The Legendre SL Problem and Spherical Harmonics PhL 3.19.10

This subject is a lot harder than it seems. Maybe this time I will get through it.

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The Associated Legendre ODE

1. Consider the associated Legendre ODE operator from Schaum p 149.

$$L = (1-z^2)\,D^2 - 2z\,D + \{\, \nu(\nu+1) - \mu^2/(1-z)^2 \,\} \qquad \text{ODE:} \qquad Lu = 0$$

The solutions of this ODE are the usual $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ functions where ν and μ are arbitrary.

2. Suppose we now define a new differential operator,

$$L_1 = -(1-z^2)\,D^2 + 2z\,D + \mu^2/(1-z)^2 \qquad -L = L_1 - \nu(\nu+1)$$

We can then write our ODE Lu=0 in the following form:

$$L_1 u(z) = \nu(\nu+1)u(z) = \lambda\, u(z) \qquad \lambda = \nu(\nu+1)$$

It is exactly the same ODE, but it is now written in the form of an **eigenvalue equation** for the differential operator L_1.

3. We know that z = ±1 are regular singular points of either ODE (as is z = ∞).

The Associated Legendre Sturm-Liouville Problem

We can now define an **Associated Legendre Sturm-Liouville** ODE problem as follows:

$$L_1 u(z) = \nu(\nu+1)u(z) \qquad u(-1) = \text{finite} \quad u(1) = \text{finite.} \qquad \text{interval } (-1,1)$$

I have now studied the behavior of the Q and P functions at z = ± 1 in a separate document titled "Legendre functions evaluated at ...". In all situations we find that either F = 0 at an endpoint or F = ∞, with one exception which is $P_0^0(1) = 1$. [Here F stands for either P or Q]. We also learned that in the case

that Q = 0 at an endpoint, Q is basically the same as P. Therefore, we need only worry about the P functions. We also learned that these results are valid for but off-the-cut and on-the-cut P functions simply because things are either 0 or ∞, so the on-the-cut linear combination makes no difference. But in our SL problem, we expect to have real solutions to the problem, so we assume on-the-cut Bateman functions, at least so far.

I will now quote the essential P results from that document:

the P(+1) finite cases: $P_\nu^\mu(1) = 0$ in these cases [with exception $P_0^0(1) = 1$] , else $P_\nu^\mu(1) = \infty$

ν = anything:

Re(μ) < 0 and μ ≠ integer

μ = integer $P_\nu^\mu(1) = \delta_0^\mu$

the P(-1) finite cases: $P_\nu^\mu(-1) = 0$ in these cases, else $P_\nu^\mu(-1) = \infty$

ν = integer:

Re(μ) > 0 and μ ≠ integer

μ = 1,2,3.... // in this case, $P_\nu^\mu(z) \equiv 0$ for $-\mu \le \nu < \mu$

μ = -1, -2, -3 and ν = integer and ![$-|\mu| \le \nu < |\mu|$] // ie, ! [$\mu \le \nu < -\mu$]

μ = 0

spectrum: μ = integer and ν = |μ|, |μ|+1, ... can use either $P_\nu^\mu(z)$ or $P_\nu^{|\mu|}(z)$

$P_\nu^\mu(z) \equiv 0$: μ = 1,2,3.... ν = integer (-μ,μ-1) as noted 5 lines above

ν = special integrally-spaced values shown here:

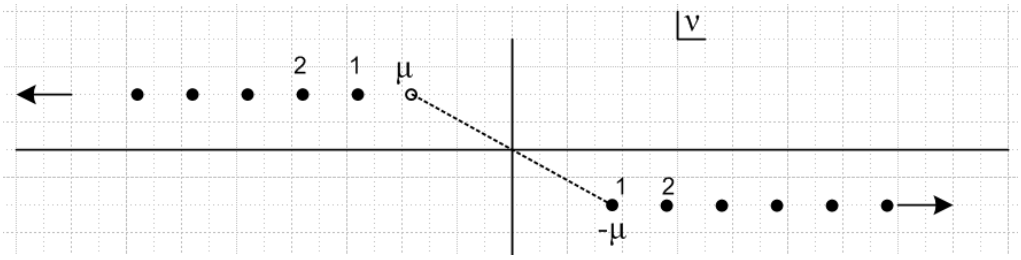
Re(μ) < 0 and μ ≠ integer and [ν = μ-J or ν = -μ+I]

