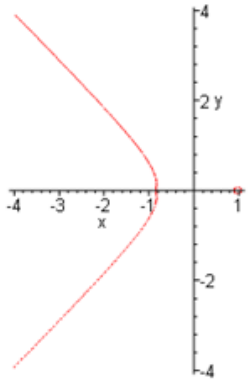
How Olver Obtains a large ν form for P and Q PhL 2.13.11

1. Bateman. If you try to find this large ν limit using Bateman's suggested (44), the thing is invalid near z = 1 so it is not useful to me. Here is how that goes:

Recall the general form of

$$F(a,b,c,z) = 1 + ab/c\ z/1! + a(a+1)b(b+1)/[c(c+1)]\ z^2/2! + \ldots$$

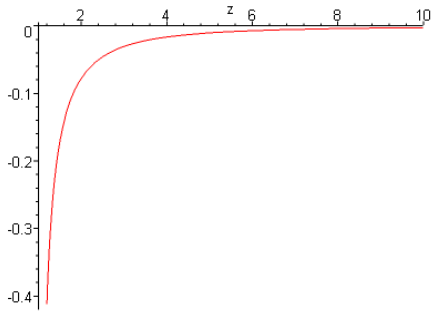
If we can find a form in which ν only appears in the c parameter, then that form IS a large ν asymptotic expansion. For Q, the only such form is (44) !! Bateman mentions this on page 62 in the large ν paragraph. But here is the convergence region for (44):

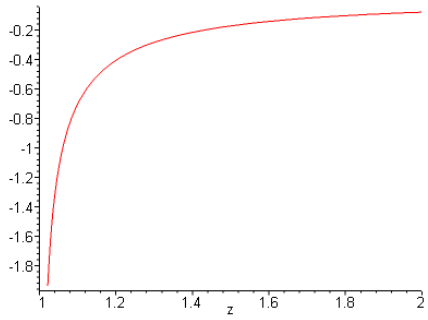


$(-z+\sqrt{z^2-1})/[2\sqrt{z^2-1}]$ (26) (44) right try z = 0

$(2\sqrt{z^2-1})/(-z+\sqrt{z^2-1})]$ (29) (47) left and z = 1try z=0

Notice the very strange contour around the point z = 1!!! We can get more detail on that by plotting the function shown above f := (-z+sqrt(z^2-1))/(2*sqrt(z^2-1)); for a few ranges of z>1





We can see from the right plot that we hit f = -1 and some value like z = 1.1.

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fsolve(f+1,z = 1.02..1.2);
                                    1.060660172
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[If you manually solve you find z = $3/\sqrt{8}$ = the number above.]

Therefore, our form (44) does not converge below z = 1.06, so our large ν expansion will be no good as we get close to z=1, horrors!!! And (47) does not have ν isolated into the c position. Therefore, we have no forms at all that give us what we want!!

2. How Olvers does it

We start on his page 463 (djvu 477) which is in Chapter 12. Warning: every chapter has its own equation (1.01), so when you hunt back, be sure you stay in Chapter 12!

He defines u = n+1/2 where n is what appears in the Legendre equation.

12.1 The associated Legendre equation is given by

$$(1-x^2)\frac{d^2L}{dx^2} - 2x\frac{dL}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}L = 0; \qquad (12.01)$$

see Chapter 5, (12.02). We seek asymptotic solutions of this differential equation for *large positive values of the degree n, and fixed real values of the order m*, neither *n* nor *m* necessarily being an integer. *Throughout the present section the argument x is assumed to be real.*

$$w = (x^2-1)^{1/2}L.$$

In 12.04 he seems to define ζ this way

$\zeta^{1/2} = \cosh^{-1}x$ where x is the usual x, for the case $x \geq 1$

He converts the Legendre equation to some other form. Olver is following a more general approach he started with generic f(x) and g(x) appearing in a certain ODE form which Legendre fits, see page 438 (d 452) The new ODE is this

$$W \equiv \left(\frac{\zeta}{x^2-1}\right)^{1/4} w = \{\zeta(x^2-1)\}^{1/4} L$$

satisfies the equation

$$\frac{d^2W}{d\zeta^2} = \left\{\frac{u^2}{4\zeta} + \frac{m^2-1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\right\} W, \tag{12.06}$$

