Coulomb Wave Functions PhL 12.15.08

Summary of this document 1
1. Relation to the Coulomb-potential radial Schrodinger equation. 4
2. Frobenius Solution to get indices 5
3. The F_L Solution 5
4. Show that the CWF equation reduces to the Confluent HG equation: 6
5. Computing the large-ρ limit of F_L : determining the value of C_L. 8
6. The G_L Solution. 11
7. Bateman to the rescue? 13
Reference: J. Math. Phys. 40, 6145 (1999); DOI:10.1063/1.533083 14
8. Goldberger Watson to the rescue? 14
Whitaker Functions 14
Goldberger Watson Comments 16

Milton Abramowitz did this Coulomb WF A&S Chapter 14 himself, page 538.

Summary of this document

1. First we show that the A&S Coulomb Wave Function equation (CWFE) in variable ρ and function w(ρ) is equivalent to the radial Schrodinger equation (RSE) for a V(r) ~ 1/r "Coulomb" central potential, in variable r and function χ(r), where the true SE radial function is $R_\ell(r) = \chi_\ell(r)/r$. The connection is that χ(r) = w(ρ) with no multiplying functional factors, where ρ = kr with ħk being the momentum of the colliding (or orbiting) particle of reduced mass μ. Whereas the RSE has a χ'(r) term, the CWFE has no w'(ρ) term. The dimensionless parameter η appearing in the A&S ODE is the same as n in Schiff, namely, $\eta = n = (\mu/\hbar^2)\, ZZ'e^2/k$.

2. Next, we note that the CWFE has a regular singularity at ρ=0. We attempt a **Frobenius** series solution at this point ρ=0 and compute the **indices** to be r = -L and L+1 (L = ℓ, both used below). AS claim that the CWFE has an irregular singularity at ρ = ∞ and is therefore not a **Fuchsian** ODE. We do not carry through the Frobenius method for recursion relations for reasons soon to be seen.

We know on general grounds that an ODE with a regular singularity at ρ=0 will have two independent solutions of the form $w_1 = \rho^{L+1}$*Mclaurin and $w_2 = w_1(\rho)\ln(\rho) + \rho^{-L}$ *Maclaurin' where w_1 converges around ρ=0 and the w_2 does not. This is the general notion for me of 1st and 2nd kind solutions. Both solutions are formally "regular" but AS refer to w_2 as "the irregular" solution just to distinguish the two solutions.

3. We then examine the first CWFE solution which is written $w_1 \sim \mathbf{F}_L = C_L\, \rho^{L+1}\, \Phi_L(\rho)$ with Φ being the Mclaurin series and C_L a special normalizing constant. Instead of solving for Φ directly, we digress as follows.

4. We prove that the CWFE is convertible to the Confluent Hyper Geometric Equation (CHGE). As above, the CWFE involves w(ρ) while the CHGE involves y(x). We find that the connection is then given

by $w(\rho) = x^{\ell+1}e^{-x/2}\ y(x)$ where x = 2iρ, a = L+1-iη and c = 2L+2. Note that in this case, there *are* multiplying functional factors. We use Maple to compute the required second derivative $w'' = \partial_\rho{}^2\ [x^{\ell+1}e^{-x/2}\ y(x)\ ]$ as part of this proof. If we take y(x) = M(a,c,x) [AS] = Φ(a,c,x) [Bateman], and knowing this is a Mclaurin series in x, we identify $F_L = C_L\ \rho^{L+1}\ \Phi_L(\rho) = C_L\ \rho^{L+1}\ e^{-x/2}\ \Phi(a,c,x)$, where we now have two different functions with the name Φ. We can conclude that $\Phi_L(\rho)$ is the Mclaurin series for the product of functions $e^{-x/2}\ \Phi(a,c,x)$, where again x=2iρ. For the record, then

$$w_1(\rho) \sim F_L(\rho) = C_L\ \rho^{L+1}\ \Phi_L(\rho) = C_L\ \rho^{L+1}\ e^{-i\rho}\ \Phi(L+1-i\eta,\ 2L+2,2i\rho) \qquad \rho = kr$$

where we see this Φ is evaluated at a purely imaginary argument. Since we know all about Φ from Bateman, we are happy to have this connection between F_L and Φ. By the way, since the CHGE has all real constants, $w_1^*(\rho)$ is also a solution,

$$w_1^*(\rho) \sim F_L^*(\rho) = C_L\ \rho^{L+1}\ \Phi_L^*(\rho) = C_L\ \rho^{L+1}\ e^{+i\rho}\ \Phi(L+1+i\eta,\ 2L+2,-2i\rho) \qquad \rho = kr$$

and this solution has e^{+ikr} dependence and appears in Schiff page 142 equation 21.18.

5. We then want to know the **large-ρ behavior of $F_L(\rho)$**. We know from Bateman that the way you do this is by using the relation Φ ~ Ψ + Ψ ' (shorthand for a long equation), and then insert the asymptotic forms for Ψ and Ψ '. When we do this lengthy calculation, we find that if we set the real normalizing constant $C_L(\eta)$ to a certain value, the limit is extremely simple:

$$F_L(\rho) \rightarrow \sin\theta_L = \sin\left[\rho - \eta\ln(2\rho) - L\pi/2 + \sigma_L\right] \text{ where } \sigma_L = \arg\Gamma(L+1+i\eta)$$

This limit is of course of interest in the Coulomb scattering problem. Each partial wave L has a different phase which comes from - Lπ/2 + σ_L where this latter symbol is called the **"Coulomb Phase Shift".**

6. We then study the second solution called G_L. We expect to see it as $w_2 = F_L \ln(\rho) + \rho^{-L}$ Mclaurin, but this is not G_L. Instead G_L is defined very strangely as:

$$G_L(\rho) = 2\eta/C_o{}^2(\eta)\ \left[\ F_L \ln(\rho) + \{\ln(2) + q_L(\eta)/\ p_L(\eta)\}\ F_L\ \right] + \rho^{-L}\ D_L(\eta) \text{ Mclaurin}$$

where we have *added in* a certain multiple of F_L. The reason for the obscure "constant" functions of η is not given and seems totally mysterious as this point. The Mclaurin series is called ψ(ρ) with some recursion relation for its coefficients. With this definition, we have no idea how we might compute the large-ρ limit, but A&S claim that, amazingly, it is this:

$$G_L(\rho) \rightarrow \cos\theta_L = \cos\left[\rho - \eta\ln(2\rho) - L\pi/2 + \sigma_L\right] \text{ where } \sigma_L = \arg\Gamma(L+1+i\eta)$$