spectrum: ν = -μ, -μ+1,... and use $P_\nu^\mu(z)$

Comments supporting the above. If we know that μ = integers, which is the case in many PDE problems, then P(1) = finite for all ν and we get no conditions on ν. But in order for P(-1) = finite, ν is restricted to a set of integers. Due to ν symmetry, we should include only positive ν values. For μ = 0,1,2... ν = 0,1,2... . But when μ is a negative integer μ = -|μ|, then we get ν = |μ|, |μ|+1, ... as the set of ν. We can summarize:

μ = 0,1,2...	ν = 0,1,2....	forced by P(-1) = finite
μ = -1,-2...	ν = |μ|, |μ|+1, ...	forced by P(-1) = finite

There is one *other* SL possibility. If Re(μ) < 0 and μ ≠ integer , we are OK at P(1) and we are also OK at P(-1) *provided* ν assumes one of the special values shown above. Here is a picture showing the location of the ν spectral values (black filled circles) for a given complex μ with Re(μ) < 0.



left string: $\nu = \mu - J$ right string: $\nu = -\mu + I$

The upper left string is really the $\nu \rightarrow -\nu-1$ symmetry string. For example, upper left point 1 is the symmetry value for the lower right point 1. We do not want to include P functions twice in a spectral situation, so we keep only the right side string.

We see that our case here really includes the case where μ = negative integer. So we could combine our two cases like this:

$P_\nu^\mu(-1) = 0$ in the following two situations:

1. $\mathrm{Re}(\mu) < 0$ $\nu = -\mu + I$ $I = 0,1,2....$
2. $\mu = 0,1,2...$ $\nu = I$

Examples and the Alternate P definitions.

Example 1: Suppose we have a φ situation which allows μ = integers as solutions.

$$\Phi_\mu(\varphi) = A_\mu \sin(\mu\varphi) + B_\mu \cos(\mu\varphi)$$

That is to say, the set of eigenvalues for the φ problem is "μ = all integers". In the Laplace PDE linked $z = \cos\theta$ SL problem which we are now studying, we see that for $\mu = 0,1,2,3$ we have $P_\nu^\mu(z)$ for any integer ν. We first remove redundancy by limiting this to $\nu = 0,1,2...$ Second, we note that $P_\nu^\mu(z) \equiv 0$ for $\nu = 0,1..\mu-1$ so in fact the spectrum is $\nu = \mu$, $\mu+1$ and so on. On the other hand for $\mu =$ -1, -2, -3 we know directly that the spectrum is then $\nu = -\mu$, $-\mu+1$, etc. For ν in the range $\nu = 0,1,...-\mu-1$, $P_\nu^\mu(-1) = \infty$ which is why we exclude such $P_\nu^\mu(z)$ from the spectrum. So the "correct" thing to do is say $\nu = |\mu|,|\mu|+1$, etc and this handles both cases. This is what Stakgold does on p 395.

You might then ask if it is correct to include all integer μ values in the spectrum of the PDE problem. The answer in general is yes, you must include them all. You could use $P_\nu^{|\mu|}(z)$ for either sign of μ if you want, and as Stakgold does. The reason you need both signs of μ is that $\Phi_\mu(\varphi) \neq \Phi_{-\mu}(\varphi)$ in the general case shown above. If we had $A_\mu = 0$, then we could and should throw out either the positive or negative values of μ since in that case $P\Phi$ would not be independent in the two cases.

Alternate P Definition for Certain Integral ν,μ values. We just showed above that $\nu = |\mu|,|\mu|+1...$ is the correct spectrum for this SL problem. *However*, some people prefer to say the spectrum is $\nu = 0,1,..$ etc and artificially *define* $P_\nu^\mu(z) \equiv 0$ for $\nu = 0,1.. -\mu-1$ when $\mu =$ -1, -2, -3 etc. I find this immensely confusing, but people do it, and Maple supports it. The idea is that we already have $P_\nu^\mu(z) \equiv 0$ truly for $\mu = 1,2,3...$ and $\nu = 0,1.. \mu-1$, and this sort of matches this fact for negative integral μ values.

