The Gamma Function PhL 1.10.10

(1) Euler's Integral: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ $\mathrm{Re}(z) > 0$

(2) Recursion: $\Pi(z) \equiv \Gamma(z+1) = z\ \Gamma(z)$ $\Gamma(n+1) = n!$ $\Gamma(n) = (n-1)!$
$n! = n\ (n-1)!$

(3) Extension to Re(z)<0: $\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z)$ $\Gamma(z+1)\Gamma(-z) = -\pi/\sin(\pi z) = z\,\Gamma(z)\,\Gamma(-z)$
" Euler's Reflection Formula" $\Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$

(4) Pole residues: $1/(2\pi i) \oint_{z=-n} dz\,\Gamma(z) = (-1)^n/n!$ // CCW loop

(5) Special values: $0! = \Gamma(1) = 1$ $\Gamma(1/2) = \sqrt{\pi}$ $\Gamma(3/2) = \sqrt{\pi}/2$ $\Gamma(-1/2) = -2\sqrt{\pi}$

$\Gamma(m+1/2) = \sqrt{\pi}\ 2^{-m}\,(2m-1)!!$ m=1,2,3...
$\Gamma(-m+1/2) = \sqrt{\pi}\ 2^m\,(-1)^m\,[(2m-1)!!]^{-1}$ m=1,2,3... 5!! = 5*3*1 6!! = 6*4*2

(5a) More double factorials:

$$n!! = \left(\frac{2}{\pi}\right)^{\frac{1-\cos\pi n}{4}} 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)$$

n even or odd

$n!! = 2^{n/2}\Gamma(n/2+1)$ $(n-1)!! = 2^{n/2}\,\Gamma(n/2+1/2)/\sqrt{\pi}$ n even
$n!! = 2^{n/2}\Gamma(n/2+1)\,(2/\pi)^{1/2}$ $(n-1)!! = 2^{(n-1)/2}\Gamma(n/2+1/2)$ n odd

$\sqrt{\pi}\ /[\Gamma(n/2+1)\,\Gamma(1/2 - n/2)] = (-1)^{n/2}\,(n-1)!!\,/\,n!!$ n even Smythe #1
$\Gamma(1/2 + n/2)\,/\,\Gamma(1+n/2) = (2/\sqrt{\pi})\ (n-1)!!\,/\,n!!$ n odd Smythe #2

(6) $|\Gamma(ix)|^2 = \pi/[x\ \mathrm{sh}(\pi x)]$ // just $-\pi/\sin(\pi z) = z\,\Gamma(z)\,\Gamma(-z)$ with $z = ix$

(7) Stirling: $\Gamma(x+1) = x! = \sqrt{2\pi x}\ x^x\ e^{-x}\ \{\ 1 + 1/(12x) + 1/(288x^2) + ...\ \}$

(8) Duplication formula: $2^{2x-1}\,\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\ \Gamma(2x)$

(9) Real analytic: $\Gamma(z^*) = [\Gamma(z)]^*$

(11) Flip Rule: $\Gamma(a)/\Gamma(b) = \Gamma(1-b)\sin(\pi b)\,/\,[\Gamma(1-a)\sin(\pi a)\ ]$

example 1: Pochhammer symbol $(b)_n = \Gamma(b+n)/\Gamma(b) = (-1)^n\,\Gamma(1-b)/\Gamma(1-b-n)$
example 2: $\Gamma(x)/\Gamma(x-n) = (-1)^n\,\Gamma(n+1-x)/\Gamma(1-x)$

(12) Digamma function: $\psi(z) = \Gamma'(z)/\Gamma(z) = (\ln\Gamma(z))'$ // see Bateman p 16 for rules about

A derivation of the Euler Reflection Formula

I do this because this is such an important Gamma formula.