in which

$$\psi(\zeta) = \frac{4m^2-1}{16}\left(\frac{1}{x^2-1} - \frac{1}{\zeta}\right). \tag{12.07}$$

where note that the top line is not two separate equations! So W is a rescaled version of the normal Legendre solution L. He turns his crank and ends up finally with this for Q

$$Q_n^m(\cosh\xi) = \frac{e^{m\pi i}u^m}{1+\delta_{2p+1}}\left(\frac{\xi}{\sinh\xi}\right)^{1/2}\left\{K_m(u\xi)\sum_{s=0}^{p}\frac{A_s^m(\xi^2)}{u^{2s}}\right.$$
$$\left. - \frac{\xi}{u}K_{m+1}(u\xi)\sum_{s=0}^{p-1}\frac{B_s^m(\xi^2)}{u^{2s}} + \eta_{2p+1,2}(u,\xi^2)\right\}, \tag{12.13}$$

where $u = n+\frac{1}{2}$,

where we are in the large n limit! But this is not very useful unless we know what those A and B coefficients are. Also, δ and η are certain quantities for which he presents upper bounds only. His corresponding P formula is this

$$P_n^{-m}(\cosh\xi) = \frac{1}{u^m}\left(\frac{\xi}{\sinh\xi}\right)^{1/2}\left\{I_m(u\xi)\sum_{s=0}^{p}\frac{A_s^{-m}(\xi^2)}{u^{2s}}\right.$$
$$\left. + \frac{\xi}{u}I_{m-1}(u\xi)\sum_{s=0}^{p-1}\frac{B_s^{-m}(\xi^2)}{u^{2s}} + \eta_{2p+1,1}(u,\xi^2)\right\}, \tag{12.16}$$

In his general theory on page d 454 he has

$$B_s(\zeta) = -A_s'(\zeta) + \frac{1}{\zeta^{1/2}}\int_0^\zeta \{\psi(v)\,A_s(v) - (v+\tfrac{1}{2})\,A_s'(v)\}\frac{dv}{v^{1/2}}, \tag{2.12}$$

$$A_{s+1}(\zeta) = vB_s(\zeta) - \zeta B_s'(\zeta) + \int \psi(\zeta)\,B_s(\zeta)\,d\zeta. \tag{2.13}$$

where prime means derivative. You start with $A_0(\zeta) = 1$ (constant). Then the first equation gives you B_0 as a pretty simple integral. You put that into the second equation to get A_1, and iterate in that manner. You can see that only the ψ function appears here, so this is a general class of coefficients, and it shows up in his adjusted ODE solution. For the Legendre case he rewrites them as

$$B_s^m(\zeta) = -A_s^{m\prime}(\zeta) + \frac{1}{\zeta^{1/2}} \int_0^\zeta \left[\frac{4m^2-1}{16} \left\{ \operatorname{csch}^2(v^{1/2}) - \frac{1}{v} \right\} A_s^m(v) - \left(m + \frac{1}{2} \right) A_s^{m\prime}(v) \right] \frac{dv}{v^{1/2}}, \tag{12.11}$$

$$A_{s+1}^m(\zeta) = m\{B_s^m(\zeta) + B_s^m(0)\} - \zeta B_s^{m\prime}(\zeta) + \frac{4m^2-1}{16} \int_0^\zeta \left\{ \operatorname{csch}^2(v^{1/2}) - \frac{1}{v} \right\} B_s^m(v)\, dv, \tag{12.12}$$

where again prime means derivative and you start with $A_0(\zeta) = 1$. Amazingly he does not state B_0 so we have to compute it like this

$B_0^m(\zeta) = 1/\sqrt{\zeta} \int_0^\zeta dv/\sqrt{v}\, [\, (4m^2-1)/16 * \{ \operatorname{csch}^2\sqrt{v} - 1/v\}$

$= (4m^2-1)/16 * 1/\sqrt{\zeta} \int_0^\zeta dv/\sqrt{v}\, \{ \operatorname{csch}^2\sqrt{v} - 1/v\}$

It must be that the B series is less important in the limit. Yes, both B series are down by $1/u = 1/n = 1/v$ for large values. So this then is where the AS results come from! I have verified that AS quoted them exactly right!