_k} \right) \right). \end{aligned} \quad (20)$$

Assuming $|z| \ll |\nu|$ and keeping the dominant terms, it comes out for the z -zeros of $K_\nu(z)$

$$z_n \simeq e^{-i\pi/2} 2\nu \exp \left(-1 - i(n - \tfrac{1}{4})\pi/\nu \right). \quad (21)$$

Obviously, since we are considering values of ν in the first quadrant, only large positive values of the integer n are compatible with the assumption $|z| \ll |\nu|$. Moreover, for a given value of $|\nu|$, the approximation (21) loses accuracy as $\arg \nu$ goes from $\pi/2$ towards 0 and becomes completely useless for real ν . A numerical comparison of the values obtained with Eqs. (18) and (21) for moderate values of s and n shows that both labels coincide.

4 Zeros of $K_\nu(z)$ for intermediate values of $|\nu|$.

We have used a numerical procedure to explore the behaviour of the z -zeros of $K_\nu(z)$, as a function of ν , for moderate values of $|\nu|$. In order to simplify the presentation of the results, two different situations have been considered: (i) variable $|\nu|$ with fixed $\arg \nu$, and (ii) variable $\arg \nu$ keeping $|\nu|$ fixed. The resulting trajectories of the first, second, and third zeros are represented in Figures 1, 2, and 3, respectively. For the evaluation of $K_\nu(z)$, expression (3) has been used. More sophisticated algorithms can be found in the literature [2,9,13,14,15,16,17,29,30,31,32], but either they have been conceived only for particular (real or pure imaginary) values of the variable or the order, or they become unnecessarily complicated for our purpose of giving a merely qualitative description of the zeros. For the location of these, the Newton method has been used. Let us mention, however, other existing techniques like, for instance, the auxiliary tables for the evaluation of the ν -zeros of $K_\nu(z)$, for z real or pure imaginary, proposed by Cochran and Hoffspiegel [4], the finite element approximation method, applied by Leung and Ghaderpanah [22] to a very precise determination of the zeros of $K_n(z)$, and a procedure, applied by Segura [28] to the computation of zeros of Bessel and other special functions, that uses fixed point iterations and does not require the evaluation of the functions.