7. Bateman produces a similar complicated function like G_L in his section on "the logarithmic case" on page 261. Bateman is discussing that the "second kind" solution is peculiar because the argument c in Φ(a,c,x) is a positive integer, a fact related (I think) to the fact that ρ=0 is a regular singular point. You see this same ln(x) factor in Bateman's result and you see the correct Mclaurin series, and thoughtfully

Bateman outlines how this complex formula can be derived. So the AS G_L is Bateman's second solution plus an admixture of F_L.

8. At this point we are still mystified by this complicated G_L functional form that is required to obtain the simple asymptotic behavior $G_L(\rho) \rightarrow \cos\theta_L$, but then **Goldberger and Watson** come to the rescue. This rescue first requires us to accept yet another conversion of the ODE to a form called **Whittaker's equation** (WE). If y(a,c,x) is a solution of the CHGE, then the corresponding solution of the WE is given by $f(x) = e^{-x/2}\, x^{\ell+1}\, y(a,c=2\ell+2,x)$. Recall that we had $w(\rho) = x^{\ell+1}e^{-x/2}\, y(x)$ where $w(\rho)$ was for CWFE and y(x) was for CHGE. So we now have $f(x) = e^{-x/2}\, x^{\ell+1}\, y(a,c,x) = w(\rho)$. So we see that the Whittaker functions f(x) are basically the *same* as the Coulomb Wave function functions $w(\rho)$ apart from the change of argument $x = 2i\rho$, so the Whittaker functions are good to deal with. On the other hand, our information store is mostly Bateman on the Φ and Ψ functions, so we want to relate the Whitakers to these. Instead of writing Whittaker functions as f(a,c,x), the notation $f_{\kappa,\mu}(x)$ is used, where $c = 1 + 2\mu$ and $a = 1/2 - \kappa + \mu$. In our case, this means we have $\mu = \ell+1/2$ and $\kappa = i\eta$. So we then have:

$M_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2}\, \Phi(1/2 + \mu-\kappa;\, 1+2\mu;\, z) \qquad = z_1 \qquad$ AS 13.1.32

$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2}\, \Psi(1/2 + \mu-\kappa;\, 1+2\mu;\, z) \qquad = z_5 \qquad$ AS 13.1.33

as our first and second kind Whittaker functions, so we know all about these functions! GW then define the following two functions (Y* ≠ complex conjugate of Y, but will be so in the large x limit!)

$Y = \text{phase} * e^{\pi\eta/2}\, W_{k,\mu}(x)$

$Y^* = \text{phase}' * e^{\pi\eta/2}\, W_{-k,\mu}(-x)$

Then, using our known properties of Ψ to get the large-x limits of these things , we find that

$Y \rightarrow i\, e^{-i\theta}$

$Y^* \rightarrow -i\, e^{+i\theta}$ = complex conjugate of Y

Then, using the same F_L we had in A&S, we can relate this to $\Phi(a,c,x)$ and thus to $M_{\kappa,\mu}(x)$. But, we know that Bateman gives us $\Phi \sim \Psi + \Psi'$ (same shorthand as above), which means $M \sim W + W'$ and we find this very simple result

$F_L = (Y+Y^*)/2 \rightarrow \sin\theta$

where the equality is not an asymptotic limit, but an exact fact. We then DEFINE G_L to be this

$G_L = (Y-Y^*)/2i \rightarrow \cos\theta \qquad$ so $\qquad Y = F_L + iG_L$ and $Y^* = F_L - iG_L$

where again the equality is not an asymptotic limit, but an exact fact. We now have an exact expression for G_L in terms of our familiar Ψ functions. In our case with c = integer, each Ψ function include both a $\Phi(a,c,x)\ln(x)$ term and a series term as in Bateman page 261. This then leads to the huge messy form of G_L as given in A&S. The point is that we define G_L as shown above so it has the limit $\cos\theta$, and then it comes out being the big mess shown in A&S when we just evaluate it as per Bateman. Then as shown

above we find that $F_L + iG_L = Y \sim W_{k,\mu}(x) \sim \Psi$ and are not surprised to find a simple integral representation for $F_L + iG_L$ as in AS.

The GW versions of F and G have an extra factor of $\sqrt{2/\pi}$ and $-\sqrt{2/\pi}$ which I have removed in the above discussion. Their method of analysis comes from a paper of Yost, Wheeler and Bright in 1936, so the Whittaker functions must have been in use by that time. Original Whittaker and Watson was 1909 (?).

So here is what we have learned. The F_L function is normalized so that it $\rightarrow \sin\theta$ in the large r limit appropriate for scattering. The "second solution" G_L is then carefully selected so it $\rightarrow \cos\theta$ in this same limit.

1. Relation to the Coulomb-potential radial Schrodinger equation.

The ODE is the radial SE using a Coulomb Potential 1/r. It is not the equation for $R_\ell(r)$ but rather for $\chi_\ell(r) = r\, R_\ell(r)$. This ODE appears on page 117 of Schiff as 19.2. There is no first derivative, sort of a rarity. We might as well write it out:

$$\chi'' + [k^2 - U(r) - \ell(\ell+1)/r^2]\,\chi = 0 \qquad U(r) = 2\mu/\hbar^2\, V(r) \qquad V(r) = ZZ'e^2/r \qquad k^2 = 2\mu E\hbar^2$$

where for proton-proton scattering you would put Z = Z' = +1, and $\ell = 0,1,2,3...$ Milton writes this as

$$w'' + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2]\, w = 0$$

so we have to do some more scaling to get into his form where $k^2 = 1$. Try $\rho = kr$ = dimensionless, and $w(\rho) = \chi(r)$ so that [and $r = \rho/k$ and $1/r = k/\rho$]

