Pulling the z=1 Legendre Cut off to the Right PhL 1.30.10

The Visio drawings for this doc are located here: physics/EM/Smythe/ smythe oblate greens picture.vsd

1. A study of the Q_1 function and Riemann Sheets and Choices of Same. 1
(a) Paradox. 1
(b) Paradox Resolution. 2
(b1) Wind around to stay on the principle sheet. 2
(b2) Go south through the cut instead. 3
(c) Digression on a simpler problem. 3
(b2 continued): going south. 3
(d) Obviously, this is a painful way to do things, 4
(e) Question: Are these equal, or are they not equal: $Q_1(z)_{0,0R}$ and $Q_1(-z)_{0,0R}$?? 5
(f) Now let's try $z = i\zeta$ and see if we can get happiness with that. 6
(g) Conclusions of the Previous Very Long Section about Q_1 8
2. Given $f(z)_{0,0L}$, how do you find $f(z)_{0,0R}$? 9
Prescription #1 : 11
Prescription #2 : Alternatively, just change the angle parameterization and then the prescription is to simply do nothing. 12
Example 1: f(z) = Q1(z) 12
3. Question: How do we apply our prescription to a Bateman formula. For example,. 12
4. Finding the large-z behavior of $Q_\nu^\mu(z)_{0,1L}$ 13
Backstep 1. 16
So here is My Big Result 18

1. A study of the Q_1 function and Riemann Sheets and Choices of Same.

(a) Paradox. This is crucial to understand, and there is a huge crevasse looming here into which I don't want to fall. We know there is a horizontal branch cut running z = +1 off to the left. We shall now set up a **paradox** and then resolve it. We know various Q function facts: [let $z = i\zeta$ with $\zeta > 0$, then z is above the cut]

$$Q_\nu^\mu(-z) = -Q_\nu^\mu(z)\, e^{\pm i\pi\nu} \quad \Rightarrow \quad Q_\nu^\mu(-z) = -Q_\nu^\mu(z)\, e^{+i\pi\nu} \qquad Q_\nu^\mu(-i\xi) = -Q_\nu^\mu(i\xi)\, e^{+i\pi\nu}$$

$$\Rightarrow Q_\nu(-z) = -Q_\nu(z)\, e^{+i\pi\nu} \quad \Rightarrow \quad Q_1(-z) = -Q_1(z)\, e^{+i\pi} = Q_1(z)$$

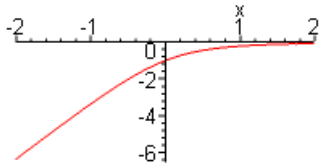
$$\Rightarrow Q_1(-i\zeta) = Q_1(i\zeta)$$

This suggests that $Q_1(i\zeta)$ is symmetric under $\zeta \to -\zeta$. On the other hand, we know that

$$Q_1(j\zeta) = \zeta \cot^{-1}(\zeta) - 1 \qquad Q_1(z) = (z/2) \ln [\,(z+1)/(z-1)] - 1$$

which is blatantly NOT symmetric under $\zeta \rightarrow -\zeta$

```
plot(x*arccot(x)-1, x=-2..2);
```



(b) Paradox Resolution. So there is our "paradox". The resolution is this: the Legendre fact $Q_\nu^\mu(-z) = -Q_\nu^\mu(z)\, e^{\pm i\pi\nu}$ applies to the Q function on a certain Riemann sheet (call it the principle sheet), but when we write $Q_1(j\zeta) = \zeta \cot^{-1}(\zeta) - 1$, as we go south, we are *slipping through the cut* onto a different Riemann sheet, so below the border we have two different animals.

(b1) First animal: Wind around to stay on the principle sheet. To see how this happens, we use the log form shown above and we shall "wind" around the branch point z = +1 only:

$$Q_1(z) = (z/2)\ln\left[(z+1)/(z-1)\right] - 1 = (z/2)\ln(z+1) - (z/2)\ln(z-1) - 1$$

Start with $z-1 = |z-1|e^{+i\pi}$ which is just above the cut. Then

$$Q_1(z+) = (z/2)\ln(z+1) - (z/2)\ln(|z-1|e^{+i\pi}) - 1$$

$$= (z/2)\ln(z+1) - (z/2)\ln(|z-1| + i\pi) - 1$$

$$= (z/2)\ln(z+1) - (z/2)\ln(|z-1|) + i\pi(z/2) - 1$$

Now if we wind CW around z = 1 in such a way as to remain on this same principle sheet and just move below the cut, we then get

$$Q_1(z-) = (z/2)\ln(z+1) - (z/2)\ln(|z-1|) - i\pi(z/2) - 1$$

the $i\pi$ part has changed sign. So we are always here on the principle sheet, and we can then look at our Legendre rule which says $Q_1(-z) = Q_1(z)$ or $Q_1(-x-i\varepsilon) = Q_1(x+i\varepsilon)$. Evaluating each:

$$Q_1(x+i\varepsilon) = (x/2)\ln(x+1) - (x/2)\ln(|x-1|) + i\pi(x/2) - 1$$

$$Q_1(-x-i\varepsilon) = (-x/2)\ln(-x+1) - (-x/2)\ln(|-x-1|) - i\pi(-x/2) - 1$$

$$= -(x/2)\ln(-x+1) + (x/2)\ln(|x+1|) + i\pi(x/2) - 1$$

$$= (x/2)\ln(|x+1|) - (x/2)\ln(-x+1) + i\pi(x/2) - 1$$

But on the cut we know x+1 > 0 so remove | | and we see that $Q_1(-x-i\varepsilon) = Q_1(x+i\varepsilon)$ as predicted.