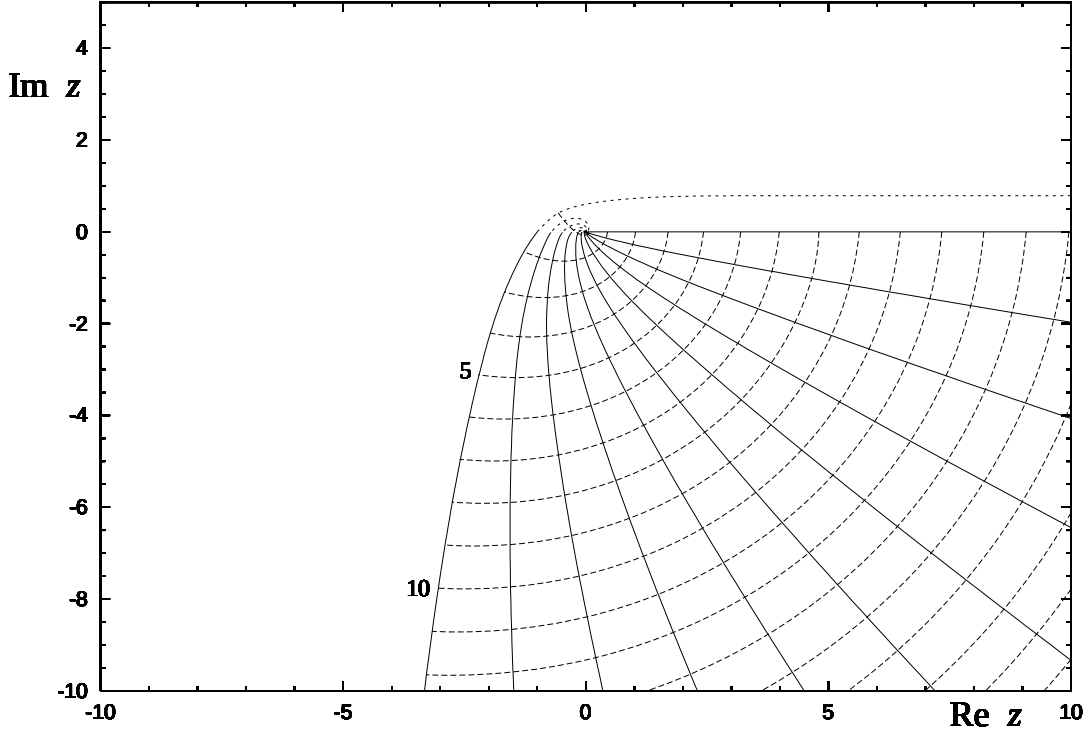


Fig. 1. Trajectories of the first zero, z_1 , of $K_\nu(z)$. The continuous lines represent the trajectories followed by z_1 as ν varies whereas $\arg \nu$ remains fixed and equal to $m\pi/20$, with $m = 0$ (first trajectory from the left), $1, 2, \dots, 10$ (horizontal trajectory). The continuous lines turn into short-dashed ones as the zero leaves the Riemann sheet $-\pi < \arg z \leq \pi$. The trajectory corresponding to real ν ($m = 0$) goes to $(+\infty, \pi/4)$, with $\arg z_1 \rightarrow -2\pi$, for $\nu = 1/2$. The nearly circular dashed arcs represent the trajectories of z_1 for fixed integer $|\nu|$ and varying $\arg \nu$ going from 0 to $\pi/2$. The trajectories corresponding to $|\nu| = 5$ and 10 are explicitly indicated.

A double precision FORTRAN code was used to compute the expression

$$\mathcal{E}_\nu(z) = \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1-\nu)_k} - (z/2)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(1+\nu)_k} \quad (22)$$

whose zeros coincide with those of the right hand side of Eq. (3). The two series were summed term by term until the absolute value of the ratio of the last term to the partial sum turned out to be less than 10^{-18} . A warning message was foreseen for the case of the number of summed terms becoming larger than 100, but this limit was never reached. Fortunately, zeros at large values of $|z|$ occur for values of ν either of large modulus or in the neighbourhood of $n + \frac{1}{2}$, $n = 0, 1, 2, \dots$. In the first case, the series in (22) converge reasonably, from the numerical point of view; in the second case, analytical methods can be used to approximate the zeros. Typical runs of our procedure can be seen in Table 1, which shows intermediate results in the determination of the first, second and third zeros of $K_\nu(z)$, with $|\nu| = 21$, $\arg \nu = 7\pi/20$. Initial approximate values

