Deriving the Bateman P and Q table forms PhL 2.24.10

Bateman on pages 124 to 139 provides 18 different expressions each for P and Q. Each group of 18 can be divided into three groups of six. The "analysis" for the Q table entries is similar to that for the P, and we only talk about the P table entries below.

We will first discuss the first group of six "forms" (as I like to call the table entries) which arise from the triple linear transformations known as the "Kummer relations", then the last two groups of 6 involve the quadratic hypergeometric transformations which are discussed later.

The first six P expressions come from the linear Kummer transformations of the HGF. 1

Derivation of Bateman P table form (15) starting with (14) [Bateman p 124] 2

Comments regarding deriving (33) from (32) for the Q function 3

Quadratic Transformations 4

Which quadratic transformations give the other P forms in the Bateman P table? 5

Derivation of table entry P (20) from (14) 5

Derivation of the last group of six P table entries: Derivation of (28) from (14) 6

Summary to this Point 7

Attempt to prove that the two terms in each table entry separately solve the Legendre equation. 7

Conjectured proof for why each term in the Bateman table satisfies the Legendre equation. 10

Appendix A: (Quadratic) Derivation of (28) from (27) on page 65 (28) is also p 111 (3). 10

We have to start off in the Bateman HGF chapter.

The first six P expressions come from the linear Kummer transformations of the HGF.

Consider the information on pages 105-108. We start with F(a,b,c,z) = u_1. There are 6 forms for the "argument" of F which appear here are these: (there is a fancy name for these which I forget)

z
z/(z-1)
1-z
1-1/z
z^{-1}
1/(1-z)

We know that all the six u_i solve the HG ODE. Therefore, when we look at something like the Kummer relation p 107 (33), we may conclude that each of the three terms involved in any triple Kummer relation is *separately* a solution of the HF ODE. This idea is a major theme of this write-up. Each term in a Kummer relation solves the same HGE.

Now, let's change the names of the above arguments, and then let's assume that w = (1-z)/2 since this is what appears in our "definition" of the P function, p 124 (14). Maple then computes the expressions in the middle column:

w	(1-z)/2	(14)
w/(w-1)	(z-1)/(z+1)	(16)
1-w	(1+z)/2	(15)
1-1/w	(z+1)/(z-1)	(17)
w^{-1}	2/(1-z)	(19)
1/(1-w)	2/(1+z)	(18)

Given (14) for P, we may derive the other five expressions for P using selected triple Kummer relations. This is why all the other entries for P have two terms instead of one term as (14) has. This is not quite as automated as it seems, so we will do an example: we will derive (15) from (14). The reader can then derive any of the other four forms using the same method. I think the example reveals all the "tricks" that arise. I must say, it takes more work that I thought it would.

Derivation of Bateman P table form (15) starting with (14) [Bateman p 124]

So we start with (14) as the definition of P, and we want to derive (15). We need a Kummer relation which has this form:

$$F(....(1-z)/2\ ) = g_1(z)\ F_2(.....\ (1+z)/2\ ) + g_2(z)\ F_2(.....\ (1+z)/2\ )$$

To find the relation, we consult Bateman pages 105-108 where we back-shuffle the arguments as shown above. For example, we look for a Kummer relation of this form

$$\underset{u_1}{F(....(w)\ )} = g_1(w)\ \underset{u_2}{F_2(.....\ 1-w\ )} + g_2(w)\ \underset{u_6}{F_2(.....\ 1-w\ )}$$

We locate this as the "126" relation which is page 107 (33). It has the form

$$u_1 = \text{gammas} * u_2 + \text{gammas} * u_6$$

Some of the relations also have phases, but this one did not. [These single-signed phases arise when one loops around a branch point to generate some of the Kummer relations, I think.] We then go to page 105 and select (1) for u_1, but then we have two choices for u_2 and also for u_6. As a guide, we notice that the F's in (15) have exactly the same first two parameters as the term in (14). This suggests that we take the first u_2 expression since a,b are the same (and hopefully c will turn out right). Then for u_6 we take the second option (22), and things will hopefully work out. In terms of w, then , our "126" says this:

$$F(-\nu,\nu+1;\ 1-\mu;\ w) = g(a,b,c) * F(a,b,a+b+1-c;\ 1-w) + g'(a,b,c) * w^{1-c}\ (1-w)^{c-a-b} F(1-a,1-b,c+1-a-b;\ 1-w)$$

where the g functions are gamma function sets. Now we set a = -ν, b=ν+1 and c = 1-μ and hope:

$$F(-\nu,\nu+1;\ 1-\mu;\ w) = g(a,b,c) * F(-\nu,\ \nu+1,\ 1+\mu;\ 1-w) + g'(a,b,c) * w^{\mu}\ (1-w)^{-\mu}\ F(1+\nu,-\nu,\ 1-\mu;\ 1-w)$$