(b2) Second animal: Go south through the cut instead. But now, instead of "winding z around" the branch point at z = +1 (which kept us on the principle sheet), suppose we move right through the cut, heading south. How do we "do this" ?

$$Q_1(z) = (z/2)\ln[\,(z+1)/(z-1)] - 1$$

If we want to maintain our notation $z-1 = |z-1|e^{+i\theta}$, we would have to have $\theta > \pi$, and in fact we would have to have θ in $(\pi,3\pi)$ in this parameterization (for the new sheet). The issue is what to do with the ln(z-1) term.

(c) Digression on a simpler problem. So consider a simpler problem: f(z) = ln(z) with cut pulled to the left as usual. If we want to go south through that cut, we have $z = |z|e^{+i\theta}$ with θ in $(\pi,3\pi)$ for the sheet we arrive at. But write as $z = |z|e^{+i(\theta-2\pi)}\, e^{+i2\pi} = |z|e^{+i(\theta')}\, e^{+i2\pi}$ where θ' in range $(-\pi,\pi)$. Then

$$\ln(z)_{\mathbf{first}} = \ln(\,|z|e^{+i(\theta')}\, e^{+i2\pi}\,) = \ln(|z|) + i\theta' + i2\pi = \ln(z)_{\mathbf{principle}} + i2\pi\,.$$

where we call our new sheet the "first" and our original sheet the "principle".

Now go back to the principle sheet and f(z) = ln(z) = ln(R) + iθ. The domain is the entire z plane, but the range is just the **horizontal strip** centered on the x axis running from -iπ to +iπ. So our "first" range is the "next strip up".

(b2 Second animal continued): going south. So let's apply this to $Q_1(z)$ and see what happens:

$$Q_1(z)_{\mathbf{principle}} = (z/2)\ln(z+1)_{\mathbf{principle}} - (z/2)\ln(z-1)_{\mathbf{principle}} - 1$$

$$Q_1(z)_{\mathbf{first}} = (z/2)\ln(z+1)_{\mathbf{principle}} - (z/2)[\,\ln(z-1)_{\mathbf{principle}} + i2\pi\,] - 1$$

$$= (z/2)\ln(z+1) - (z/2)[\,\ln(z-1) + i2\pi\,] - 1$$

where an "unmarked function" means it is IT's principle sheet. Then here is our result

$$Q_1(z)_{\mathbf{first}} = Q_1(z)_{\mathbf{principle}} - i\pi z$$

so there is a very significant difference between $Q_1(z)_{\mathbf{first}}$ and $Q_1(z)_{\mathbf{principle}}$! It is not just some simple sign or phase difference. We could generalize this as follows:

$$Q_1(z)_{0,N;L} = Q_1(z)_{0,0;L} - i\pi N z$$

where we indicate the principle sheet by 0,0 (these being winding numbers about z=-1 and z=+1), and the capital letter L reminds us that the z=1 cut is taken to the Left. . So now we know $Q_1(z)_{0,N;L}$ on lots of sheets, where N = any integer, positive or negative. We could generalize this further to include winding around the z = -1 branch point, but let's not do that right now.

Now back to this notation

$Q_1(z)_{0,1;L} = Q_1(z)_{0,0:L} - i\pi z$

We know that $Q_1(z)_{0,0L} = Q_1(-z)_{0,0L}$ from our principle sheet Legendre rule. We understand that

$Q_1(z)_{0,0L} = (z/2)\ln(z+1) - (z/2)\ln(z-1) - 1$ with $(z-1) = |z-1|\, e^{i\theta}$ θ in $(-\pi,\pi)$

$Q_1(z)_{0,1L} = (z/2)\ln(z+1) - (z/2)\ln(z-1) - 1$ with $(z-1) = |z-1|\, e^{i\theta}e^{i2\pi}$ θ in $(-\pi,\pi)$

Yes, we know that $e^{i2\pi} = 1$, so this is really a bookkeeping device. It is like a distribution having a symbolic function. The "meaning" is only found when the (z-1) form is inserted into the ln function, and then there is a difference between $e^{i2\pi}$ and 1.

Since $Q_1(z)_{0,0L} = Q_1(-z)_{0,0L}$ is symmetric, it is pretty clear that $Q_1(z)_{0,1L}$is NOT symmetric.

$Q_1(z)_{0,1L} = Q_1(z)_{0,0L} - i\pi z$
$Q_1(-z)_{0,1L} = Q_1(-z)_{0,0L} + i\pi z$

$Q_1(-z)_{0,1L} = Q_1(z)_{0,1L} + 2i\pi z$

So, in our application, we want to use $Q_1(z)_{0,0L}$ above the cut, but $Q_1(z)_{first}$ as we dive down through the cut. Both functions $Q_1(z)_{0,1L}$ and $Q_1(z)_{0,0L}$ have a jump at the cut, but if we do a hand off at the border we go smoothly from $Q_1(z)_{0,0L}$ to $Q_1(z)_{0,1L}$ as we pass through. In other words, we will have $Q_1(x+i\varepsilon)_{0,0L} = Q_1(x-i\varepsilon)_{0,1L}$.