$$\partial_\rho\, w(\rho) = \partial_r \chi(r)\, dr/d\rho = 1/k\, \partial_r \chi(r) \qquad \text{or} \qquad w' = \chi'/k \qquad w'' = \chi''/k^2$$

so the second equation then becomes

$$\chi''/k^2 + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2]\,\chi = 0$$
$$\chi'' + [k^2 - 2\eta k^2/\rho - \ell(\ell+1)k^2/\rho^2]\,\chi = 0$$
$$\chi'' + [k^2 - 2\eta k/r - \ell(\ell+1)1/r^2]\,\chi = 0$$

which we compare to our starting point of

$$\chi'' + [k^2 - (2\mu/\hbar^2)\, ZZ'e^2/r - \ell(\ell+1)/r^2]\,\chi = 0$$

This comparison then shows that we need

$$2\eta k = (2\mu/\hbar^2)\, ZZ'e^2 \quad \Rightarrow \quad \eta = (\mu/\hbar^2)\, ZZ'e^2/k$$

Now recall from Schiff's collision chapter that we used

$$n = (\mu ZZ'e^2/\hbar^2)/k \qquad \text{// dimensionless, real, not an integer}$$

so we then just have $\eta = n$. When we find a solution $w(\rho)$, we say

$\chi(r) = w(kr)$ $\quad$ $R_\ell(r) = \chi_\ell(r) / r = w_\ell(kr)/r$

and n just appears in the result.

So, now that we know the connection to the SE and the Coulomb Problem, what are the solutions?

2. Frobenius Solution to get indices

If we try Frobenius like so

$w = \rho^a \Sigma_{k=0} a_k \rho^k$ $\quad$ $w'' + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2] w = 0$

we get this result

$$\Sigma_{k=0}\, a_k \{ (k+a)(k+a-1)\, \rho^{k+a-2} + \rho^{k+a} - 2\eta\, \rho^{k+a-1} - \ell(\ell+1)\, \rho^{k+a-2} \} = 0$$
$$\Sigma_{k=0}\, a_k\, \rho^{k+a-2} \{ (k+a)(k+a-1) + \rho^2 - 2\eta\, \rho - \ell(\ell+1) \} = 0$$

Question: what is the coefficient in this sum for the power ρ^{a-2} ? Only two terms in {..} are present for this power, and if we want this coefficient to be zero, we have to set

$(a)(a-1) = \ell(\ell+1)$ $\quad$ $\Rightarrow$ $\quad$ $a = \ell+1$ or $a = -\ell$

These are the "indices" as AS say in the first paragraph.

3. The F_L Solution

Notice that $\rho = 0$ is a "regular singular point" since at worst a double pole in the w factor. According to my general ODE notes, at such a point there should be two solutions of the following form:

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (c_0 \neq 0)\,, \qquad (2)$$

$$w_2(z) = g w_1(z) \ln(z - z_0) + (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n \quad (d_n \neq 0)\,, \qquad (3)$$

where in our case we will have $\rho_1 = \ell+1$ and $\rho_2 = -\ell$ and $z_o = 0$. We see from AS that

$w_1(\rho) = \rho^{\ell+1} \Phi_\ell(\rho)$

where Φ is a series like that shown above. We have to juggle things a bit however by changing the summation variable

$$\Phi_\ell(\rho) = \Sigma_{n=0} a_n \rho^n = \Sigma_{k=\ell+1} a_{k-\ell-1} \rho^{k-\ell-1}$$

then he defines

$$A^\ell_k = a_{k-\ell-1}$$

so in any event, we have our first "regular" solution which AS write this way

$$F_L = C_L \rho^{L+1} \Phi_L(\rho)$$

The next thing they do is select a specific value for C_L, but no reason is given! But I think I can now trace this down. First, our main connection between the coulomb wave function and the confluent is this:

$$F_L(\rho) = C_L \rho^{L+1} e^{-i\rho} M(L+1-i\eta, 2L+2; 2i\rho) \qquad \rho = kr$$

You have to wonder where that factor $e^{-i\rho}$ is coming from. Suppose we define $x = 2i\rho$, then we would have on the right above (ignoring constant factors)

$$x^{L+1} e^{-x/2} M(L+1-i\eta, 2L+2; x)$$

4. Show that the CWF equation reduces to the Confluent HG equation:

So maybe I should do the equation conversion myself! (2 hour penalty!) The CWF equation is this:

$$w'' + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2] w = 0$$

The CHG equation is this

$$xy'' + (c-x)y' - ay = 0$$

We can *try* in the CWF equation this form

$$w(\rho) = x^{\ell+1} e^{-x/2} y(x) \qquad x = 2i\rho$$

We need the second derivative of this thing, so make Maple compute it. But be careful:

$$\partial_\rho = d\rho/dx \, \partial_x = (2i) \partial_x$$

$$\partial_\rho^2 = (2i)^2 \partial_x^2 = -4 \partial_x^2$$

$$w(\rho)'' = -4\,\partial_x^2\,[x^{\ell+1}\,e^{-x/2}\,y(x)]$$

Our CWF equation then becomes this:

$$w'' + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2]\,w = 0$$

$$-4\,\partial_x^2\,[x^{\ell+1}\,e^{-x/2}\,y(x)] + [1 - 2\eta/\rho - \ell(\ell+1)/\rho^2]\,x^{\ell+1}\,e^{-x/2}\,y = 0$$

But replace $1/\rho = 2i/x$ so the above is then

$$-4\partial_x^2\,[x^{\ell+1}\,e^{-x/2}\,y(x)] + [1 - 4i\eta/x + 4\,\ell(\ell+1)/x^2]\,x^{\ell+1}\,e^{-x/2}\,y = 0$$

$$-\partial_x^2\,[x^{\ell+1}\,e^{-x/2}\,y(x)] + [1/4 - i\eta/x + \ell(\ell+1)/x^2]\,x^{\ell+1}e^{-x/2}\,y = 0$$