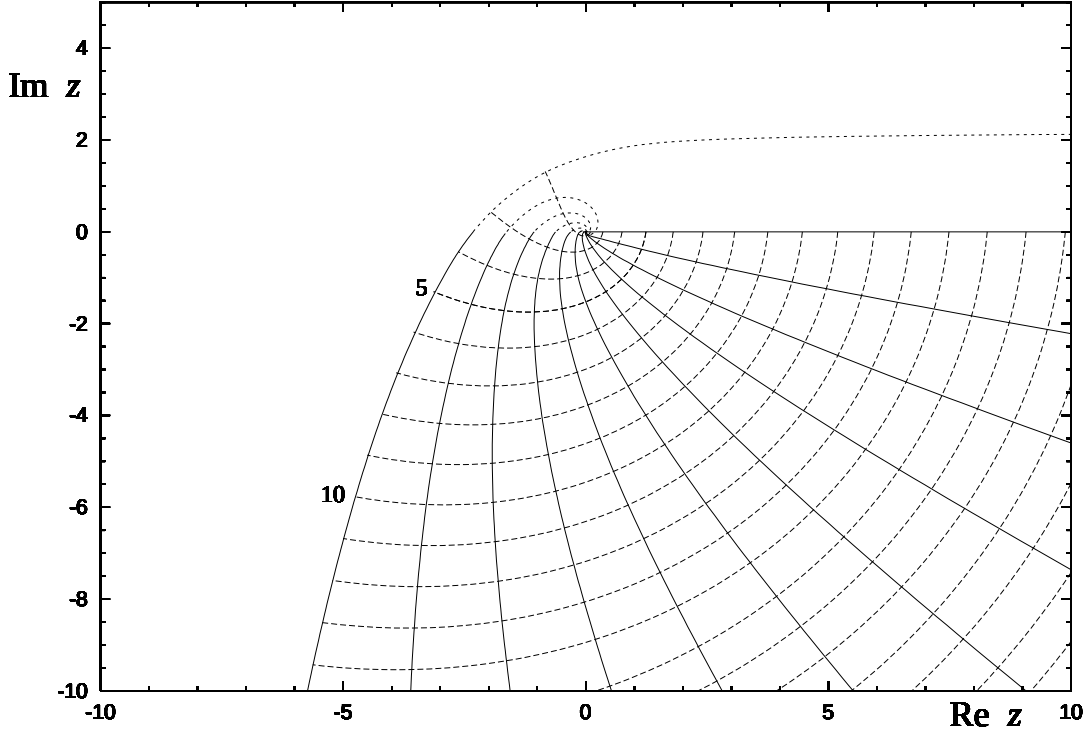


Fig. 2. Trajectories of the second zero, z_2 , of $K_\nu(z)$. The same convention as in Fig. 1 has been adopted to represent the behaviour of z_2 . The trajectory for real ν goes to $(+\infty, 3\pi/4)$, with $\arg z_2 \rightarrow -2\pi$, for $\nu = 3/2$.

of the zeros, for a given ν , were obtained by extrapolation of the trajectories followed by the zeros as ν varies. Besides the smallness of the values of the real and imaginary parts of $\mathcal{E}_\nu(z)$, their changes of sign in the successive steps of the Newton method make us to be confident about our results.

Our figures show that for large $|\nu|$, with $\arg \nu \in (0, \pi/2)$, the zeros lie far from the origin in the half plane $-\pi < \arg z < 0$, according to Eq. (18). When $|\nu|$ decreases, they approach the origin but, before reaching it, they cross the semiaxis $\arg z = -\pi$, leaving in this way the principal Riemann sheet $-\pi < \arg z \leq \pi$, for a certain $|\nu_{\text{crit},s}|$ whose value depends on $\arg \nu$ and on the order s of the zero z_s . The smaller $\arg \nu$ (between 0 and $\pi/2$) and the higher the label s of the zero, the larger the value of $|\nu_{\text{crit},s}|$. (In the case of real ν , it can be shown that $\nu_{\text{crit},s} = 2s - 1/2$.) Therefore, except for the case of pure imaginary ν , the number of zeros of $K_\nu(z)$ in the principal Riemann sheet is finite: only those zeros with label $s = 1, 2, \dots$ such that $|\nu_{\text{crit},s}| < |\nu|$, for the corresponding value of $\arg \nu$, remain in that principal Riemann sheet. If $|\nu|$ decreases again, being $\arg \nu \neq 0$, the zeros approach the origin spiraling clockwise infinitely, according to Eqs. (11) and (12). The case of real ν deserves a particular consideration.