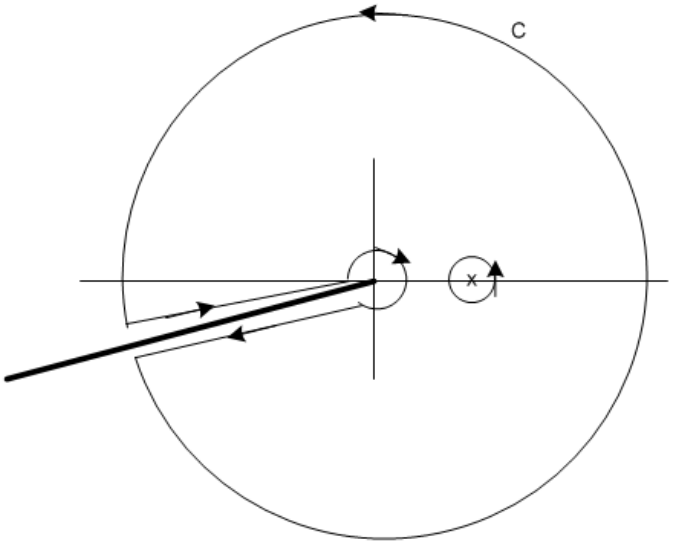
Summary: By considering the integral I shown below, evaluated two different ways, we are able to show that

$$\pi / \sin(\pi x) = \int_0^\infty dr\, r^{x-1}/(r+1)$$

Then from Bateman we identify the integral on the right above with $\Gamma(x)\Gamma(1-x)$, completing the derivation for real x. We then repeat the proof for complex z, or just continue the result off the real axis. This plan was presented in Special functions By George E. Andrews, Richard Askey, Ranjan Roy (Google scan).

So, consider this integral: $I = \int_C dz\, z^{x-1}/(z-1)$ $0 < x < 1$

Where the contour C is as shown:



This huge double loop contour can be deformed to be a simple CCW loop picking up the pole residue, and that gives

$$I = 2\pi i\ (1)^{x-1} = 2\pi i$$

As usual, we set $z = r\, e^{i\theta}$ and on loops we have $dz = r\, e^{i\theta}\, i d\theta$. Either loop integral has this form:

$$\int d\theta\, r\, e^{i\theta}\, i\, (r\, e^{i\theta})^{x-1} / (r\, e^{i\theta} - 1) = i\, r^x \int d\theta\, e^{i\theta(x-1)} / (r\, e^{i\theta} - 1)$$

For the inner loop, as $r \to 0$ we get

loop integral $= - i\, r^{x} \int d\theta\, e^{i\theta(x-1)} = r^{x} f(x)$

Since $x > 0$, this integral is 0.

For the outer loop, as $r \to \infty$ we get

loop integral $= i\, r^{x} \int d\theta\, e^{i\theta(x-1)}/(re^{i\theta}) = i\, r^{x-1} \int d\theta\, e^{i\theta(x-2)} = r^{x-1} g(x)$

Since $x < 1$, this integral is 0 (the great circle)

We are then left with just the integral of the discontinuity across the cut. Above the cut we have

$z = r\, e^{i\pi}$ above cut
$z = r\, e^{-i\pi}$ below cut

So the cut integral is this [now we have $z = r\, e^{\pm i\pi}$ so $dz = dr\, e^{\pm i\pi} = -dr$]

$$\int_{-\infty}^{0} dz\, \mathrm{disc}[\, z^{x-1}]/(z-1)$$

$$\int_{\infty}^{0} (-dr)/(-r-1) * [\, (r\, e^{i\pi})^{x-1} - (r\, e^{-i\pi})^{x-1}\,]$$

$$= - \int_{0}^{\infty} dr/(r+1) * r^{x-1} [\, e^{i\pi(x-1)} - e^{-i\pi(x-1)}\,]$$

$$= - 2i\sin(\pi[x-1]) \int_{0}^{\infty} dr/(r+1) * r^{x-1}$$

$$= + 2i\sin(\pi x) \int_{0}^{\infty} dr/(r+1) * r^{x-1}$$

Thus we have shown that

$$2\pi i = 2i\sin(\pi x) \int_{0}^{\infty} dr/(r+1) * r^{x-1}$$

$$\pi/\sin(\pi x) = \int_{0}^{\infty} dr\, r^{x-1}/(r+1) \qquad (**)$$

Now Bateman shows in full detail on page 9 that the Beta function is given by this integral

$$\int_{0}^{\infty} ds\, s^{x-1}/(s+1)^{x+y} = \Gamma(x)\Gamma(y)/\Gamma(x+y) \qquad // = B(x,y)$$