Here is an example using $P_1^{-2}(z)$. This function is perfectly well defined and has finite values for all z except it blows up at z = -1. The alternate definition is to simply declare that $P_1^{-2}(z) \equiv 0$ since this is a "nonsense" pairing of ν,μ. This then matches the *true* fact that $P_1^{+2}(z) \equiv 0$. I will now show Maple evidence of this "alternate definition".

First, here is evidence that $P_1^{-2}(z)$ blows up at z = -1:

```
LegendreP(1,-2.00000001,-.9999999999);
```

$$-.3333333720\,10^{10} + 52.35988364\,I$$

```
LegendreP(1.00000001,-2,-.9999999999);
```

$$-.3333333272\,10^{10}$$

Second, here is evidence that $P_1^{-2}(z)$ is finite at a non-singular value of z:

```
LegendreP(1,-2.000000001,0.4);
```

$$-.1714285712 + .2692793700\,10^{-9}\,I$$

```
LegendreP(1,-2.00001,0.4);
```

$$-.1714264061 + .2692759690\,10^{-5}\,I$$

Notice that the finite value -.17 is "stable" when $\mu = -2 + \varepsilon$. And third, here is evidence that Maple uses the alternate definition if you set $\mu = -2$ exactly:

```
LegendreP(1,-2.0000,0.4);
```

$$0$$

```
LegendreP(1,-2.0000,-1);
```

$$0$$

Example 1A: Suppose we know that our φ solution is even. Then $\Phi_\mu(\varphi) = B_\mu \cos(\mu\varphi)$ and our spectrum can be either 0,1,2... or 0,-1,-2... . The usual thing to do is keep only $\mu = 0,1,2..$ and then we can start the ν spectrum either at $\nu = 0$ or at $\nu = \mu$ and we don't have to deal with the artificially defined P situations.

Example 2: The famous orange slice type problem. We find that $\Phi_\mu(\varphi) = A_\mu \sin(\mu\varphi)$ $\mu = \pm n\pi/\alpha$ where α is the angle of the slice (want V = 0 on both slice faces). In this case, we must take $\mu = 0, -\pi/\alpha, -2\pi/\alpha$ ... because we need $\mathrm{Re}(\mu) < 0$ for the non-zero eigenvalues so that $P(1) \neq \infty$. Then our ν spectrum is $\nu = -\mu+I$ and is thus different for each μ value. So let's write $\mu_k = -k\pi/\alpha$ $k = 0,1,2..$ Then $\nu = k\pi/\alpha + I$.

Let's go look this up in Stakgold and make sure it comes out this way! Well, in Stak is was an exercise 6.32 and the solution was not given and I sort of winged it. I added a note just now.

So a main point is that ν = integers is NOT the spectrum for the orange slice problem! I don't think Smythe does this problem, not sure about M&F. Could write $\Sigma_{\nu+\mu\ =0}^{\infty}\ \Sigma_{\mu\ =-k\pi/\alpha\ \ k\ =\ 1..\infty}$

Orthogonality and Completeness for our Two SL Cases:

The functions appearing here are the Bateman on-the-cut P functions, except where otherwise noted.

The Integer μ Case:

The first spectral case listed above is this:

spectrum: μ = integer and $\nu = |\mu|, |\mu|+1, \ldots$ can use either $P_\nu^\mu(z)$ or $P_\nu^{|\mu|}(z)$

$$\int_{-1}^{1} dz\, P_\nu^\mu(z) P_{\nu'}^\mu(z) = \delta_{\nu,\nu'}\, K_\nu^\mu \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty}(1/K_\nu^\mu)\, P_\nu^\mu(z')\, P_\nu^\mu(z) = \delta(z'-z) \qquad K_\nu^\mu = (\nu+1/2)^{-1}\, f(\nu,\mu)$$