We want to claim that somehow this mess reduces to this form:

$$xy'' + (c-x)y' - ay = 0$$

Now for Maple, I define things as follows:

$$f = \partial_x^2\,[x^{\ell+1}\,e^{-x/2}\,y(x)];$$
$$g = [1/4 - i\eta/x + \ell(\ell+1)/x^2]\,x^{\ell+1}e^{-x/2}y(x)$$

Then our equation above says

$$-f + g = 0$$

But then I scale things like this

$$f1 = f/\,(x^{\ell+1}\,e^{-x/2})$$
$$g1 = g/\,(x^{\ell+1}\,e^{-x/2})$$

and our equation is then

$$-f1+g1= 0$$

Here is the entire Maple calculation based on this:

```
f := factor(diff(x^(L+1)*exp(-x/2)*y(x),x,x));
```

$$f := \frac{1}{4}\,\frac{x^{(L+1)}\,e^{\left(-\frac{1}{2}x\right)}\left(4L^2 y(x)+4L\,y(x)-4L\,x\,y(x)-4y(x)\,x+8L\,x\left(\frac{\partial}{\partial x}y(x)\right)+8\left(\frac{\partial}{\partial x}y(x)\right)x+y(x)\,x^2-4\left(\frac{\partial}{\partial x}y(x)\right)x^2+4\left(\frac{\partial^2}{\partial x^2}y(x)\right)x^2\right)}{x^2}$$

```
f1 := f/(x^(L+1)*exp(-x/2));
```

$$f1 := \frac{1}{4}\,\frac{4L^2 y(x)+4L\,y(x)-4L\,x\,y(x)-4y(x)\,x+8L\,x\left(\frac{\partial}{\partial x}y(x)\right)+8\left(\frac{\partial}{\partial x}y(x)\right)x+y(x)\,x^2-4\left(\frac{\partial}{\partial x}y(x)\right)x^2+4\left(\frac{\partial^2}{\partial x^2}y(x)\right)x^2}{x^2}$$

```
g1 := ( 1/4 -I*n/x + L*(L+1)/(x^2) )*x^(L+1)*exp(-x/2)*y(x)/(x^(L+1)*exp(-x/2));
```

$$g1 := \left(\frac{1}{4}-\frac{I\,n}{x}+\frac{L(L+1)}{x^2}\right)y(x)$$

```
h := x*simplify(- f1 + g1);
```

$$h := L\,y(x)+y(x)-2L\left(\frac{\partial}{\partial x}y(x)\right)-2\left(\frac{\partial}{\partial x}y(x)\right)+\left(\frac{\partial}{\partial x}y(x)\right)x-\left(\frac{\partial^2}{\partial x^2}y(x)\right)x-I\,y(x)\,n$$

```
collect(%,diff);
```

$$(-2+x-2L)\left(\frac{\partial}{\partial x}y(x)\right)+L\,y(x)+y(x)-I\,y(x)\,n-\left(\frac{\partial^2}{\partial x^2}y(x)\right)x$$

```
collect(%,y);
```

$$(L+1-I\,n)\,y(x)+(-2+x-2L)\left(\frac{\partial}{\partial x}y(x)\right)-\left(\frac{\partial^2}{\partial x^2}y(x)\right)x$$

The last line is then our final equation which is then:

(L+1-iη)y + (x-2L-2)y' - xy" = 0

xy" + (2L+2 - x) y' - (L+1-iη)y = 0

xy" + (c-x)y' -ay = 0

We conclude that

a = L+1-iη c = 2L+2

So our solution to the original CWF equation will be this:

$x^{\ell+1}\ e^{-x/2}$ M(L+1-iη; 2L+2; x)

But then we replace x = 2iρ and we have

$\rho^{L+1}\ e^{-i\rho}$ M(L+1-iη; 2L+2; 2iρ)

where I throw out a constant factor. If we multiply this by a "normalizing constant" C_L we get exactly the function A&S show for F_L on page 538.

5. Computing the large-ρ limit of F_L : determining the value of C_L.

F_L is defined as follows on p 538 (with $C_L(\eta)$ as stated in 14.1.7)

$$F_L = C_L(\eta)\, \rho^{L+1}\, e^{-i\rho}\, M(L+1-i\eta;\, 2L+2;\, 2i\rho) \qquad a = L+1-i\eta \qquad b = 2L+2 \qquad b-a = L+1+i\eta$$

What is the large-ρ limit of this thing? This is in Bateman but let's use AS p 508 top instead (for the limit of the confluent function M). Note that we are just using the usual Bateman idea that $\Phi = A\Psi + B\Psi'$ to get this limit! We take the upper sign in 13.5.1 because our arg(z) = π/2 which is in the first range.

$$M(L+1-i\eta;\, 2L+2;\, 2i\rho) / \Gamma(2L+2) \rightarrow e^{+i\pi(L+1-i\eta)} / \Gamma(L+1+i\eta)\, (2i)^{i\eta-L-1}\, (\rho)^{i\eta-L-1}$$

$$+\ e^{2i\rho} / \Gamma(L+1-i\eta)\, (2i)^{-L-1-i\eta}\, (\rho)^{-L-1-i\eta}$$

so we get

$$F_L / \Gamma(2L+2) \rightarrow C_L(\eta)\ \rho^{L+1}\ e^{-i\rho}\ \{\ 1/\Gamma(L+1+i\eta) * e^{+i\pi(L+1-i\eta)}\ (2i)^{-L-1+i\eta}\, (\rho)^{-L-1+i\eta}$$

$$+\ 1/\Gamma(L+1-i\eta) * (2i)^{-L-1-i\eta}\, (\rho)^{-L-1-i\eta}\ \ e^{2i\rho}\ \ \}$$

$$= C_L(\eta)\ \{\ e^{+i\pi(L+1-i\eta)} / \Gamma(L+1+i\eta) * (2i)^{-L-1+i\eta}\ (\rho)^{+i\eta}\ \ e^{-i\rho}$$

$$+\ 1/\Gamma(L+1-i\eta) * (2i)^{-L-1-i\eta}\, (\rho)^{-i\eta}\ \ e^{i\rho}\ \ \}$$