Therefore if we set x+y = 1 so that y = 1-x we have

$$\int_0^\infty dr\, r^{x-1}/(r+1) = \Gamma(x)\Gamma(1-x)$$

Notice that for $0 < x < 1$, this integral converges, so that is not an issue. Therefore, using (**), we have just derived the pretty famous Euler Reflection Formula for the Gamma function

$$\pi / \sin(\pi x) = \Gamma(x)\Gamma(1-x)$$

Our entire derivation goes through as long as $0 < \mathrm{Re}(x) < 1$ to get a complex result. Or we can take the real result and continue it off the real axis. In any event, our result is this:

$$\pi / \sin(\pi z) = \Gamma(z)\Gamma(1-z)$$

or

$$\Gamma(1-z) = \pi / [\, \Gamma(z) \sin(\pi z)\,] \qquad 1-z = -z' \text{ then } z' \to z \text{ to get:}$$

or

$$\Gamma(-z) = -\pi / [\, \Gamma(1+z) \sin(\pi z)\,] \qquad (*)$$

We can define $\Gamma(z)$ by the usual integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \mathrm{Re}(z) > 0$$

and then the formula (*) tells us the meaning of the Gamma function everywhere in the left have z plane. In particular, we can see that it has poles at z = 0, -1, -2 and so on from the $\sin(\pi z)$ function, and we could easily compute the residues of those poles (as on first page above).

Derivation of Smythe #1 and finding values for $P_n(0)$:

This arises in the context of computing $P_n(0)$. We so this with Bateman (22) which says

$$P_n(0) = \sqrt{\pi} / [\, \Gamma(1/2-n/2)\Gamma(1+n/2)\,]$$

If n is 1, 3, 5, the first gamma is $\Gamma(0)$, $\Gamma(-1)$, etc and has poles, so $P_n(0) = 0$ when n is odd.

For <u>n even</u>, we can apply one result above [the rest of this section is only value for n = even]

$$n!! = 2^{n/2}\Gamma(n/2+1)$$

which handles the second gamma so we have

$$P_n(0) = \sqrt{\pi}\, 2^{n/2} / [\, \Gamma(1/2-n/2)\, n!!\,]$$

Now we have to get rid of the -n/2 using

$$\Gamma(1/2\text{-}n/2)\Gamma(n/2+1/2) = \pi / \sin(\pi[1/2\text{-}n/2]) = \pi/\cos(-n\pi/2) = \pi/\cos(n\pi/2) = \pi / (-1)^{n/2}$$
$$z \qquad 1\text{-}z$$

which tells us that

$$\Gamma(1/2\text{-}n/2) = \pi\,(-1)^{n/2} / \Gamma(n/2+1/2)$$

But for n even we have a rule above which says

$$(n\text{-}1)!! = 2^{(n-1)/2}\Gamma(n/2+1/2)\,(2/\pi)^{1/2} = 2^{n/2}\,\Gamma(n/2+1/2)/\sqrt{\pi}$$

$$\Rightarrow\ 1/\,\Gamma(n/2+1/2) = 2^{n/2}/\,[\sqrt{\pi}\,(n\text{-}1)!!]$$

so we then have

$$\Gamma(1/2\text{-}n/2) = (-1)^{n/2} / \Gamma(n/2+1/2)$$

$$= \pi\,(-1)^{n/2}\ 2^{n/2}/\,[\sqrt{\pi}\,(n\text{-}1)!!] = \sqrt{\pi}\,(-1)^{n/2}2^{n/2} / (n\text{-}1)!!$$