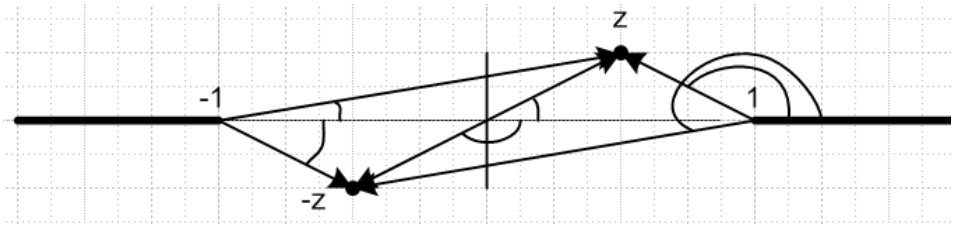
(d) Obviously, this is a painful way to do things, where we have to indicate with a subscript which sheet we are "on" with our function. Another way to work is as follows: "rotate" the cut from z = +1 CCW around its branch point until the cut points to the right. This then "exposes" the region we want in a continuous fashion. Obviously doing this causes a rearrangement of the "sheet structure", since we have sort of gone half a flight of stairs down on a spiral staircase. We can then REDEFINE what we mean by our principle and other sheets in this manner:

$Q_1(z)_{0,0R} = (z/2)\ln(z+1) - (z/2)\ln(z-1) - 1$ with $(z-1) = |z-1|\, e^{i\theta}$ θ in $(0,2\pi)$
with $(z+1) = |z+1|\, e^{i\varphi}$ φ in $(-\pi,\pi)$

We have now reparameterized (z-1) in terms of an angle that runs 0 to 2π, which seems completely unambiguous to me graphically speaking, no need to draw it. Now there is no cut between -1 and 1, and this new way of defining "the principle sheet" is very useful for a problem (like one involving an oblate hyperboloid) where we want continuity in the region (-1,1).

The "Legendre rules" have to be modified for this new way of thinking. And I don't think the modified rules will be the same as the rules for Bateman's on-the-cut functions, so now we have a whole third set of rules to worry about! But the rules will be easy to derive because all we have to do is examine any $(z-1)^{\alpha}$ factors and process them accordingly. [ha!]

We would like now to compare $Q_1(z)_{0,0R}$ to $Q_1(-z)_{0,0R}$. So what then is the meaning of "-z" ? Now I guess it is time for a picture:



where you see the angles used to parameterize the two points z and -z . Relative to the origin, we might say $z = R\, e^{i\psi}$ and $-z = R\, e^{-i(\pi-\psi)}$ if we choose ψ in (-π,π). Then $-z = R\, e^{i\psi}\, e^{-i\pi}$

[This stuff is unbelievably slippery and treacherous, like making a sculpture out of mercury. At every turn, something bad wants to happen, high vigilance is required.]

(e) Question: Are these equal, or are they not equal: $Q_1(z)_{0,0R}$ and $Q_1(-z)_{0,0R}$??

Here is an argument that says they are exactly the same:

$Q_1(-z) = (-z/2) \ln [\, (-z+1)/(-z-1)] - 1$
$= (-z/2) \ln [\, (z-1)/(z+1)] - 1$ // applied (-1)/(-1) inside the log
$= (+z/2) \ln [\, (z+1)/(z-1)] - 1$ // inverted the log
$= Q_1(z)$

But this is wrong! [I said it was slippery.] The problem involves those (-1) factors. In doing so, we make this claim:

$$(-z+1) = -\,(z-1)$$

which certainly looks reasonable. *But we are inside a log, so things are delicate.* We can see the angle in the picture for the vector (-z+1) which is (-z -(-1)) which is the lower left arrow whose angle let us denote as $-\theta_1$ where $\theta_1 > 0$. And we can see the angle for (z-1) which is the upper right arrow whose angle let us call θ_2. Then we have

$(z-1) = |z-1|\, e^{i\theta 1}$ and $(-z+1) = |-z+1|\, e^{-i\theta 2} = |-z+1|\, e^{-i(\pi-\theta 1)} = |z-1|\, e^{i\theta 1}\, e^{-i\pi}$

$\Rightarrow \quad (-z+1) = (z-1)\, e^{-i\pi}$

But we cannot willy-nilly set $e^{-i\pi} = -1$ because we are inside a log and there is going to be a difference between $e^{-i\pi}$ and $e^{+i\pi}$ and we have to keep track of the distinction.

We could have arrived at the conclusion $(-z+1) = (z-1)\, e^{-i\pi}$ at once by inspection of the picture. If we start with arrow (z-1), we have to wind it clockwise by π to get to the arrow (-z+1). We could translate one of the vectors and put them at the same origin for this comparison.

Now about the claim, used in our little naive processing above, that

$$(-z-1) = -\,(z+1)$$

These involve the other two parallelogram angles in the picture. We want to write

$$(-z-1) = (z+1)\, e^{\mathbf{phase}}$$

So start with arrow (z+1) which is top left, rotate it to get arrow (-z-1)which is bottom right. The required angle is $+\pi$ so we have

$$(-z-1) = (z+1)e^{+i\pi}$$

Now let's to back and redo our steps more carefully:

$Q_1(-z)_{0,0R} = (-z/2)\ln[(-z+1)/(-z-1)] - 1$ // we set $z\to -z$ inside our $Q_1(z)$

$= (-z/2)\ln[(z-1)\, e^{-i\pi}/(z+1)e^{+i\pi}] - 1$ // install precision results from above

$= (-z/2)\ln[(z-1)/(z+1)e^{+i2\pi}] - 1$ // put entire phase down

$= (+z/2)\ln[(z+1)e^{+i2\pi}/(z-1)] - 1$ // use $\ln(1/z) = -\ln(z)$

$= (z/2)\ln[(z+1)/(z-1)] - 1 + (z/2)\, i2\pi$

$= Q_1(z)_{0,0R} + 2\pi i\,(z/2)$

so

$$Q_1(-z)_{0,0R} = Q_1(z)_{0,0R} + 2\pi i\,(z/2)$$

Remember that all these Q's are really $Q_{0,0R}$ on our newly defined principle sheet. So now we see after doing things with precision instead of sloppily that $Q_1(z)_{0,0R}$ is NOT symmetric. So we now have the answer to our posed question:

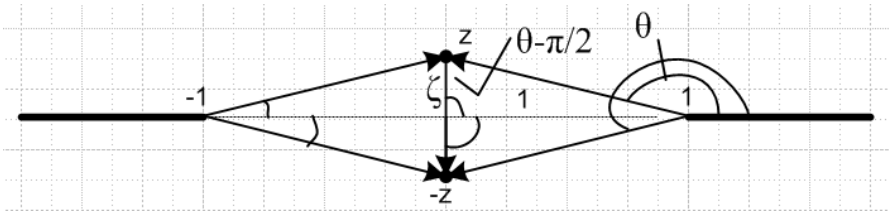
answer: $Q_1(z)_{0,0R}$ and $Q_1(-z)_{0,0R}$ are NOT equal

The trick here is that *inside a log or a power expression*, we have to parameterize quantities with great care relative to branch points, keeping angles in legal range.

(f) Now let's try $z = i\zeta$ and see if we can get happiness with that. Go back to

$Q_1(z)_{0,0R} = (z/2)\ln[(z+1)/(z-1)] - 1$ with $(z-1) = |z-1|\, e^{i\theta}$ θ in $(0,2\pi)$

with $(z+1) = |z+1|\, e^{i\varphi}$ φ in $(-\pi,\pi)$



where we always intend to be on the new 0,0 principle sheet. We have

$$Q_1(z)_{0,0R} = (z/2)\ln[|z+1|\, e^{i\varphi}/|z-1|\, e^{i\theta}] - 1 = (z/2)\ln[|z+1|\, e^{i\varphi}\, e^{-i\theta}/|z-1|] - 1$$

$$= (z/2)\ln[|z+1|/|z-1|] - 1 - (z/2)i(\theta-\varphi) = (z/2)\ln[|z+1|/|z-1|] - 1 - (z/2)i(2\theta-\pi)$$

and the picture shows that $\theta+\varphi = \pi$ so can write $\theta-\varphi = \theta - (\pi-\theta) = 2\theta-\pi$. Now

$$|z+1|^2 = |i\zeta+1|^2 = (i\zeta+1)(-i\zeta+1) = \zeta^2 + 1$$

$$|z-1|^2 = |i\zeta-1|^2 = (i\zeta-1)(-i\zeta-1) = \zeta^2 + 1$$

Obviously these distances are the same, just from the picture, so we have

$$\ln [|z+1|/|z-1|] = \ln(1) = 0$$

and

$$Q_1(z)_{0,0R} = -1 -(z/2)i(2\theta-\pi) = -1 - (i\zeta/2)\, i\, (2\theta-\pi) = -1 + \zeta\,(\theta-\pi/2)$$

The angle $\theta-\pi/2$ appears in our picture above, and it is clear that

$$(\theta-\pi/2) = \sin^{-1}[1/(\zeta^2+1)] = \cos^{-1}[\zeta/(\zeta^2+1)] = \tan^{-1}(1/\zeta) = \cot^{-1}(\zeta)$$

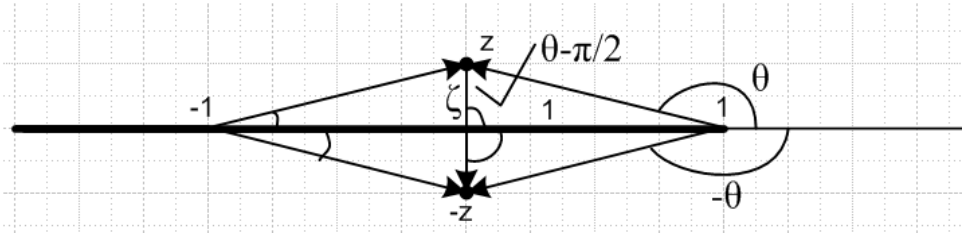
so we have then shown the famous result

$$Q_1(i\zeta)_{0,0R} = -1 + \zeta \cot^{-1}(\zeta)$$

<u>So that is how it works!</u> The entire interesting part of the result comes from the phase of our two involved arrows (z+1) and (z-1). The lengths of these arrows is the same and contributes nothing.

Notice that $Q_1(i\zeta)_{0,0R} \neq Q_1(-i\zeta)_{0,0R}$, in agreement with our general claim above.

Now what do we get for the function $Q_1(i\zeta)_{0,0L}$? Just for the record before we depart this section. We can go through the same set of steps and see where things differ. Here is the picture:



So here are the pieces. This relates the top right and lower left arrows,

was: $(-z+1) = (z-1)\, e^{-i\pi}$ is now: $(-z+1) = (z-1)\, e^{-i\pi}$ same

LL TR

We start with top right arrow and rotate CW by π. The next piece:

was: $(-z-1) = (z+1)e^{+i\pi}$ is now: $(-z-1) = (z+1)e^{-i\pi}$ not same

LR TL

This relates top left and lower right arrows. Start with top left and rotate CW by π. Now because your numerator and denominator phases are the same, they cancel and we naive argument goes through and we get the result $Q_1(-z)_{0,0L} = Q_1(z)_{0,0L}$. But what IS this function in ζ language? It can only be this:

$$Q_1(i\zeta)_{0,0L} = -1 + |\zeta| \cot^{-1}(|\zeta|)$$

We know this is right in the upper half without bars. In order to have $Q_1(-i\zeta)_{0,0L} = Q_1(i\zeta)_{0,0L}$, this is what it has to be! So again, this function is symmetric on sheet 0,0L, whereas $Q_1(i\zeta)_{0,0R}$ is not symmetric. Also, notice that $Q_1(i\zeta)_{0,0L}$ has asymptotic decay in both directions, whereas $Q_1(i\zeta)_{0,0R}$ decays going up, but in fact blows up going down!