and, lo and behold, the two F functions have exactly the parameters we see in (15). Now we replace w = (1-z)/2 so the "out front powers" become : [note that (1-w) = (1+z)/2)]

$$w^{\mu}\,(1-w)^{-\mu} \;=\; (1-z)^{\mu}\,(1+z)^{-\mu} \;=\; (z+1)^{-\mu}\,(z-1)^{\mu}\,e^{\mp i\pi\mu}$$

where we had to flip the (1-z) factor around using Bateman p 123 (12), and this shows how those mysterious phase factors sometimes appear! So our equation above is now

$$F(-\nu,\nu+1;\,1-\mu;\,(1-z)/2) \;=\; g(a,b,c)\,*\,F(-\nu,\,\nu+1,\,1+\mu;\,(1+z)/2)$$
$$+\; g'(a,b,c)\,*\,(z+1)^{-\mu}\,(z-1)^{\mu}\,e^{\mp i\pi\mu}\,F(1+\nu,-\nu,\,1-\mu;\;(1+z)/2)$$

Now we grab our expression (14) as our starting point

$$P_{\nu}^{\mu}(z) = (z+1)^{\mu/2}(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\,F(-\nu,1+\nu,\,1-\mu,\,(1-z)/2) \qquad (14)$$

and we stuff in F from above to get,

$$P_{\nu}^{\mu}(z) = (z+1)^{\mu/2}(z-1)^{-\mu/2}\;[1/\Gamma(1-\mu)]\,g(a,b,c)\;F(-\nu,\,\nu+1,\,1+\mu;\,(1+z)/2)$$
$$+\;(z+1)^{\mu/2}(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\;\;g'(a,b,c)\,*\,(z+1)^{-\mu}\,(z-1)^{\mu}\,e^{\mp i\pi\mu}\,F(1+\nu,-\nu,\,1-\mu;\;(1+z)/2)$$

$$=\;(z+1)^{\mu/2}\;\;(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\,g(a,b,c)\;\;F(-\nu,\,\nu+1,\,1+\mu;\,(1+z)/2)$$
$$+\;(z+1)^{-\mu/2}(z-1)^{\mu/2}\;\;[1/\Gamma(1-\mu)]\;\;g'(a,b,c)\;\;\;e^{\mp i\pi\mu}\;\;\;F(-\nu,\,\nu+1,\,1-\mu;\;(1+z)/2)$$

and how you see all the features of table entry (15) starting to appear. Notice now the first factors have their signs reversed, just as in (15). The only thing remaining is fiddling with the gamma functions. From our "126" formula we had

$$g(a,b,c) = \Gamma(c)\Gamma(c-a-b)/\,[\,\Gamma(c-a)\Gamma(c-b)] \;=\Gamma(1-\mu)\Gamma(-\mu)/\,[\,\Gamma(1+\nu-\mu)\Gamma(-\mu-\nu)]$$

$$g'(a,b,c) \;=\Gamma(c)\Gamma(a+b-c)/\,[\,\Gamma(a)\Gamma(b)] \;=\Gamma(1-\mu)\Gamma(\mu)/\,[\,\Gamma(-\nu)\Gamma(\nu+1)] = -\,\sin(\pi\nu)\Gamma(1-\mu)\Gamma(\mu)/\,\pi$$

where we used the gamma rule $\Gamma(\nu+1)\Gamma(-\nu) = -\,\pi/\sin(\pi\nu)$ on the second line. When we now install our g and g', the $\Gamma(1-\mu)$ factors cancel in both terms, and we get

$$P_{\nu}^{\mu}(z) = \;\;(z+1)^{\mu/2}\;\;(z-1)^{-\mu/2}\;\;\Gamma(-\mu)/\,[\,\Gamma(1+\nu-\mu)\Gamma(-\mu-\nu)]\;\;\;F(-\nu,\,\nu+1,\,1+\mu;\,(1+z)/2)$$
$$+\;(z+1)^{-\mu/2}(z-1)^{\mu/2}\;\;[-\,\sin(\pi\nu)\Gamma(\mu)/\,\pi]\;\;\;\;e^{\mp i\pi\mu}\;\;\;F(-\nu,\,\nu+1,\,1-\mu;\;(1+z)/2)$$

and this is seen to exactly match (15).

Comments regarding deriving (33) from (32) for the Q function

The first six entries in the Q table mimic the first six entries in the P table, showing the same sequence of arguments as we listed above at the start, beginning with (1-z)/2.

Probably we would not derive things in this order, but if we wanted to, we could pretty much repeat what we did above. The main difference is that in (32) we have two terms, not one term, but the F functions are very similar and have those same a and b parameters. Much of the second term is the same as the first term with $\mu \rightarrow -\mu$. So each of the terms in (32) will generate two terms and then we will have four terms, but magically these will combine together and we end up with the two terms in (33). It happens that both terms in (33) have dual-signed phase factors. I have not done this derivation in detail, but I know it works because it would not be in the table if it did not work!