and so we end up with

$$P_n(0) = \sqrt{\pi}\,2^{n/2} / [\,\Gamma(1/2\text{-}n/2)\,n!!\,]$$

$$= \sqrt{\pi}\,2^{n/2}\,(n\text{-}1)!!/\,[\sqrt{\pi}\,(-1)^{n/2}2^{n/2}\ \ n!!\,]$$

$$= (-1)^{n/2}(n\text{-}1)!!/\,n!! \qquad\qquad n \text{ even}$$

which is the correct result says Smythe. We can summarize by saying

$$\sqrt{\pi}\ /\,[\,\Gamma(1/2\text{-}n/2)\Gamma(1+n/2)\,] = (-1)^{n/2}(n\text{-}1)!!/\,n!! \qquad \text{// Smythe \#1}$$

Derivation of Smythe #2 and finding values for $Q_n(0+)$: // off the cut Q !

For $Q_n(0\pm)$ we use Bateman p 134 (40). Negative integers in the a,b position just cause F truncation, fine. Just set F=1 on both terms. We then have:

$$Q_n(0\pm) = \sqrt{\pi}\,2^{-1}\Gamma(1/2 + n/2)\,(\pm j)^{-n-1}/\,\Gamma(1+n/2) + 0$$

If we are coming down from above, the expo sign is – so we have $e^{-i\pi n/2} = (-j)^n$ and then

$$Q_n(0+) = (+j)^{-n-1}2^{-1}\sqrt{\pi}\ \Gamma(1/2 + n/2) / \Gamma(1+n/2) \qquad \text{// valid for all n}$$

For n even, we use our first n!! line on the front page, which says

$n!! = 2^{n/2}\Gamma(n/2+1)$ $\quad$ $(n-1)!! = 2^{n/2}\,\Gamma(n/2+1/2)/\sqrt{\pi}$ $\quad$ n even

and which tell us that

$\Gamma(n/2+1) = 2^{-n/2}\, n!!$

$\Gamma(n/2+1/2) = \sqrt{\pi}\ (n-1)!!\ 2^{-n/2}$

$\Gamma(1/2 + n/2) / \Gamma(1+n/2) = \sqrt{\pi}\ (n-1)!!\ 2^{-n/2}\ /\ 2^{-n/2}\, n!! = \sqrt{\pi}\,(n-1)!! / n!!$

and therefore

$Q_n(0+) = (+j)^{-n-1}2^{-1}\sqrt{\pi}\ \Gamma(1/2 + n/2) / \Gamma(1+n/2)$

$= (+j)^{-n-1}2^{-1}\sqrt{\pi}\,\sqrt{\pi}\,(n-1)!! / n!! = (+j)^{-n-1}2^{-1}\pi\ (n-1)!! / n!!$ $\quad$ n even

But if n is even we have

$(+j)^{-n} = (-j)^{n} = 1$ for n = 0, -1 for n = 2, and so on $\quad\Rightarrow\quad$ $(+j)^{-n} = (-1)^{n/2}$

so that $(+j)^{-n-1} = (-1)^{n/2}/j = -j\ (-1)^{n/2}$ and then

$Q_n(0+) = (-1)^{n/2}\,(-j\pi/2)\ (n-1)!! / n!!$ $\quad$ n even $\quad$ // agrees with Smythe

Above we found that

$P_n(0) = (-1)^{n/2}(n-1)!!/ n!!$ $\quad$ n even

so we can say that

$Q_n(0+) = (-j\pi/2)\ P_n(0)$ $\quad$ n even // agrees with Smythe

For n odd, we start over with our odd n data from above

$Q_n(0+) = (+j)^{-n-1}2^{-1}\sqrt{\pi}\ \Gamma(1/2 + n/2) / \Gamma(1+n/2)$ $\quad$ // valid for all n