Here are a few more facts:

$$Q_1(i\zeta)_{0,0R} = -1 + \zeta \cot^{-1}(\zeta) \qquad \text{for all real } \zeta$$

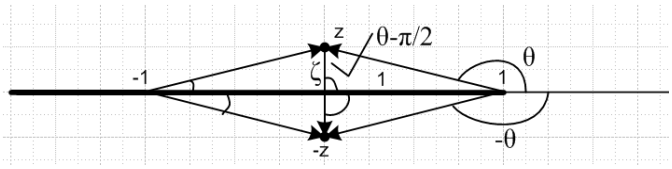
$$Q_1(-i\zeta)_{0,0R} = Q_1(i\zeta)_{0,0R} + 2\pi i\,(i\zeta/2) = -1 + \zeta \cot^{-1}(\zeta) - \pi\zeta$$

$$= -1 + \zeta\,[\cot^{-1}(\zeta) - \pi]$$

(g) Conclusions of the Previous Very Long Section about Q_1

There are two different ways to define the "principle sheet" of the $Q_1(z)$ function. They are really defined by the way the z=1 cut is "pulled away" from its branch point. We present (picture just happens to show z on the imaginary axis, but z could be anywhere) a summary below. In both cases, one can write the function as $Q_1(z) = (z/2) \ln [(z+1)/(z-1)] - 1$, but the *meaning* of the function (its actual numerical values at values of z on the sheet of interest) are not the same in the lower half plane, and they differ by more than just a phase.

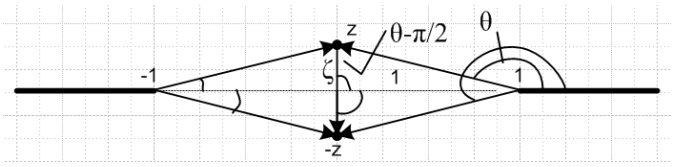
$Q_1(z)_{0,0L}$: Bateman rules Apply



$Q_1(i\zeta)_{0,0L} = -1 + |\zeta| \cot^{-1}(|\zeta|)$

$Q_1(-z)_{0,0L} = Q_1(z)_{0,0L}$

$Q_1(z)_{0,0R}$: Bateman rules do not Apply



$Q_1(i\zeta)_{0,0R} = -1 + \zeta \cot^{-1}(\zeta)$

$Q_1(-z)_{0,0R} = Q_1(z)_{0,0R} + 2\pi i\,(z/2)$

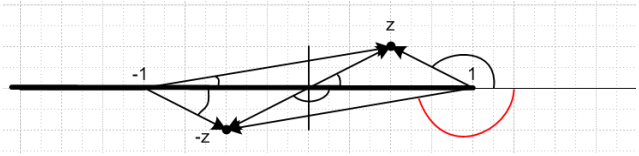
The 0,0 notation refers to winding numbers relative to the branch points at z = -1 and z = +1. We set these both to zero for our "principle sheet" in the two cases. Angles are measured as shown in the two cases. Angles relative to the left branch point are the same in both cases.

The two functions agree in the upper half plane, but disagree quite violently in the lower half plane. We notice, for example, that $Q_1(i\zeta)_{0,0L}$ decays as ζ^{-1-1} in both up and down directions, in accordance with the Bateman rule on this subject, whereas $Q_1(i\zeta)_{0,0R}$ blows up in the down direction, and thus has totally different asymptotic properties in that direction.

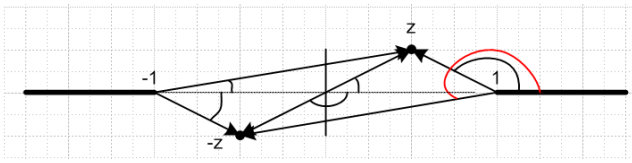
The violent difference between the two functions arises from a simple phase factor inside the log, which causes an extra term that gets multiplied by the outside factor (z/2).

2. Given $f(z)_{0,0L}$, how do you find $f(z)_{0,0R}$?

By the notation used in this question, I am assuming that f(z) has branch points at z = ± 1, and that R means we just rotate the right one around CCW by π. The main fact here is how you parameterize the quantity (z-1) when z is in the lower half plane! To wit:



L situation



R situation

The only thing different here is that in the L situation, a point in the lower half plane has a negative angle in the range (0,-π), while in the R situation that same point has a positive angle in the range (π,2π).

Note added 2.11.10: In the upper picture, the red angle is some θ_L where $\theta_L < 0$. In the lower picture we have some positive angle θ_R. It seems pretty clear that $-\theta_L + \theta_R = 2\pi$. So $\theta_R = \theta_L + 2\pi$. This would then be a "rule" for processing a vector z-1. In the L world you have $(z-1)_L = |z-1|\, e^{i\theta_L}$ and in the R world you have $(z-1)_R = |z-1|\, e^{i\theta_R}$. Then we can say $(z-1)_R = |z-1|e^{i(\theta_L + 2\pi)} = (z-1)_L\, e^{i2\pi}$. This then should be a rule for converting formulas from L to R world. More appears below on this subject. This all is true only when Im(z) < 0. When Im(z) > 0, we have $(z-1)_R = (z-1)_L$ as shown by black angles.