This needs some processing at this point. Write

$$i^\alpha = (e^{i\pi/2})^\alpha = e^{i\pi\alpha/2} \qquad \Rightarrow \qquad (i)^{-L-1+i\eta} = e^{i\pi(-L-1+i\eta)/2} = e^{-i\pi(L+1-i\eta)/2}$$
$$(i)^{-L-1-i\eta} = e^{i\pi(-L-1-i\eta)/2} = e^{-i\pi(L+1+i\eta)/2}$$

Then have

$$F_L / \Gamma(2L+2) = C_L(\eta)\ \{\ e^{+i\pi(L+1-i\eta)} / \Gamma(L+1+i\eta) * (2)^{-L-1+i\eta}\ (\rho)^{+i\eta}\ \ e^{-i\rho}\ e^{-i\pi(L+1-i\eta)/2}$$

$$+\ 1/\Gamma(L+1-i\eta) * (2)^{-L-1-i\eta}\, (\rho)^{-i\eta}\ \ e^{i\rho}\ \ e^{-i\pi(L+1+i\eta)/2}\}$$

$$= C_L(\eta)\ \{\ \ 1/\Gamma(L+1+i\eta) * (2)^{-L-1+i\eta}\ (\rho)^{+i\eta}\ \ e^{-i\rho}\ e^{+i\pi(L+1-i\eta)/2}$$

$$+\ 1/\Gamma(L+1-i\eta) * (2)^{-L-1-i\eta}\, (\rho)^{-i\eta}\ \ e^{i\rho}\ \ e^{-i\pi(L+1+i\eta)/2}\}$$

$$= 2\, C_L(\eta)\, \mathrm{Re}\ \{\ \ 1/\Gamma(L+1+i\eta) * (2)^{-L-1+i\eta}\ (\rho)^{+i\eta}\ \ e^{-i\rho}\ e^{+i\pi(L+1-i\eta)/2}\ \ \}$$

where in the second last line we have A + A* inside the {..}, hence 2 Re(A). Now we have to identify the precise phase inside

$2^{i\eta} = e^{i\eta \ln(2)}$ $\rho^{i\eta} = e^{i\eta \ln(\rho)}$ $e^{+i\pi(-i\eta/2)} = e^{\pi\eta/2}$

so we have

$= 2\, C_L(\eta)\, \mathrm{Re}\, \{\ 1/\Gamma(L+1+i\eta) * (2)^{-L-1}\ e^{i\eta \ln(2)}\, e^{i\eta \ln(\rho)}\ e^{-i\rho}\ e^{+i\pi(L+1)/2}\ e^{\pi\eta/2}\ \}$

$= 2\, C_L(\eta)\, \mathrm{Re}\, \{\ 1/\Gamma(L+1+i\eta) * (2)^{-L-1}\ e^{i\eta \ln(2) + i\eta \ln(\rho)\ -i\rho + i\pi(L+1)/2}\ e^{\pi\eta/2}\ \}$

$= 2\, C_L(\eta)\, \mathrm{Re}\, \{\ [\ 1/\Gamma(L+1+i\eta) * (2)^{-L-1}\, e^{\pi\eta/2}\ ]\ [e^{i\eta \ln(2\rho)\ -i\rho + i\pi(L+1)/2}\ ]\ \}$

where we have now isolated the phasor factor. Now one more step: write

$\Gamma(L+1+i\eta) = |\,\Gamma(L+1+i\eta)\,|\, e^{i\sigma}$ $\sigma = \arg\,\{\,\Gamma(L+1+i\eta)\}$ then have

$= C_L(\eta)\, \mathrm{Re}\, \{\ [\ 1/|\Gamma(L+1+i\eta)| * (2)^{-L}\, e^{\pi\eta/2}\ ]\ [e^{i\eta \ln(2\rho)\ -i\rho + i\pi(L+1)/2 - i\sigma}\ ]\ \}$

where we move the σ to the phase, and cancel factors of 2. But now the first [..] is real, so have

$= C_L(\eta)\ [\ 1/|\Gamma(L+1+i\eta)| * (2)^{-L}\, e^{\pi\eta/2}\ ]\ \cos[\ \eta \ln(2\rho) - \rho + \pi(L+1)/2 - \sigma\ )]$

Now we see that if C_L is defined as in 14.1.7

$C_L = 2^L\, e^{-\pi\eta/2}\, |\Gamma(L+1+i\eta)|/\,\Gamma(2L+2)$

We then find that

$F_L = \Gamma(2L+2) * \{2^L\, e^{-\pi\eta/2}\, |\Gamma(L+1+i\eta)|/\,\Gamma(2L+2)\} * [\ 1/|\Gamma(L+1+i\eta)| * (2)^{-L}\, e^{\pi\eta/2}\ ] * \cos[...]$

$= \cos[\varphi]$ where $\varphi = \eta \ln(2\rho) - \rho + \pi(L+1)/2 - \sigma = -\theta + \pi/2$

Now use the fact that $\cos(\pi/2 - \theta) = \sin(\theta)$ and we get our final answer

$F_L = \sin(\theta)$ $\theta = \rho - \eta \ln(2\rho) - L\pi/2 + \sigma$ $\sigma = \arg\,\Gamma(L+1+i\eta)$

and THIS is exactly the reason for setting C_L as shown ! Then the limit has unity coefficient and is just a sine! And of course in scattering theory it is this large ρ = kr limit that we really care about.

Is my result consistent with 14.5.1 of AS? Let's look at this in more detail. His expressions g and f (not to be confused with mine above!) are series in 1/ρ so I want to keep only the first term. The leading term in the f series is $f_0 = 1$. The leading term in the g series is $g_1 = b_0 \sim 1/\rho$. So the real first term of the large ρ limit from 14.5.1 is just this:

$F_L = \sin\theta_L = \sin\,[\rho - \eta \ln(2\rho) - L\pi/2 + \sigma_L\ ]$ where $\sigma_L = \arg\,\Gamma(L+1+i\eta)$

and luckily this is exactly my result.

6. The G_L Solution.

Recall that when you have a regular singular point at ρ=0, there are two regular solutions that I quoted above from my cribbed web ODE notes.