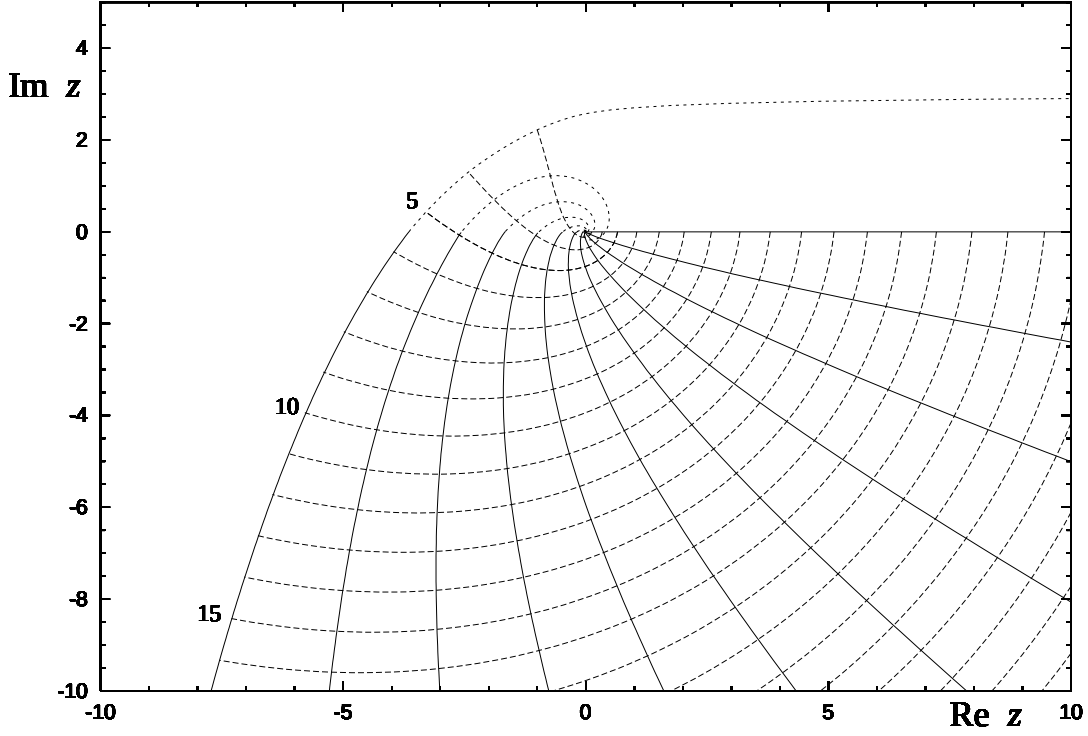


Fig. 3. Trajectories of the third zero, z_3 , of $K_\nu(z)$. The lines represent the behaviour of z_3 with the same convention as in Fig. 1. The trajectory for real ν goes to $(+\infty, 5\pi/4)$, with $\arg z_3 \rightarrow -2\pi$, for $\nu = 5/2$.

5 Zeros of $K_\nu(z)$ for real ν .

Let us now concentrate on the particular case of ν being real. According to the second of Eqs. (1), the zeros of $K_\nu(z)$ appear in complex conjugate pairs. We refer, in what follows, only to those zeros of negative argument. Let us also recall that, for half-integer ν , $\nu = n + \frac{1}{2}$, the Macdonald function reduces to a factor, different from zero in the finite plane, times a polynomial of degree n . Therefore, $K_\nu(z)$ presents exactly n z -zeros in each one of the Riemann sheets. Of course, the locations of these zeros are the same in all sheets.

The approximate expression (18) for the first zeros of $K_\nu(z)$ ($\nu \gg 1$) is valid also in this case. Equation (21), instead, is not applicable because the assumption $|z| \ll |\nu|$, used in its derivation, is not justified. Also, Eqs. (11) and (12) are no more valid due to the possibility of $|z|$ becoming infinite for certain values of ν , a fact that invalidates Eq. (5), from which Eqs. (11) and (12) stem.

According to Eq. (2), the z -zeros of $K_\nu(z)$ are obtained from the z -zeros of $H_\nu^{(1)}(z)$ by a rotation of $-\pi/2$ in the complex z -plane. It is then immediate to deduce the behaviour of the zeros of $K_\nu(z)$ for varying real ν from a previous discussion [5,6] of the zeros of $H_\nu^{(1)}(z)$. For convenience of the reader, we recall

Table 1

Typical output of the Newton method applied to the location of the first, second and third zeros of $K_\nu(z)$, with $|\nu| = 21$, $\arg \nu = 7\pi/20$. Besides the successive approximations to the zeros, the values of $\mathcal{E}_\nu(z)$, given by Eq. (22), are shown.