Clarification is needed here. Let's talk about the L world where we measure all angles as shown in the upper picture. Then in the R world we measure all angles as shown in the lower picture. We *could* make some other convention for measuring angles in the R world but I think this is the only way that makes sense. You want to start measuring angle rotating up from the cut, not from a point to the left of z = 1 which would be immensely confusing. That is to say, you want an arrow sitting at 2:55 PM to have a small positive angle. So this is the way we will do it! [Called "method B" in "the meaning of f(-z)..."] Also, it shows that things are continuous in the area of angle = π.

So what exactly do we mean by this claim that $(z\text{-}1)_L = (z\text{-}1)_R\, e^{-i2\pi}$? Let's think about this in the context of a test function. Let $f(z) = (z\text{-}1)^{1/5}$. We would then say $f(z)_{0,0L} = (z\text{-}1)_L^{1/5}$. We want to understand the meaning of $f(z)_{0,0R}$. We want to compute this function and study the angles a bit. But first, with $f(z)_{0,0L} = (z\text{-}1)_L^{1/5}$ we understand that the angle(z-1) lies in range (-π,π), no question about that! This function has a cut going to the Left from z = +1. The function is real when z = 4 and in fact has the value $f(4)_{0,0L} = 3^{1/5}$. Now consider

$f(z)_{0,0L} \equiv (z\text{-}1)_L^{1/5}$ angle$(z\text{-}1)_L$ lies in range (-π,π)

$f(z)_{0,0R} \equiv (z\text{-}1)_R^{1/5}$ angle$(z\text{-}1)_R$ lies in range (0,2π)

For Im(z) > 0, these functions are the same. But for z = 4-iε, lets say, we have $f(4)_{0,0L} = 3^{1/5}$ still, but we have angle 2π for $f(z)_{0,0R}$ so we get $f(z)_{0,0R} = [\,3e^{i2\pi}]^{1/5} = 3^{1/5}\, e^{i2\pi/5}$, so now our two functions are not the same function. We write them both as $f(z) = (z\text{-}1)^{1/5}$ generically, but when we examine just what this means on the two different sheet situations, they are different.

So we see the difference here by using the "do nothing" method for "conversion". We just take *the same form* and use Right world angles instead of Left world angles.

The second method of conversion is this idea:

$$f(z)_{0,0R} \equiv (z\text{-}1)_R^{1/5} = [(z\text{-}1)_L\, e^{i2\pi}]^{1/5} = (z\text{-}1)_L^{1/5}\, e^{i2\pi/5}$$

In this method, we use L world angles for $(z\text{-}1)_L^{1/5}$!!!!! So for example, if z = 4-iε, the L world angle for that is 0, so we get $3^{1/5}\, e^{i2\pi/5}$.

The Bateman formulas have factors like $(z\text{-}1)_L^{1/5}$. You can convert a Bateman formula to the R world using either of the above methods.

This entire discussion is repeated below.

So let's first consider a point z in the upper half plane such that $(z\text{-}1) = Re^{i\theta}$. We know that angle θ is in the range (0,π). Then we have

$(z\text{-}1)_{0,0R} = (z\text{-}1)_{0,0L}$ Im(z) > 0

They are exactly the same.

Now, imagine both pictures above with the labels z → -z everywhere, no other change in the picture. Then we have our z in the lower half plane. In this case we again imagine $(z\text{-}1)_L = Re^{i\theta}$, where now for the top L picture, θ is in the range (0,-π). Looking at the pictures, we can see that in the R world, this same vector (z-1) has an angle ψ in the range (π,2π). And we see that -θ + ψ = 2π, so ψ = θ + 2π. Then we have $(z\text{-}1)_R = Re^{i\psi} = Re^{i\theta}\, e^{i2\pi} = (z\text{-}1)_L\, e^{i2\pi}$. Thus we have shown that

$(z\text{-}1)_{0,0R} = (z\text{-}1)_{0,0L}\, e^{i2\pi}$ Im(z) < 0

If we consider z in the range -1 ≤ Re(z) ≤ 1, then the function $(z\text{-}1)_{0,0L}$ has a discontinuity as you cross the cut, whereas $(z\text{-}1)_{0,0R}$ has no discontinuity. In fact we have

$$(x+i\varepsilon-1)_{0,0L} = (x+i\varepsilon-1)_{0,0R}$$

$$(x-i\varepsilon-1)_{0,0L} = (x-i\varepsilon-1)_{0,0R}\ e^{-i2\pi}$$

As before, this is all in symbolic form and becomes significant when we put (z-1) into a power or log. Thus we have for a power α :

$$(x+i\varepsilon-1)^{\alpha}_{0,0L} = (x+i\varepsilon-1)^{\alpha}_{0,0R}$$

$$(x-i\varepsilon-1)^{\alpha}{}_{0,0L} = (x-i\varepsilon-1)^{\alpha}{}_{0,0R}\ e^{-i2\pi\alpha}$$

$$\text{disc}[(z-1)^{\alpha}_{0,0L}] = (x+i\varepsilon-1)^{\alpha}_{0,0L} - (x-i\varepsilon-1)^{\alpha}_{0,0L} = (x-1)^{\alpha}_{0,0R}\ [\ 1 - e^{-i2\pi\alpha}\ ]$$