Quadratic Transformations

This large subject is addressed in three separate parts of the huge Bateman chapter on the HGF: page 64, 92 and 110. The formulas are in the p 110-113 area. Bateman notes that the quadratic transformations are only valid when the a,b,c of F(a,b,c,z) satisfy **certain conditions**. (There are other special conditions for having a cubic transformation.)

On page 64 Bateman quotes four quadratic transformations which are "the fundamental formulas of Gauss and Kummer". In three of these, the C parameter is related to the A and/or B parameters, but in the fourth (26) C is independent. I mention this for the following reason: the basic P representation (14) has C unrelated to A and B. However, that fourth (26) quadratic equation is not quite in the right form for use with our P, and it has the wrong arguments anyway. Bateman then goes on to state a fifth formula (28) which is the one I derive in Appendix A as an "exercise". This formula is restated as p 111 (3).

All I do is start with (27) and prove (28). I accept (27) as true in the limited region you read about below. Bateman gives a method of proving (27) in the text on page 65.

We now glance at the Bateman page 82 text on this subject. All he says here is that you can understand the quadratic and cubic transformations in terms of the P function. He does not prove or quote proofs of the P function equations he shows here.

Then on page 110 it seems strangely that Bateman starts all over again on this subject. Now he lists off not four, but 6 equations which are called "special transformations". He says that ALL quadratic transformations can be obtained from these 6 plus the usual linear Kummer relations. Three of the six equations are the same as three of the four equations back on page 64, while the other three are new. Only the 6th equation has the property mentioned above that C is decoupled from A and B.

Bateman then goes on to provide a huge list of Goursat's quadratic transformations, and then certain of his cubic transformations. Goursat's name is associated with this stuff for the following reason:
(typo: adment should say admet, admits for solution)

BIBLIOGRAPHY

I. Original Works. Goursat's doctoral thesis was "Sur l'équation différentielle linéaire qui adment pour intégrate la série hypergéometrique," in Annales scientifiques de l'École normale superieure, **10** , supp. (1881). 3-142. The

Amazingly, I found this in PDF at an obscure server, it is 148 pages long.
[http://gallica.bnf.fr/ark:/12148/bpt6k99753h . I looked through it, but the formulas are presented in a super dense notation with parameters A,B,C,ℓ,m,λ,μ and so on which are not defined in an obvious place, and the table is highly encoded. And of course it is in French, which slows me down a bit. So we are grateful to the Bateman folks for extracting the results!

In Appendix A I do a certain derivation of one formula from another. When I started on this, I thought this was going to be the quadratic transformation that let you derive the quadratic P table forms from the

linear ones, but it turns out that it is a *different* quadratic formula that does this! But as I did my derivation, curious details emerged which I think are worth keeping, hence we have Appendix A. One curious detail is that in certain cases where you have a relation between two F functions, the relation might not be valid in the entire intersection of the regions in which the separate F functions converge! I think this curious behavior is specific to some of the quadratic and cubic formulas.

Which quadratic transformations give the other P forms in the Bateman P table?

Page 112 (22) looks promising because the LHS is F(a, 1-a, c, w). We can set a = -ν, so 1-a = 1+ν, and set c = 1-μ and then we have a match. We then have to set w = (1-z)/2 which causes the right side F to have argument $1-z^2$ which is one of the forms in the table. So let's try it. Here are some calculations we will need:

$(1-w) = (1/2)(z+1)$

$a = -\nu$ $\qquad$ $1-a = 1+\nu$ $\qquad$ $c = 1-\mu$

$c/2-a/2 = 1/2-\mu/2+\nu/2$

$c/2+a/2-1/2 = 1/2 -\mu/2 - \nu/2 - 1/2 = -\mu/2 - \nu/2$ $\qquad$ $1-c = \mu$

$$F(a, 1-a, c,w) = (1-w)^{c-1} F(c/2-a/2, c/2+a/2-1/2; c; 4w(1-w))$$

$$F(a, 1-a, c, (1-z)/2) = 2^{1-c}(z+1)^{c-1} F(c/2-a/2, c/2+a/2-1/2; c; 1-z^2)$$

$$F(-\nu, 1+\nu, 1-\mu, (1-z)/2) = 2^{\mu}(z+1)^{-\mu} F(1/2-\mu/2+\nu/2, -\mu/2 - \nu/2; 1-\mu; 1-z^2) \qquad (*)$$

I am happy to report that the RHS F function matches P (20) ! So, let's formally derive (20) from (14):