2.4. Regular Solution. Regular Singularities

When the coefficients $p(z)$ and $q(z)$ of the equation

$$w'' + pw' + qw = 0 \tag{1}$$

satisfy certain conditions, the Laurent series in (15), (16) or (17) of the last section happen to contain only a finite number of negative power terms. By extracting the lowest negative power factor and combining it in the factor $(z - z_0)^\rho$, we find

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (c_0 \neq 0) , \tag{2}$$

$$w_2(z) = g w_1(z) \ln(z - z_0) + (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n$$
$$(d_n \neq 0) , \tag{3}$$

where (3) includes the cases (16) and (17) of the last section. Solutions (2) and (3) are called *regular solutions*; they are characterized by the fact that the series in them are Taylor series. Substituting such solutions in (1), we obtain

So we expect our second regular solution to have this form:

$$w_{2,L} = k \{ F_L \ln(\rho) + \rho^{-L} \Sigma_{n=0} c_n \rho^n \}$$

If we shift the summation index to k = -L + n, the series can be written

$$\Sigma_{n=0} c_n \rho^n = \Sigma_{n=-L} c_{k+L} \rho^{k+L} \equiv \Sigma_{n=-L} a_k^L \rho^{k+L} = \psi \qquad \text{of 14.1.17}$$

So then we have

$$w_{2,L} = k \{ F_L \ln(\rho) + K \rho^{-L} \psi \}$$

where we add constant K which just redefines the series coefficients. We can always add some F_L to this thing and still have a second independent solution, so do this

$$G_L = w_{2,L} + k [\ln(2) + q/p] F_L = k F_L [\ln(2\rho) + q/p] + k K \rho^{-L} \psi$$

We now want to set these values for the constants

$k = 2\eta/C_o^2(\eta)$ // our same C_L thing, see 14.1.8 for this particular one

$kK = D \quad \Rightarrow \quad$ second term $= D\rho^{-L}\psi \equiv \theta_L$

Then we have

$$G_L = \{ 2\eta/C_o^2(\eta)\} \; F_L [\ln(2\rho) + q/p] \; + \theta_L$$

and where q and p and D are certain given expressions. So we have justified 14.1.14 from first principles on a general ODE. This is in fact the second "regular solution" but they like to call it "the irregular Coulomb Wave Function" to distinguish if from F_L which is the "regular Coulomb Wave Function" .

Now the next question of course is why do they select all these strange "constants"? It must involve some limit. Notice that the series ψ is not just some arbitrary series, it is a special series that solves the ODE!! We know that in the large ρ limit, we have $F_L \rightarrow \sin(\theta_L)$. so the big question is finding the limit of the series ψ! In other words, we have

$$G_L \rightarrow \{ 2\eta/C_o^2(\eta)\} \; \sin(\theta_L) [\ln(2\rho) + q/p] \; + \lim[\theta_L]$$

More theory is needed here so that we could justify the integral representations shown on page 539, and then these could be used to find the limit of G_L. In the end, I have no doubt whatsoever that all the tricky functions are chosen for exactly one reason: to get this limiting result

$G_L \rightarrow \cos(\theta_L)$ as shown in 14.5.2

which then is a nice match for

$F_L \rightarrow \sin(\theta_L)$

In fact, looking at the integral representations, we might say $H_L = F_L + iG_L$ and the limit we must have that $H_L \rightarrow \cos + i \sin = 1$! It would probably not be too hard to show this from 14.3.1, but how do you derive 14.3.1 ?? We know from our ODE1 theory that the wronskian for the CWF equation should be a constant, and we see that this comes out 1 in 14.2.4. Getting this to be 1 might explain some of the complicated factors which appear in the definition of G_L.

Is this going to be in Whittaker and Watson which I ordered yesterday? Here is their index:"

PART I. THE PROCESSES OF ANALYSIS

CHAPTER		PAGE
I	Complex Numbers	3
II	The Theory of Convergence	11
III	Continuous Functions and Uniform Convergence . . .	41
IV	The Theory of Riemann Integration	61
V	The fundamental properties of Analytic Functions; Taylor's, Laurent's, and Liouville's Theorems	82
VI	The Theory of Residues; application to the evaluation of Definite Integrals	111
VII	The expansion of functions in Infinite Series	125
VIII	Asymptotic Expansions and Summable Series	150
IX	Fourier Series and Trigonometrical Series	160
X	Linear Differential Equations	194
XI	Integral Equations	211

PART II. THE TRANSCENDENTAL FUNCTIONS

XII	The Gamma Function	235
XIII	The Zeta Function of Riemann	265
XIV	The Hypergeometric Function	281
XV	Legendre Functions	302
XVI	The Confluent Hypergeometric Function	337
XVII	Bessel Functions	355
XVIII	The Equations of Mathematical Physics	386
XIX	Mathieu Functions	404
XX	Elliptic Functions. General theorems and the Weierstrassian Functions	429
XXI	The Theta Functions	462
XXII	The Jacobian Elliptic Functions	491
XXIII	Ellipsoidal Harmonics and Lamé's Equation	536
	APPENDIX	579
	LIST OF AUTHORS QUOTED	591
	GENERAL INDEX	595

[NOTE. The decimal system of paragraphing, introduced by Peano, is adopted in this work. The integral part of the decimal represents the number of the chapter and the fractional parts are arranged in each chapter in order of magnitude. Thus, e.g., on pp. 187, 188, § 9·632 precedes § 9·7 because $9{\cdot}632 < 9{\cdot}7$.]

Probably it is NOT there, so I would need another Bateman like source on this subject.

These functions are used in nuclear collision analysis, and AS give one Bloch source for same, but they give no nice information source!

7. Bateman to the rescue?

Recall we have

$$F_L = C_L(\eta)\,\rho^{L+1}\,e^{-i\rho}\,M(L+1-i\eta;\,2L+2;\,2i\rho) \qquad a = L+1-i\eta \qquad c = 2L+2$$

Let's look again at page 260 where he discusses the special case where c = 2,3,4.... which is exactly our situation!! In this case, he directly states the second solution on page 261 as (13), and Bateman explains how you derive this result from an earlier integral representation. So I could probably make Bateman and A&S agree on this log solution thing. But that does not give me an explanation of the amazingly simple integral representation shown on page 539 as 14.3.1.