And we have for a log:

$$\log(x+i\varepsilon-1)_{0,0L} = \log(x+i\varepsilon-1)_{0,0R}$$

$$\log(x-i\varepsilon-1)_{0,0L} = \log(x-i\varepsilon-1)_{0,0R} - i2\pi$$

$$\text{disc}[\ \log((z-1)_{0,0L}] = +i2\pi$$

So here is a proposed prescription to move from one world to the other:

$$(z-1)_{0,0R} = (z-1)_{0,0L} \qquad \text{for Im}(z) > 0$$
$$(z-1)_{0,0R} = (z-1)_{0,0L}\ e^{i2\pi} \qquad \text{for Im}(z) < 0$$

So this is the idea of the "half staircase". In the upper half plane we make no change, meaning winding-number 0, but in the lower half plane we have winding-number = 1 for (z-1). So we could rephrase the above this way:

$$(z-1)_{0,0R} = (z-1)_{0,0L} \qquad \text{for Im}(z) > 0$$
$$(z-1)_{0,0R} = (z-1)_{0,1L} = (z-1)_{0,0L}\ e^{i2\pi} \qquad \text{for Im}(z) < 0$$

Now suppose we have some formula which involves (z-1) which we interpret as $(z-1)_{0,0L}$. This might be some factor or F argument that appears in a Bateman relation. If we want to know what the corresponding formula is in our R world, we make the replacement $(z-1)_{0,0R} = (z-1)\ e^{i2\pi}$ but only when Im(z) < 0. So, again, if (z-1) appears in a formula, our replacement is $(z-1) \rightarrow (z-1)\ e^{i2\pi}$ when Im(z) < 0, but our replacement is $(z-1) \rightarrow (z-1)$ if Im(z) > 0.

Prescription #1 : To convert a formula from 0,0L to 0,0R:

$(z-1) \rightarrow (z-1)$ Im(z) > 0
$(z-1) \rightarrow (z-1)\ e^{i2\pi}$ Im(z) < 0 // ie, $(z-1)_R = (z-1)_L\ e^{i2\pi}$ as in note above

After the prescription is applied, we use the 0,0L parameterization of the angles (CORRECT!!!!)

This same prescription would of course convert n,mL to n,mR. It is just a simple situation that I have made tremendously complicated by being confused and stupid.

Factors like z or (z+1) undergo no change as part of this prescription.

Prescription #2 : Alternatively, just change the angle parameterization and then the prescription is to simply do nothing.

In section (f) above, we took this approach to show that

$$Q_1(i\zeta)_{0,0R} = -1 + \zeta \cot^{-1}(\zeta) \qquad \text{for all } \zeta \text{ in } (-\infty,\infty)$$

Example 1: f(z) = $Q_1(z)$ Consider,

$$Q_1(z)_{0,0L} = (z/2) \ln[(z+1)/(z-1)] - 1$$

Then we may calculate $Q_1(z)_{0,0R}$ as follows:

$$Q_1(z)_{0,0R} = (z/2) \ln[(z+1)/(z-1)_R] - 1 = (z/2) \ln[(z+1)/(z-1)_L] - 1 = Q_1(z)_{0,0L} \qquad \text{Im}(z) > 0$$

$$Q_1(z)_{0,0R} = (z/2) \ln[(z+1)/(z-1)_R] - 1 = (z/2) \ln[(z+1)/\{(z-1)_L\, e^{i2\pi}\}] - 1$$

$$= (z/2) \ln[(z+1)/\{(z-1)_L\}] - 1 + (z/2) \ln[e^{-i2\pi}]$$

$$= (z/2) \ln[(z+1)/\{(z-1)_L\}] - 1 + (z/2)(-2i\pi)$$

$$= (z/2) \ln[(z+1)/\{(z-1)_L\}] - 1 - i\pi z$$

$$= Q_1(z)_{0,0L} - i\pi z \qquad \text{Im}(z) < 0$$

which agrees with our efforts above.

3. Question: How do we apply our prescription to a Bateman formula?

For example,

$$Q_1(z)_{0,0L} = 2^{-2} \sqrt{\pi}\, \Gamma(2)/\Gamma(5/2)\, (1/z^2)\, F(3/2, 1; 5/2; 1/z^2) \qquad |z| > 1 \quad \text{// Bateman p122(5)= p134(41)}$$

It would seem that the prescription is "do nothing" because (z-1) does not appear anywhere. But we know that is wrong for the following reason. We know that for large z, $Q_1(z)_{0,0L} \rightarrow 1/z^2$ in all directions , but on the other hand we know that $Q_1(z)_{0,0R} \rightarrow z$ for a ray in the lower half plane. So perhaps we get a contradiction because we the above formula does not really apply in the neighborhood of z = 1.

We know that, in other Bateman forms, we often encounter F of the form F[a,b;c;(1-z)/2] and here we DO see our (z-1) expression and maybe we can "work with this" to learn something. When we wind our

$$F[a,b;c;-(z-1)/2] \;\rightarrow\; F[a,b;c;-(z-1)/2 * e^{i2\pi}] \qquad \text{for } Imz < 0$$

$$F[a,b;c;s] \;\rightarrow\; F[a,b;c;s\, e^{i2\pi}] \qquad \text{or } Ims > 0 \qquad s = -(z-1)/2$$

So what happens in an F function when we wind the argument once around the origin like this? Well, we know that F[a,b;c; s] is *analytic* in a unit disc around the origin, so for |s| < 1, such a wind we know will do nothing at all. In z-space this disk is radius 2 about point z = 1. This just reminds us that the function F[a,b;c; - (z-1)/2] is analytic in large disk around z = 1, and does not have a branch point at z = 1. So if we do a winding of +1, this function does not change at all. (!!)