$n!! = 2^{n/2}\Gamma(n/2+1)\,(2/\pi)^{1/2}$ $\quad$ $(n-1)!! = 2^{(n-1)/2}\Gamma(n/2+1/2)$ $\quad$ n odd

and which tell us that

$\Gamma(n/2+1) = (2/\pi)^{-1/2}\ 2^{-n/2}\, n!!$

$\Gamma(n/2+1/2) = 2^{-(n-1)/2}(n-1)!!$

$\Gamma(1/2 + n/2) / \Gamma(1+n/2) = 2^{-(n-1)/2}(n-1)!! \ / \ (2/\pi)^{-1/2}\, 2^{-n/2}\, n!!$

$= 2^{-n/2+1/2+1/2+n/2}\ \ (n-1)!! / [\ n!!\ \sqrt{\pi}\ ] \ = \ 2\ (n-1)!! / [\ n!!\ \sqrt{\pi}\ ]$

which we restate for copying to the front page

$\Gamma(1/2 + n/2) / \Gamma(1+n/2) = (2/\sqrt{\pi}\ )\ \ (n-1)!! / n!!$ n odd Smythe #2

and therefore

$Q_n(0+) = (+j)^{-n-1} 2^{-1} \sqrt{\pi}\ \ \Gamma(1/2 + n/2) / \Gamma(1+n/2)$

$= (+j)^{-n-1} 2^{-1}\ \sqrt{\pi}\ \ * 2\ (n-1)!! / [\ n!!\ \sqrt{\pi}\ ] \ = \ (+j)^{-n-1}\ (n-1)!! / n!!$

But now n is odd, so we have

$(+j)^{-n-1}$ = : n = 1 -1
n = 3 +1 etc

$(+j)^{-n-1} = (-1)^{(n+1)/2}$

and our final result is

$Q_n(0+) = (-1)^{(n+1)/2}\ (n-1)!! / n!!$ n = odd // agrees with Smythe

Summary of our results for P and Q at 0:

$P_n(0) =$ 0 n odd
$P_n(0) =$ $(-1)^{n/2}(n-1)!! / n!!$ n even

$Q_n(0+) = (-1)^{(n+1)/2}\ (n-1)!! / n!!$ n odd
$Q_n(0+) = (-1)^{n/2}\ (-j\pi/2)\ \ (n-1)!! / n!!$ n even

$Q_n(0+) = (-j\pi/2)\ P_n(0)$ n even

$P_n(0) / Q_n(0+) \ = \ (-j\pi/2)^{-1} = 2j/\pi$

[I think the above stuff is superseded by Legendre doc prop. -- 1.27.11]

Notes

(1) Recall from above that

$\Gamma(z+1)\Gamma(-z) = -\pi/\sin(\pi z) = z\,\Gamma(z)\,\Gamma(-z)$

If we set $z = (-1/2+i\tau)$ we get

$\Gamma(-1/2+i\tau+1)\Gamma(-[-1/2+i\tau]) = -\pi/\sin(\pi[-1/2+i\tau]) = (-1/2+i\tau)\,\Gamma(-1/2+i\tau)\,\Gamma(-[-1/2+i\tau])$

But we know that

$-\sin(\pi[-1/2+i\tau]) = \sin(\pi[1/2-i\tau]) = \cos(i\pi\tau) = \cosh(\pi\tau)$

Thus the above becomes

$\Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau) = (-1/2+i\tau)\,\Gamma(-1/2+i\tau)\,\Gamma(1/2-i\tau)$

So here then are two useful results, the first being the less obvious

$|\Gamma(1/2+i\tau)|^2 = \Gamma(1/2+i\tau)\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$

$(-1/2+i\tau)\,\Gamma(-1/2+i\tau)\,\Gamma(1/2-i\tau) = \pi/\cosh(\pi\tau)$

Here is Maple confirmation of the first result above

```
GAMMA(1/2+I*tau)*GAMMA(1/2-I*tau);
```

$$\Gamma\left(\frac{1}{2}+I\tau\right)\Gamma\left(\frac{1}{2}-I\tau\right)$$

```
f :=simplify(%);
```

$$f := -\frac{\pi}{\sin\left(\frac{1}{2}\pi\,(-1+2\,I\tau)\right)}$$

```
combine(f,trig);
```

$$\frac{\pi}{\cosh(\pi\,\tau)}$$