I just found this interesting paper abstract;

"The motivation for the present paper lies in the fact that the literature concerning the Coulomb wave functions $F_L(\varepsilon,\rho)$ and $G_L(\varepsilon,\rho)$ is a **jungle** in which it may be hard to find a safe way when one needs general formulas for the Coulomb wave functions with complex values of the variable ρ and the parameters L and ε. For the Coulomb wave functions and certain linear combinations of these functions **we discuss** the connection with the Whittaker function, the Coulomb phase shift, Wronskians, reflection formulas (L-->-L-1), integral representations, series expansions, circuital relations (ρ-->$\rho e^{+/-i\pi}$) and asymptotic formulas on a Riemann surface for the variable ρ. The parameters L and ε are allowed to assume complex values."

Reference: J. Math. Phys. 40, 6145 (1999); DOI:10.1063/1.533083

So there would be a place to look. A web scan shows that CWF is a very obscure topic, this function does not have a classical history behind it.

8. Goldberger Watson to the rescue?

Whitaker Functions

Once again, from AS we had:

$$F_L = C_L(\eta)\,\rho^{L+1}\,e^{-i\rho}\,M(L+1-i\eta;\,2L+2;\,2i\rho)$$

Look on Bateman page 248 to see how the Whittaker function is defined:

$$M(a;c;z) = z^{-c/2}\,e^{z/2}\,M_{\kappa,\mu}(z) \qquad a = 1/2 - \kappa + \mu \qquad c = 1 + 2\mu$$

$$\mu = (c-1)/2 \qquad \kappa = 1/2 + \mu - a = 1/2 + c/2 - 1/2 - a = c/2 - a$$

This is confirmed in AS 13.1.32 which I will now derive from the above

$$M(1/2 - \kappa + \mu;\,1 + 2\mu;z) = z^{-(\mu+1/2)}\,e^{z/2}\,M_{\kappa,\mu}(z)$$

In our application we have

$$a = L+1-i\eta \qquad c = 2L+2 \qquad \mu = L+1/2 \qquad \kappa = L+1 - (L+1-i\eta) = i\eta$$

so we have this connection

$$M(L+1-i\eta;\ 2L+2;z) = z^{-L-1}\, e^{z/2}\, M_{\kappa,\mu}(z)$$

$$M(L+1-i\eta;\ 2L+2;2i\rho) = (2i\rho)^{-L-1}\, e^{i\rho}\, M_{\kappa,\mu}(2i\rho)$$

Then we can make this connection:

$$F_L = C_L(\eta)\, \rho^{L+1}\ e^{-i\rho}\ \{\ (2i\rho)^{-L-1}\, e^{i\rho}\, M_{\kappa,\mu}(2i\rho)\ \}\ = C_L(\eta)\ \{\ (2i)^{-L-1} M_{\kappa,\mu}(2i\rho)\ \}$$

$$= C_L(\eta)\, 2^{-L-1}\, e^{-i\pi(L+1)/2}\ M_{i\eta,L+1/2}\,(2i\rho)$$

so I think the Whittaker of imaginary argument is our baby, just a constant out in front, and this function has more information! But Bateman only has a few integral representations! GW (181) does sort of confirm that $F_L \sim M_{i\eta,L+1/2}\,(2i\rho)$ where $z = 2i\rho$. In (183) you see that GW have normalized their F and G functions slightly differently from AS.

Now look at Goldberger Watson page 260 where we see just this value of μ. GW have switched the roles of symbols κ and k which adds some confusion. I will translate:

(174) $\kappa = i\,\eta$ // GW way $k = i\xi$

(172C) $\eta = M_r e_1 e_2/k$ $M_r = \mu$ = reduced mass $e_1 = Ze$ $e_2 = Z'e$
// so GW must be setting $\hbar = 1$

Now, what are two independent solutions?

$$M(1/2 - \kappa + \mu;\ 1 + 2\mu;z) = z^{-(\mu+1/2)}\ e^{z/2}\, M_{\kappa,\mu}(z)$$

$a = 1/2 - \kappa + \mu$ $c = 1 + 2\mu$ $c\text{-}a = 1/2 + \kappa + \mu$

Consider this thing:

$$M(1/2 + \kappa + \mu;\ 1 + 2\mu;-z) = (-z)^{-(\mu+1/2)}\ e^{-z/2}\, M_{-\kappa,\mu}(-z)$$

If we think of $M(1/2 - \kappa + \mu;\ 1 + 2\mu;z) = M(a,c,z) = y_1$ of Bateman page 252, then we would say that

$$y_1 = M(1/2 - \kappa + \mu;\ 1 + 2\mu;z)$$

$$y_3 = e^z\, M(1/2 + \kappa + \mu;\ 1 + 2\mu;-z)$$

Now for non-integral c, these two solutions are the same by Kummer's transform page 253. But in our case, $c = 2L+2 = 2,4,6....$ and so is a positive integer and we are in exactly the case discussed page 260.

But all we learn there is that y_2 does not exist. I suspect Kummer's is still true somehow, we we then have

$$M_{\kappa,\mu}(z) = z^{+(\mu+1/2)}\, e^{-z/2}\, M(1/2 - \kappa + \mu;\, 1 + 2\mu; z) = z^{+(\mu+1/2)}\, e^{-z/2}\, y_1$$

$$M_{-\kappa,\mu}(-z) = (-z)^{+(\mu+1/2)}\, e^{+z/2}\, M(1/2 + \kappa + \mu;\, 1 + 2\mu; -z) = (-z)^{+(\mu+1/2)}\, e^{+z/2}\, e^{-z}\, y_3$$

$$= (-z)^{+(\mu+1/2)}\, e^{-z/2}\, y_3 = (-z)^{+(\mu+1/2)}\, e^{-z/2}\, y_1$$

so if Kummer is still true, these are NOT lin indeps. Ah, but GW are using W, not M!