$\Re z_1$	$\Im z_1$	$\Re \mathcal{E}_\nu(z_1)$	$\Im \mathcal{E}_\nu(z_1)$
14.00000000	− 8.60000000	− .127330E+01	− .817656E−01
14.02983390	− 8.68222831	.154771E+00	.778107E−01
14.02406924	− 8.67520518	.105223E−01	.979004E−03
14.02389450	− 8.67463545	− .594713E−04	.105606E−04
14.02389461	− 8.67463884	.429746E−07	− .440585E−08
$\Re z_2$	$\Im z_2$	$\Re \mathcal{E}_\nu(z_2)$	$\Im \mathcal{E}_\nu(z_2)$
10.90000000	− 8.00000000	− .151232E+01	− .835798E+00
10.98486986	− 7.94986297	− .232404E+00	.536993E−01
11.00073780	− 7.95926858	.207686E−01	− .296534E−01
10.99984823	− 7.95696547	.191900E−03	− .214639E−03
10.99983888	− 7.95694790	− .278481E−06	.649298E−06
10.99983889	− 7.95694795	.698624E−11	− .156303E−10
$\Re z_3$	$\Im z_3$	$\Re \mathcal{E}_\nu(z_3)$	$\Im \mathcal{E}_\nu(z_3)$
8.50000000	− 7.00000000	.741241E+00	.345969E+01
8.60000000	− 6.92107409	.978987E+00	.293843E+01
8.70000000	− 7.02107409	.658986E+00	.247911E+01
8.80000000	− 7.12107409	.598150E+00	.174818E+01
8.88169635	− 7.22107409	.831925E+00	.871598E+00
8.80046094	− 7.31734050	− .282066E+00	.243279E+00
8.82858965	− 7.32026067	.229670E−01	.745812E−01
8.82891727	− 7.32660986	.257068E−04	− .695341E−03
8.82889674	− 7.32655881	− .673649E−06	− .726223E−05
8.82889659	− 7.32655825	.622240E−10	.575485E−09

here, without demonstration, the main results of that discussion translated to those zeros in the lower half plane $\Im z < 0$. To be specific, let us focus on the trajectory followed by a zero, z_s , as ν decreases from $+\infty$ to 0.

For very large ν , z_s is given approximately by Eq. (18). It lies far from the origin in the quadrant $-\pi < \arg z < -\pi/2$. As ν decreases, it moves upwards

and to the right, reaching the horizontal semiaxis $\arg z = -\pi$ for $\nu = 2s - 1/2$. Then, it makes a nearly circular quarter of a turn, crosses the vertical semiaxis $\arg z = -3\pi/2$ for $\nu = s - 1/3$, and goes nearly horizontally towards $(+\infty, (2s - 1)\pi/4)$ as $\nu \rightarrow s - 1/2$ from above.

The behaviour of z_s for $\nu < s - 1/2$ is not uniquely defined. In fact, the implicit function $z_s(\nu)$ defined by the condition $K_\nu(z) = 0$ presents a logarithmic branch point at $\nu = s - 1/2$, and the values of z_s for $\nu < s - 1/2$ depend on the chosen branch. One can pass continuously from the values of z_s for $\nu > s - 1/2$ to those for $\nu < s - 1/2$ if one avoids the branch point by the usual procedure of adding a small imaginary part to ν as its real part crosses the value $s - 1/2$. But the resulting values of z_s for $\nu < s - 1/2$ depend on the sign of that imaginary part. Since this ambiguity in the values of z_s has been thoroughly discussed in Ref. [6], in the context of the z -zeros of $H_\nu^{(1)}(z)$, we do not consider necessary to pursue further.

Besides those described above, $K_\nu(z)$ presents an infinite set of zeros near and at the left of the vertical semiaxis $\arg z = -3\pi/2$ for every integer $\nu = n$ [33, p. 513]. As ν increases, they approach that semiaxis, cross it when $\nu = n + 1/3$, and move nearly horizontally to the right, going to infinity as $\nu \rightarrow n + 1/2$, which is a logarithmic branch point for the position of the zeros as a function of ν , as it was shown in Refs. [5,6] for the zeros of $H_\nu^{(1)}(z)$. The zeros make a discontinuous jump of $\pm i\pi/2$, according to how the branch point is circumvented, move nearly horizontally to the left, reach the vertical semiaxis $\arg z = -3\pi/2$ when $\nu = n + 2/3$, and go to the left, to end, for $\nu = n + 1$, at positions intermediate between those that they occupied for $\nu = n$. A figure illustrating that behaviour can be obtained by rotating Fig. 1 in ref. [5] by an angle of $-\pi/2$.

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