Derivation of table entry P (20) from (14)

$$P_\nu^\mu(z) = (z+1)^{\mu/2}(z-1)^{-\mu/2} [1/\Gamma(1-\mu)] F(-\nu,1+\nu, 1-\mu, (1-z)/2) \qquad (14)$$

and we now use (*) to replace the F

$$P_\nu^\mu(z) = (z+1)^{\mu/2}(z-1)^{-\mu/2} [1/\Gamma(1-\mu)] \{2^{\mu}(z+1)^{-\mu} F(1/2-\mu/2+\nu/2, -\mu/2 - \nu/2; 1-\mu; 1-z^2)\}$$

$$= 2^{\mu} (z+1)^{-\mu/2}(z-1)^{-\mu/2} [1/\Gamma(1-\mu)] F(1/2-\mu/2+\nu/2, -\mu/2 - \nu/2; 1-\mu; 1-z^2) \qquad (20)$$

and we have derived (20) from (14). This has been "a long time coming", at least a full day to get to this point from starting this document. Notice in passing that I have written the correct complex outside function of z, which is put into "shorthand" in (20). I discuss this important detail elsewhere. The point being that $(z^2-1)^\alpha$ and $(z+1)^\alpha(z-1)^\alpha$ are *different* complex functions and it is the second that is implied. If you don't believe this, evaluate both sides in Maple for various z and see.

So, finally we have "made the connection" from the P table linear forms to one of the quadratic forms for P. This is of course a "single term" table entry for P. We can then use the Kummer linear relations to get many of the other quadratic forms in the Bateman table, since we have (see top of this doc!)

z^2	(22)
$z^2/(z^2-1)$	(25)
$1-z^2$	(20)
$1-1/z^2$	(24)
z^{-2}	(23)
$1/(1-z^2)$	(21)

Derivation of the last group of six P table entries: Derivation of (28) from (14)

There are still 6 more entries in the P table with more elaborate arguments. To handle these, we would try first to derive (28) which has a single term, then it would generate the remaining 5 by a list of related arguments similar to what I show above. A good-looking candidate quadratic transformation is page 112 (25) which might cut the mustard. The LHS is F(a, 1-a, c, -w) and the right side argument is

$$\text{arg} = 4\, w^{1/2}(w+1)^{1/2} \,/\, [\,(1+w)^{1/2} + w^{1/2}]^2$$

If we set -w = (1-z)/2 , we find first from Maple that

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w := (z-1)/2;
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$$w := \frac{1}{2}z - \frac{1}{2}$$

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4*sqrt(w)*sqrt(w+1) / (sqrt(w+1)+sqrt(w))^2:simplify(%);
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$$4\,\frac{\sqrt{2z-2}\,\sqrt{2z+2}}{(\sqrt{2z+2}+\sqrt{2z-2})^2}$$

which we evaluate by hand to set

$$= 2\sqrt{z^2-1} \,/\, [\, z + \sqrt{z^2-1}\,\,]$$

which is indeed the argument of table entry (28). In order to derive (28), we will need these calculations:

$$1+w \;= 2^{-1}(z+1)$$

$$(1+w)^{1/2} + w^{1/2} \;=\; 2^{-1/2}\,(\,\sqrt{z+1} \;+\sqrt{z-1}\,)$$

$$[(1+w)^{1/2} + w^{1/2}]^2 \;=\; 2^{-1}\,(\,\sqrt{z+1} \;+\sqrt{z-1}\,) \;= z + \sqrt{z^2-1}$$

then our quadratic formula page 112 (25) says this:

$$F(a,1-a,c,-w) = (1+w)^{c-1}\,((1+w)^{1/2} + w^{1/2})^{2-2a-2c}$$
$$F(c+a-1,c-1/2;2c-1;\, 4\, w^{1/2}(w+1)^{1/2} \,/\, [\,(1+w)^{1/2} + w^{1/2}]^2\,)$$

Let's get rid of all the w's in favor of z's using the results above

$$F(a,1-a,c,(1-z/2)) = 2^{1-c}(z+1)^{c-1}\,(z+\sqrt{z^2-1}\,)^{1-a-c}$$
$$F(c+a-1,\ c-1/2\quad 2c-1;\quad 2\sqrt{z^2-1}\,/\,[\,z+\sqrt{z^2-1}\ ]\,)$$

Now recall our result earlier computed,

$a = -\nu$ $\qquad 1-a = 1+\nu$ $\qquad c = 1-\mu$ $\qquad 1-c = \mu$

$c/2-a/2 = 1/2-\mu/2+\nu/2$

$c/2+a/2-1/2 = 1/2 -\mu/2 - \nu/2 - 1/2 = -\mu/2 - \nu/2$ $\qquad 1-a-c = 1+\nu-1+\mu = \nu+\mu$

so we then get (25) appearing in this form

$$F(-\nu,1+\nu,\ 1-\mu,\ (1-z/2)) = 2^{\mu}(z+1)^{-\mu}\,(z+\sqrt{z^2-1}\,)^{\nu+\mu}$$
$$F(-\nu-\mu,\ 1/2-\mu\ ;\ \ 1-2\mu\ \ ;\ \ 2\sqrt{z^2-1}\,/\,[\,z+\sqrt{z^2-1}\ ]\,) \qquad \text{p 112 (25)}$$