Now from AS 13.1.33 we have

$$M_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2}\, \Phi(1/2 + \mu - \kappa;\, 1 + 2\mu;\, z) = z_1 \qquad \text{AS } 13.1.32$$
$$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2}\, \Psi(1/2 + \mu - \kappa;\, 1 + 2\mu;\, z) = z_5 \qquad \text{AS } 13.1.33$$

and this agrees with Bateman p 264. Bateman refers to these solutions as $z_1$and z_5 as you see. Bateman lists off other solutions, but does not say which pairs are linearly independent, and the two solutions that GW are dealing with are z_6 and z_8 which are W's. If these were M's, they would be about the same due to Kummer's which is Bateman page 264 (7).

But really $W_{\kappa,\mu}(z)$ and $W_{-\kappa,\mu}(-z)$ are like $\Psi(1/2 + \mu - \kappa;\, 1 + 2\mu;\, z)$ and $e^z\, \Psi(1/2 + \mu + \kappa;\, 1 + 2\mu;\, -z)$ which are in fact y_5 and y_7 on Bateman page 258, and these are linearly independent, and we even get the Wronskian of these two functions as $K_{57} \neq 0$ QED.

Conclusion: I am now happy with GW's claim in their (175) as a general solution! Then (176a) of GW is really Bateman p 259 (7).

Goldberger Watson Comments // starting on page 260

Equation (171) is the confluent HG equation which has no first derivative term. He recasts it into Whitaker's standard form (173) using μ and κ constants instead of a,b where there is a change in functional form as noted in Bateman page 248. The solution to (171) are the Φ and Ψ of Bateman, but the solutions to (173) are the Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ which "correspond" to Φ and Ψ .

Equation (176a) is what Bateman p 259 (7) looks like if you put it in terms of the M and W functions. Equation (176b) then must has our friend $\Phi = M$ in the square bracket as the usual confluent HG power series.

We know that we can write asymptotic forms for Φ, in fact I just did that above in this document. This was based on the known large-z forms of Ψ. GW write down these two limits in terms of W in equations (177).

Next, they define two new functions called $Y_\ell(z)$ and $Y_\ell^*(z)$ where * does not mean CC. These Y's are just constants times the two W's of interest. The phase in (179) is the same as the phase σ_L I used in my computation above. We then get limits for Y and Y* as in (180) and we notice that both Y and Y* are *unimodular* functions in this limit, ie, they are just phases. So in the asymptotic limit (only), they are in fact CC of each other! Write them in this way

$Y \rightarrow i\, e^{-i\theta}$ $\quad\quad$ $Y^* \rightarrow -i\, e^{+i\theta}$

Now we back up a step and replace the W's in (176a) with Y and Y* and we get then the very simple result (181). So fine, we then have this result:

F_ℓ = what must be $\sqrt{2/\pi}\ F_L(AS) = 1/2\ \sqrt{2/\pi}\ [\ Y + Y^*]$

Thus we would find that

$F_L(AS) = 1/2\ [Y + Y^*] = 1/2\ [i\, e^{-i\theta} \ \ -i\, e^{+i\theta}] =$

$= 1/2\ [i\ (C\text{-}iS) - i(C+iS)] = 1/2\ [\ 2S] = S = \sin(\theta)$

The equality here would apply in general, but only in the limit are Y and Y* cc's.

Now (182) says

G_ℓ = what must be $-\sqrt{2/\pi}\ G_L(AS). = \sqrt{2/\pi}\ /(2i)\ *[\ Y^* - \ Y]$

Thus, we would find that

$G_L(AS) = 1/(2i)\ [\ Y\text{-}Y^*] = 1/(2i)\ [i\, e^{-i\theta} + i\, e^{+i\theta}] = 1/2[e^{-i\theta} + e^{+i\theta}] = 1/2*2C$

$= \cos(\theta).$

in agreement with everything I have done above.

Now comes our little bonus. we can write:

$F_L(AS) = 1/2\ [Y + Y^*]$
$G_L(AS) = 1/(2i)\ [\ Y\text{-}Y^*]$

$Y + Y^* = 2\ F_L(AS)$
$Y - Y^* = 2i\ G_L(AS)$

Add and subtract these last two lines to get

$Y = F_L(AS) + i\ G_L(AS)$
$Y^* = F_L(AS) - i\ G_L(AS)$

Thus, the integral representation in AS 14.3.1 is no longer a mystery. It is just an integral representation for $Y \sim W \sim \Psi$ so this thing is just a version of Bateman page 255 (2) where the integration variable t has been scaled z $t_{Bateman} = t_{AS}$. Then 14.3.2 is just that for a different Ψ, no problemo.

Comment: notice that the large-r limits of the solutions $F_L(AS)$ and $G_L(AS)$ are just sine and cosine functions of arguments which involve both kr and ln(kr), which is the slightly strange part, and the phase

dependence on L is two fold. One is the $-\pi L/2$ term you see in there, and the other is that σ_L which is arg($\Gamma(L+1+i\eta)$) which is a function of both L and energy $\eta = n$.

It is not clear yet exactly how you set up the Coulomb scattering problem, but you can see that these functions are going to be very relevant! So I think GW have made a contribution to my picture here, and of course W&W will also when that book arrives from Amazon.

GW quote their stuff from a paper by Yost, Wheeler and Bright in 1936. Remember that AS started in 1964, many years later, so you would think they would have cleaned things up a little better than this. From the GW presentation, we see that G_L is just ~ (Y-Y*) which is the difference between two scaled Ψ functions, but in AS we see G_L instead as some horrendous monster with 5 auxiliary functions! This is just the way it is when you write it in the official manner that you are supposed to write the second solution of an ODE. Again, AS have unimpressive references, 3 are to themselves.

Magnus and Oberhettinger, by the way, wrote their own German special functions books perhaps a little before the Bateman project which they were both principles in. You see references MO in GR for example. Most of GR's Whittaker references are from MO, some are from WH which is Whittaker and Watson, a few from Bateman which is their EH I. I wonder if MO continued to think of their own books as the main act, and Bateman as just some project they were paid to work on for a while.

Here is a nice list of Special Function book references:

http://www.ericweisstein.com/encyclopedias/books/SpecialFunctions.html