The parameter μ is really a "bystander parameter" here, which we have assumed is an integer of either sign. If we like, we can certainly rewrite the above replacing μ by |μ| so we are just using the positive μ part of the above claim, then we get

$$\int_{-1}^{1} dz\, P_\nu^{|\mu|}(z)\, P_{\nu'}^{|\mu|}(z) = \delta_{\nu,\nu'}\, K'^{\mu}_{\nu} \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty}(1/K_\nu^\mu)\, P_\nu^{|\mu|}(z')\, P_\nu^{|\mu|}(z) = \delta(z'-z) \qquad K_\nu^\mu = (\nu+1/2)^{-1}\, f(\nu,|\mu|)$$

As noted above, we can start the ν spectrum at 0 if we make the "alternate definition of P" at nonsense points for negative μ. Also, remember that the value of the normalization constant is not really important, the key fact is that for different ν, things are *orthogonal*. You can change the normalization constant by changing the definition of the P functions (ie, by changing their scale).

Now, we can write this another way if we want to use $P_\nu^{|\mu|}(z)$ functions. According to Bateman, we have the following rule for on-the-cut P functions

$$P_\nu^{-\mu}(z) = f(\nu,-\mu)\, P_\nu^\mu(z) \cos(\pi\mu) \qquad \text{Bateman p 144 (17)}$$

which for our integer μ situation becomes

$$P_\nu^{-\mu}(z) = f(\nu,-\mu)\, P_\nu^\mu(z)(-1)^\mu$$

(Stak stays away from the nonsense values of ν and μ where f has zeros or poles). Then we have $P_\nu^\mu(z) = (-1)^\mu\, f(\nu,\mu)\, P_\nu^{-\mu}(z)$ for negative μ and then we know that

$$P_\nu^\mu(z) = \begin{cases} P_\nu^{|\mu|}(z) & \mu > 0 \\ (-1)^\mu\, f(\nu,\mu)\; P_\nu^{|\mu|}(z) & \mu < 0 \end{cases}$$

So we can write the above as (recall that $f(\nu,-\mu) = 1/f(\nu,\mu)$) [since the P's are always bilinear, the phase factor $(-1)^\mu$ never appears anywhere]

μ > 0:

$$\int_{-1}^{1} dz\, P_\nu^{|\mu|}(z)\, P_{\nu'}^{|\mu|}(z) = \delta_{\nu,\nu'}\, K_\nu^\mu \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty}(1/K_\nu^\mu)\, P_\nu^{|\mu|}(z')\, P_\nu^{|\mu|}(z) = \delta(z'-z) \qquad K_\nu^\mu = (\nu+1/2)^{-1}\, f(\nu,\mu)$$

μ < 0:

$$\int_{-1}^{1} dz\, P_\nu^{|\mu|}(z)\, P_{\nu'}^{|\mu|}(z) = \delta_{\nu,\nu'}\, K'^{\mu}_{\nu} \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty}(1/K'^{\mu}_{\nu})\, P_{\nu}^{|\mu|}(z')\, P_{\nu}^{|\mu|}(z) = \delta(z'-z) \qquad K'^{\mu}_{\nu} = (\nu+1/2)^{-1} f(\nu,-\mu)$$

We could then combine these into a single form as follows:

$$\int_{-1}^{1} dz\, P_{\nu}^{|\mu|}(z)\, P_{\nu'}^{|\mu|}(z) = \delta_{\nu,\nu'}\, H^{\mu}_{\nu} \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty}(1/H^{\mu}_{\nu})\, P_{\nu}^{|\mu|}(z')\, P_{\nu}^{|\mu|}(z) = \delta(z'-z) \qquad H^{\mu}_{\nu} = (\nu+1/2)^{-1} f(\nu,|\mu|)$$

This is the approach taken by Stakgold on page 395. We can relate our H to his N this way:

$$2\pi H^{\mu}_{\nu} = 2\pi (\nu+1/2)^{-1} f(\nu,|\mu|) = 2\pi\, 2\, (2\nu+1)^{-1} f(\nu,|\mu|) = N_{\mu,\nu} \qquad \text{Stak A.5}$$

The non-Integer μ Case:

The second spectral case listed above is this:

ν = special integrally-spaced values shown here:
Re(μ) < 0 and μ ≠ integer and [ν = μ-J or ν = -μ+I]
spectrum: ν = -μ, -μ+1,... and use $P_{\nu}^{\mu}(z)$