So let's now derive (28):

$$P_{\nu}^{\mu}(z) = (z+1)^{\mu/2}(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\,F(-\nu,1+\nu,\ 1-\mu,\ (1-z)/2) \qquad (14)$$

$$= (z+1)^{\mu/2}(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\,\{2^{\mu}(z+1)^{-\mu}\,(z+\sqrt{z^2-1}\,)^{\nu+\mu}\}\ \ x$$
$$F(-\nu-\mu,\ 1/2-\mu\ ;\ \ 1-2\mu\ \ ;\ \ 2\sqrt{z^2-1}\,/\,[\,z+\sqrt{z^2-1}\ ]\,)$$

$$= 2^{\mu}\,(z+1)^{-\mu/2}(z-1)^{-\mu/2}\,[1/\Gamma(1-\mu)]\,(z+\sqrt{z^2-1}\,)^{\nu+\mu}$$
$$F(-\nu-\mu,\ 1/2-\mu\ ;\ \ 1-2\mu\ \ ;\ \ 2\sqrt{z^2-1}\,/\,[\,z+\sqrt{z^2-1}\ ]\,) \qquad \text{P table (28) p 129}$$

So there we are, perfect. Then those remaining 5 would be linear Kummers on this one.

Summary to this Point

So, I have now "explained" all 3*6 = 18 entries in the Bateman P table. In each group of six, we start by deriving the "single term" entry and then from that we get the other five entries from linear Kummer relations.

Attempt to prove that the two terms in each table entry separately solve the Legendre equation.

My proof here does not succeed, but I think the claim is true, I have other evidence from Maple work at least concerning certain table entries.

So here begins my attempted proof. The single term entries are of the form P = (outside factors) * F, so we know that (outside factors) * F is certainly a solution of the Legendre ODE. Write P = f(z) F. Then we are saying

$$L_{leg}\,[f(z)\,F\,] = 0$$

and we know that

$L_{HG}\, F = 0.$

where HG is some particular form of the hypergeometric ODE which has parameters which match those of the F function for P. We then apply a linear Kummer which says $F = g_1(z)F_1 + g_2(z)\, F_2$, and we showed above that all three terms separately solve the same HGE, so we have

$L_{HG1}\, F = 0 \qquad L_{HG1}\, [g_1(z)F_1] = 0 \qquad L_{HG1}\, [g_2(z)F_2] = 0$

We would like to prove these facts:

$L_{leg}\, [g_1(z)F_1] = 0 \qquad L_{leg}\, [g_2(z)F_2] = 0$

but I don't now see any obvious way to show this. We have to somehow relate these differential operators. Here are the two operators:

$L_{HG(a,b,c,z)} = z(1-z)\partial_z^2 + [\, c - (a+b+1)z\,]\, \partial_z - ab$

$L_{leg(\mu,\nu,z)} = (1-z^2)\, \partial_z^2 - 2z\, \partial_z + [\nu(\nu+1) - \mu^2/(1-z^2)]$ //Schaum p 149

I might be able to push this through, but it sounds messy. Here is a possible start. Write

$L_{HG(a,b,c,w)} = w(1-w)\partial_w^2 + [\, c - (a+b+1)w\,]\, \partial_w - ab$

Then make the substitution $w = (z-1)/2$ since this is what we see in P (14) for example,

$w = (1-z)/2 \qquad 1-w = (1+z)/2 \qquad dw = -dz/2 \qquad \partial_w = -2\partial_z$

Then we get

$L_{HG(a,b,c,(1-z)/2)} = (1-z^2)\partial_z^2 - 2[\, c - (a+b+1)(1-z)/2\,]\, \partial_z - ab$

and this at least bears some resemblance to L_{Leg}. So write

$L_{leg(\mu,\nu,z)} = (1-z^2)\, \partial_z^2 - 2z\, \partial_z + [\nu(\nu+1) - \mu^2/(1-z^2)]$

Subtract them to get

$L_{HG(a,b,c,(1-z)/2)} - L_{leg(\mu,\nu,z)}$

$= -2[\, c - (a+b+1)(1-z)/2\,]\, \partial_z - ab + 2z\, \partial_z - [\nu(\nu+1) - \mu^2/(1-z^2)] \equiv \Delta L$

so they only differ by a first order operator. Now consider the bracket

$[\, c - (a+b+1)(1-z)/2\,]$

but lets set a,b,c as we see them set in the P (14) entry, (quoting from above)