So this case is only valid for some general non-integral value of μ with Re(μ) < 0. We get:

$$\int_{-1}^{1} dz\, P_{\nu}^{\mu}(z) P_{\nu'}^{\mu}(z) = \delta_{\nu,\nu'}\, K^{\mu}_{\nu} \qquad \nu,\nu' = -\mu, -\mu+1,\ldots$$

$$\Sigma_{\nu=-\mu}^{\infty}(1/K^{\mu}_{\nu})\, P_{\nu}^{\mu}(z')\, P_{\nu}^{\mu}(z) = \delta(z'-z) \qquad K^{\mu}_{\nu} = (\nu+1/2)^{-1} f(\nu,\mu)$$

Notice that the spectral points ν = -μ+I start in the right half plane location ν = -μ and step off to the right, integrally spaced, but the values of ν themselves are NOT integers. $\Sigma_{\nu+\mu=0}^{\infty}$ is another way to write this sum.

This second case is not very well known I suspect. Here is some Maple verification. First, I show some orthogonality cases: (I use some speed up switches on the integration)

```
restart;
_EnvLegendreCut := 1..infinity:
mu := -2.3:
nu := -mu + 1:
nup := -mu + 3:
H := evalf(Int(LegendreP(nu,mu,z)*LegendreP(nup,mu,z), z = -1..1,5,_CCquad));
```

$$H := .28889\, 10^{-9}$$

```
nu := -mu + 1:
nup := -mu + 2:
H := evalf(Int(LegendreP(nu,mu,z)*LegendreP(nup,mu,z), z = -1..1,5,_CCquad));
```

$$H := .22455\, 10^{-10}$$

Then here is a case showing that the above normalization is correct:

```
mu := -2.3:
nu := -mu + 2:
nup := -mu + 2:
H := evalf(Int(LegendreP(nu,mu,z)*LegendreP(nup,mu,z), z = -1..1,5,_CCquad));
                                                    H := .00018315
K := (nu+1/2)^(-1) * GAMMA(1+nu+mu)/GAMMA(1+nu-mu):evalf(%);
                                                    .0001831475507
```

Spherical Harmonics

Stakgold.

If we follow Stakgold's approach, we have these facts:

$$\int_{-1}^{1} dz\, P_{\nu}^{|\mu|}(z)\, P_{\nu'}^{|\mu|}(z) = \delta_{\nu,\nu'}\, H_{\nu}^{\mu} \qquad \nu,\nu' = |\mu|, |\mu|+1, \ldots$$

$$\Sigma_{\nu=|\mu|}^{\infty} (1/H_{\nu}^{\mu})\, P_{\nu}^{|\mu|}(z')\, P_{\nu}^{|\mu|}(z) = \delta(z'-z) \qquad H_{\nu}^{\mu} = (\nu+1/2)^{-1}\, f(\nu,|\mu|) = N_{\mu,\nu}/(2\pi)$$

Then we could define the spherical harmonics this way (μ and ν are integers, I just keep using ν,μ)

$$Y_{\nu}^{\mu}(\theta,\varphi) \equiv P_{\nu}^{|\mu|}(z)\, e^{i\mu\varphi} \qquad \text{spectrum } \nu \geq |\mu|$$

Notice in passing that $Y_{\nu}^{-\mu}(\theta,\varphi) = Y_{\nu}^{\mu}(\theta,\varphi)^*$ for Stak's definition of the Y.

We know for the φ function that orthogonality and completeness are given by

$$\int_{0}^{2\pi} d\varphi\, e^{-i\mu\varphi}\, e^{i\mu'\varphi} = \delta_{\mu,\mu'} 2\pi$$

$$\Sigma_{\mu=-\infty}^{\infty} (1/2\pi)\, e^{-i\mu\varphi}\, e^{i\mu\varphi'} = \delta(\varphi-\varphi')$$

where I write these in a manner analogous to the way I did it for the P's. Then we can combine all this stuff and write orthogonality and completeness for these spherical harmonics:

$$\int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi\; Y_{\nu}^{\mu}(\theta,\varphi)^*\; Y_{\nu',\mu'}(\theta,\varphi) = \delta_{\nu,\nu'}\, \delta_{\mu,\mu'}\, H_{\nu}^{\mu}\, 2\pi = \delta_{\nu,\nu'}\, \delta_{\mu,\mu'}\, N_{\mu,\nu}$$

and this agrees with Stakgold p 395 E and F.