$a = -\nu$ $\quad b = 1-a = 1+\nu$ $\quad c = 1-\mu$ $\quad 1-c = \mu$

$c/2-a/2 = 1/2-\mu/2+\nu/2$ $\quad a + b + 1 = -\nu + 1+\nu + 1 = 2$ $\quad ab = (-\nu)(1+\nu)$

$c/2+a/2-1/2 = 1/2 -\mu/2 - \nu/2 - 1/2 = -\mu/2 - \nu/2$ $\quad 1-a-c = 1+\nu-1+\mu = \nu+\mu$

Then that bracket is this:

$$[\, c - (a+b+1)(1-z)/2 \,] = [1-\mu - (\,2)(1-z)/2\,] = [1-\mu - (1-z)\,] = (z-\mu)$$

So we then get

$$\Delta L = -2[\, c - (a+b+1)(1-z)/2 \,]\,\partial_z - ab + 2z\,\partial_z - [\nu(\nu+1) - \mu^2/(1-z^2)]$$

$$= -2[(z-\mu)\,]\,\partial_z - ab + 2z\,\partial_z - [\nu(\nu+1) - \mu^2/(1-z^2)]$$

$$= -2z\partial_z + 2\mu\partial_z - ab + 2z\,\partial_z - [\nu(\nu+1) - \mu^2/(1-z^2)]$$

$$= 2\mu\,\partial_z - ab - [\nu(\nu+1) - \mu^2/(1-z^2)]$$

$$= 2\mu\,\partial_z + \nu(1+\nu) - [\nu(\nu+1) - \mu^2/(1-z^2)]$$

$$= 2\mu\,\partial_z + \mu^2/(1-z^2) \equiv L_1$$

which is still a first order linear operator. But <u>at least it is something fairly simple.</u> I define it as L_1. So we now have found that

$$L_{HG(a,b,c,(1-z)/2\,)} - L_{leg(\mu,\nu,z)} = L_1$$

or

$$L_{leg(\mu,\nu,z)} = L_{HG(-\nu,\nu+1,1-\mu,(1-z)/2\,)} - L_1 \qquad L_1 = 2\mu\,\partial_z + \mu^2/(1-z^2)$$

or

$$L_{HG(-\nu,\nu+1,1-\mu,(1-z)/2\,)} = L_{leg(\mu,\nu,z)} + L_1$$

Now looking back above, recall that we said that our Kummer replacement resulted in:

$$L_{HG1}\,F = 0 \qquad L_{HG1}\,[g_1(z)F_1] = 0 \qquad L_{HG1}\,[g_1(z)F_1] = 0$$

and this HG is exactly the HG we are talking about above. So we then know that

$$L_{HG(-\nu,\nu+1,1-\mu,(1-z)/2\,)}\,[g_1(z)F_1] = 0$$

so we then know that

$$L_{leg(\mu,\nu,z)}\,[g_1(z)F_1] = -L_1\,[g_1(z)F_1]$$

which is a great improvement in our knowledge, but I don't know where to go next. It is not obvious that the RHS is going to vanish. It would seem to depend on which Kummer formula you used. But we do know these three facts:

$L_{leg(\mu,\nu,z)}\ [g_1(z)F_1] = -L_1\ [g_1(z)F_1]$
$L_{leg(\mu,\nu,z)}\ [g_2(z)F_2] = -L_1\ [g_2(z)F_2]$
$L_{leg(\mu,\nu,z)}\ [F] = -L_1\ [F]$ Kummer: $F = g_1(z)F_1 + g_2(z)F_2$

Well, this is just not panning out, like so many things I spend a half day on. So let's put it to rest for now. I think the conclusion is true, but I don't have a proof. It could possibly not be true, but in some cases I have used Maple to confirm that it is true.

Partial proof: Since $L_1 = 2\mu\,\partial_z + \mu^2/(1-z^2)$, we see that for $\mu = 0$, we have $L_1 = 0$ and therefore

$$L_{leg(\mu,\nu,z)} = L_{HG(-\nu,\nu+1,1-\mu,(1-z)/2)}$$

and therefore

$$L_{leg(\mu,\nu,z)}\ [g_1(z)F_1] = L_{HG(-\nu,\nu+1,1-\mu,(1-z)/2)}\ [g_1(z)F_1] = 0$$

so we have in this case in fact proven that each of the two terms in all the 2-term expressions for P and Q separately solve the Legendre equation! So if we are looking for counterexamples, we need $\mu \neq 0$!

Conjectured proof for why each term in the Bateman table satisfies the Legendre equation.