The corresponding completeness statement would then be:

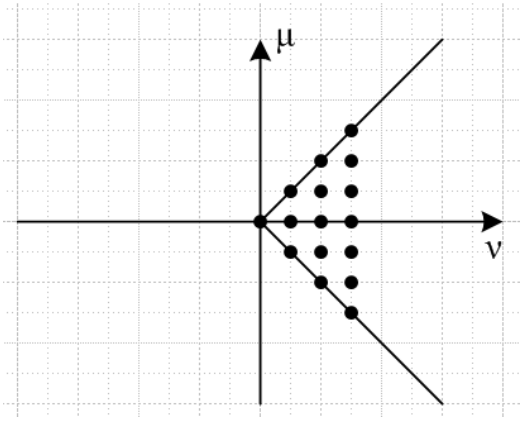
$$\Sigma_{\mu=-\infty}^{\infty} (1/2\pi)\; \Sigma_{\nu=|\mu|}^{\infty} (1/H_{\nu}^{\mu})\, Y_{\nu}^{\mu}(\theta,\varphi)^*\; Y_{\nu}^{\mu}(\theta',\varphi') = \delta(z'-z)\, \delta(\varphi-\varphi')$$

or

$$\Sigma_{\mu=-\infty}^{\infty} \Sigma_{\nu=|\mu|}^{\infty} (1/N_{\mu,\nu})\, Y_{\nu}^{\mu}(\theta,\varphi)^*\; Y_{\nu}^{\mu}(\theta',\varphi') = \delta(z'-z)\, \delta(\varphi-\varphi') = \delta(\Omega-\Omega')$$

Stak does not state completeness in this way, but you could obtain it from p 396 A and B.

Notice that Stakgold avoids having to talk about $P_\nu^\mu(z)$ when μ = negative integer. Here is a graphical way to represent the sum shown above in the completeness relation:



This shows that you could rearrange the completeness sum into its more traditional form:

$$\Sigma_{\nu=0}^{\infty} \Sigma_{\mu=-\nu}^{\nu} (1/ N_{\mu,\nu}) Y_\nu^\mu (\theta,\varphi)^* \; Y_\nu^\mu (\theta',\varphi') \;=\; \delta(z'-z)\, \delta(\varphi-\varphi') \;= \delta(\Omega-\Omega')$$

which Stak shows in page 306A.

Jackson.

We go to page 65 of green Jackson. He defines $P_\nu^\mu(z)$ for positive μ using the Rodriguez formula. This includes the Condon-Shortley convention and agrees with the Bateman on-the-cut P function. Then (3.51) is what we had above and also agrees with the Bateman on-the-cut P function. Notice that Jackson has already completely ruled out any nonsense values on either side of (3.51) in his discussion on page 64. So he really is doing $\nu = |\mu|, |\mu|+1$ etc, the same as Stakgold. So 3.51 is not meant to apply in the nonsense range!

Jackson's P orthogonality (3.52) agrees with my first form above.

Jackson then goes on to include an ugly constant in the definition of his $Y_{\ell m}$ in 3.53. This then has the benefit that we then get no constants whatsoever in the orthogonality and completeness of the Y's as he shows in 3.55 and 3.56. This then gives us the wonderful connection that $Y_{\ell m}(\theta,\varphi) = \langle\theta\varphi|\ell m\rangle$ with everything normalized in a trivial manner. The $(-1)^\mu$ phase factor appears then in 3.54 which is fine.

I have shown elsewhere, with help from Currant and Hilbert, that the $Y_{\ell m}(\theta,\varphi)$ are a complete basis for functions on the sphere. The major ingredient is that in our SL problem here we have shown that for any integer m, the functions $P_n^m(z)$ form a complete set in the z world. We know that $e^{im\varphi}$ is complete in the φ world, so the CH contribution is to show that this implies the product is complete in the product world.