I suspect that if you consider the Legendre equation all by itself, you would find sets of triple relations which are very similar to the Kummer relations for the HG function. In these triple relations, you would say that each term separately satisfied the Legendre equation. And it is some of these triple relations which make up the P and Q tables. A triple relation might have the form P = (phase)P + AP + BQ which we would get by winding around a branch point. We know in general that doing this kind of winding brings in a mixture of the two independent solutions. If we have $Q = f_1 + f_2$, then this thing would be a triple relation involving P, f_1 and f_2. The 24 Kummer triple relations come from the P function transformation rules for the HGF, and of course we can also represent the Legendre functions as P functions and we will have a similar set of 24 "Kummer" transformations, I am pretty sure.

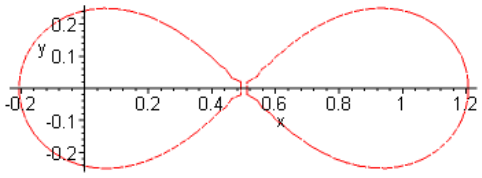
Appendix A: (Quadratic) Derivation of (28) from (27) on page 65 (28) is also p 111 (3).

This is one of what Bateman calls "six special transformations" on which all the others are based. All the others refers to the huge list that Goursat developed in his 1881 thesis,

So we start our derivation. Look at equation p 65 (27),

$F(2a,2b;\ a+b+1/2;\ w) = F(a,b;a+b+1/2;\ 4w(1-w))$ // p 65 (27)

The LHS converges inside the unit disk in w-space, while the RHS converges inside the following strange shaped curve

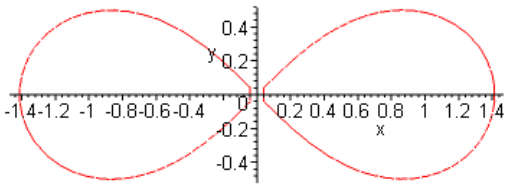


Maple reveals that the equation above is *only valid for w in the left lobe*! Bateman confirms this in his text on page 65 which I bracketed. So this is certainly a caution signal! One safe area is $|w| < .207$ which is an origin-centered circle inscribed inside the left lobe, Bateman gives the exact value of the radius. The idea is that convergence is only valid in the region which is connected to $z = 0$.

If we set $w = (z+1)/2$ [$z = 2w-1$] then we find $4w(1-w) = 1-z^2$. So rewrite the above as

$$F(2a,2b;\ a+b+1/2;\ (z+1)/2) \ = \ F(a,b;a+b+1/2;\ 1-z^2) \qquad \text{Re}(z) \le 1 \text{ only} \qquad (*)$$

The LHS has convergence in r=2 disk centered at $z = -1$. The RHS converges inside the following shape:



If we take the top lazy-8 curve and shift it to the left 1/2 unit and then scale it up by 2, we get the lower lazy-8 curve. Not surprisingly , Maple says the two sides agree only in the left lobe, which is to say, only when $z < 0$!!!! Another big caution signal saying "pay attention to this fact". In fact, Maple says we get agreement for any z such that $\text{Re}(z) \le 0$. So I have added that to the equation above. [And there is no agreement when $\text{Re}(z) > 0$].

The LHS of (*) is a solution of the HG ODE in variable $\xi = (z+1)/2$ with the given A,B,C values. Thus, so is the RHS. We now want to break the RHS into two terms using our "126" Kummer formula [which also appears as p 108 (1)]. So let's define $\psi = 1-z^2$ for the moment, and write p 108 (1) this way

$$F(a,b;C;\psi) \ = \ [\,\Gamma(C)\,\Gamma(C-a-b)/\,\Gamma(C-a)\,\Gamma(C-b)] \qquad F(a,b;\ a+b-C+1;\ 1-\psi\,)$$
$$+\,[\Gamma(C)\,\Gamma(a+b-C)/\,\Gamma(a)\,\Gamma(b)] \qquad (1-\psi)^{C-a-b} \quad F(C-a,C-b;\ C-a-b+1;\ 1-\psi\,)$$

I think the above result is valid for all ψ in the ψ-plane. I have it entered in Maple and cannot find any place where it is not true. Let's take it very slowly now. For the next step, replace $C = a+b+1/2$ since this is what we have on the RHS of (*) above (wait to see why). We will need these computations,

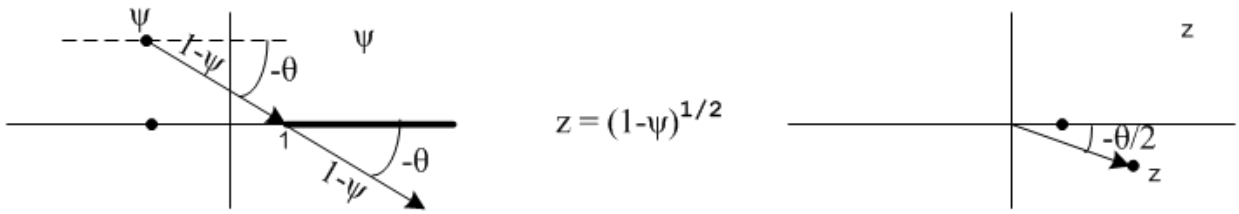
C -a-b = a+b+1/2 - a - b = 1/2 a+b-C+1 = -[C -a-b] + 1 = -[1/2] + 1 = 1/2
C-a = a+b+1/2 - a = b+1/2 C-b = a+1/2

and then we have

$$F(a,b;\, a+b+1/2;\, \psi) = [\,\Gamma(a+b+1/2)\,\Gamma(1/2)/\,\Gamma(b+1/2)\,\Gamma(a+1/2)]\; F(a,b;\, 1/2;\, 1-\psi) \quad (1)$$
$$+\; [\Gamma(a+b+1/2)\,\Gamma(-1/2)/\,\Gamma(a)\,\Gamma(b)]\; (1-\psi)^{1/2}\; F(b+1/2,\, a+1/2;\, 3/2;1-\psi)$$

Again, Maple probing suggests this is valid for any value of ψ. This is what we would expect of any of the linear Kummer relations.

In the next step we want to replace ψ by $\psi=1-z^2$ which means $1-\psi = z^2$. Now comes the tricky part. Consider the function $z = (1-\psi)^{1/2}$. Here are the complex planes:



We know there are two sheets in the ψ plane. *Normally* we pick as the sheet of interest the one which makes z be real and *positive* when ψ is to the left of the branchy point (black dots on real axes). If we take real positive values for z^2, we will certainly have $\psi = 1-z^2$ being on the real axis to the left of the branch point shown. So, if we take the "normal" sheet of $z = (1-\psi)^{1/2}$, we will obtain positive real z values when ψ wanders around to the left of the branch point. In fact, as ψ wanders over the entire ψ plane, meaning that angle θ will range from -π to π in the left picture, angle θ/2 will range from -π/2 to π/2 in the right picture, which means that for all ψ, we will get some z with Re(z) > 0, if we choose this normal branch.

However, in equation (*) above, we know we only have equality for Re(z) ≤ 0! Since we are about to use (*) in our next step, we therefore must take the *other* branch of $z = (1-\psi)^{1/2}$ which means z has negative real values for ψ to the left of the branch point. We write this as $z = -(1-\psi)^{1/2}$ so that a program like Maple will then evaluate this thing as a negative real number for example when ψ = 0.

So the upshot is this. Equation (1) above is valid for all ψ in the ψ plane. We then change variables from ψ to $z = -(1-\psi)^{1/2}$ as just described. We then have $\psi = 1-z^2$ on the left of (1), and $1-\psi = z^2$ on the right, and our result is this:

$$F(a,b;\, a+b+1/2;\, 1-z^2) = [\,\Gamma(a+b+1/2)\,\Gamma(1/2)/\,\Gamma(b+1/2)\,\Gamma(a+1/2)]\; F(a,b;\, 1/2;\, z^2) \quad (2)$$
$$+\; [\Gamma(a+b+1/2)\,\Gamma(-1/2)/\,\Gamma(a)\,\Gamma(b)]\; (-z)\; F(b+1/2,\, a+1/2;\, 3/2;\, z^2)$$

where the key idea is that we have replaced $(1-\psi)^{1/2} = -z$, not +z. Equation (2) is now valid only when Re(z) ≤ 0, and Maple confirms this, see supporting Maple file where I am doing all these things (see top of this doc for name).

Finally we use (*) to get the following, which appears as p 111 (3) in Bateman:

$$F(2a,2b;\, a+b+1/2;\, (z+1)/2) = [\,\Gamma(a+b+1/2)\,\Gamma(1/2)/\,\Gamma(b+1/2)\,\Gamma(a+1/2)]\; F(a,b;\, 1/2;\, z^2) \quad (***)$$
$$-\; [\Gamma(a+b+1/2)\,\Gamma(-1/2)/\,\Gamma(a)\,\Gamma(b)]\; z\; F(b+1/2,\, a+1/2;\, 3/2;z^2)$$

Our pathway to discovering this equation used $\mathrm{Re}(z) < 0$. But it turns out that once we have the result, it is valid for all z, and Maple *confirms* this, so there are no restrictions in Bateman.

Amidst all this detail, let's keep track of our theme idea which is that each of the three terms in the equation (***) above separately satisfies the same HGE. The reason is that this equation started as (*) which says F = F, so obviously both sides of (*) satisfy the same HGE. Then we broke the RHS of (*) into two pieces using a linear Kummer relation and we know that for any of these, all three functions solve the same HGE.

Theorem: In quadratic transformation (***) shown above, each of the three terms separately is a solution of the same HGE.