4. Finding the large-z behavior of $Q_\nu^\mu(z)_{0,1L}$

Let's consider Bateman p 130 (32) for the Q function, and just write it this symbolic way:

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0L} = A\,(z-1)^{\mu/2}\, F_1 \;+\; B\,(z-1)^{-\mu/2}\, F_2$$

where we expose the (z-1) dependence. Now let's apply our winding +1 to get

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,1L} = e^{i\pi\mu}\; A\,(z-1)^{\mu/2} F_1 \;+\; e^{-i\pi\mu}\, B\,(z-1)^{-\mu/2} F_2$$

Our first observation is that this result is no longer a Bateman Q function! So maybe we are not surprised that its asymptotic behavior has changed. The question is then this: can we somehow write the above as a linear combination of Bateman P and Q functions?

Well, to find out, we have to dive down into specific details. Let's at least write out the F arguments:

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0L} = A\,(z-1)^{\mu/2}\, F(-\nu,1+\nu;\, 1+\mu;\, (1-z)/2) \;+\; B\,(z-1)^{-\mu/2}\, F(-\nu,1+\nu;\, 1-\mu;\, (1-z)/2)$$

where A and B are functions of ν,μ and z. Bateman p 124 (14) tells us that

$$P_\nu^\mu(z)_{0,0L} = C(\nu,\mu,z)\;(z-1)^{-\mu/2}\, F(-\nu,1+\nu;\, 1-\mu;\, (1-z)/2)$$
$$P_\nu^{-\mu}(z)_{0,0L} = C(\nu,-\mu,z)\;(z-1)^{\mu/2}\, F(-\nu,1+\nu;\, 1+\mu;\, (1-z)/2)$$

Thus, it seems that we can write

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0L} = [A / C(\nu,-\mu,z)]\, P_\nu^{-\mu}(z)_{0,0L} \;+\; [B / C(\nu,\mu,z)]\, P_\nu^\mu(z)_{0,0L}$$

This must be Bateman p 140 (4). So let's just write out this (4) so things are more explicit:

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0L} = \csc(\pi\mu)\,(\pi/2)\,[P_\nu^\mu(z)_{0,0L} \;-\; f(\nu,\mu)\, P_\nu^{-\mu}(z)_{0,0L}]$$

and we assume for the moment that $\mu \neq$ integer. Now, we know from above that

$$P_\nu^\mu(z)_{0,1L} = e^{-i\pi\mu}\, P_\nu^\mu(z)_{0,0L}$$
$$P_\nu^{-\mu}(z)_{0,1L} = e^{+i\pi\mu}\, P_\nu^{-\mu}(z)_{0,0L}$$

Then we find that

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L} = \csc(\pi\mu)\,(\pi/2)\,[P_\nu^\mu(z)_{0,1L} - f(\nu,\mu)\,P_\nu^{-\mu}(z)_{0,1L}]$$

$$= \csc(\pi\mu)\,(\pi/2)\,[e^{-i\pi\mu}\,P_\nu^\mu(z)_{0,0L} - f(\nu,\mu)\,e^{+i\pi\mu}\,P_\nu^{-\mu}(z)_{0,0L}] \qquad (*)$$

and we have now achieved our goal: we have written $Q_\nu^\mu(z)_{0,1L}$ in terms of "Bateman" P and Q functions. We could now use the large z results for these P functions to find what Q does. That is a little messy, but here might be a better way. We can use p 140 (3) which says

$$\pi\, e^{i\pi\mu}\cos(\pi\nu)\,P_\nu^\mu(z)_{0,0L} = \sin\pi(\nu+\mu)\,Q_\nu^\mu(z)_{0,0L} - \sin\pi(\nu-\mu)\,Q_{-\nu-1}^\mu(z)_{0,0L}$$

$$\pi\, e^{-i\pi\mu}\cos(\pi\nu)\,P_\nu^{-\mu}(z)_{0,0L} = \sin\pi(\nu-\mu)\,Q_\nu^{-\mu}(z)_{0,0L} - \sin\pi(\nu+\mu)\,Q_{-\nu-1}^{-\mu}(z)_{0,0L}$$

Now take (*) above and insert into it the above two expressions to get

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L}$$

$$= \csc(\pi\mu)\,(\pi/2)\,[e^{-i\pi\mu}\,\{\sin\pi(\nu+\mu)\,Q_\nu^\mu(z)_{0,0L} - \sin\pi(\nu-\mu)\,Q_{-\nu-1}^\mu(z)_{0,0L}\}/[\,\pi\, e^{i\pi\mu}\cos(\pi\nu)\,]$$

$$- f(\nu,\mu)\,e^{+i\pi\mu}\,\{\sin\pi(\nu-\mu)\,Q_\nu^{-\mu}(z)_{0,0L} - \sin\pi(\nu+\mu)\,Q_{-\nu-1}^{-\mu}(z)_{0,0L}\}/[\,\pi\, e^{-i\pi\mu}\cos(\pi\nu)]\,]$$

Let's now drop the 0,0L and assume this is implied if there is no label. Then we have

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L}$$

$$= \csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,[\;e^{-i2\pi\mu}\,\{\sin\pi(\nu+\mu)\,Q_\nu^\mu(z) - \sin\pi(\nu-\mu)\,Q_{-\nu-1}^\mu(z)\}$$

$$- f(\nu,\mu)\,e^{+i2\pi\mu}\{\sin\pi(\nu-\mu)\,Q_\nu^{-\mu}(z) - \sin\pi(\nu+\mu)\,Q_{-\nu-1}^{-\mu}(z)\}\;]$$

$f(\nu,\mu) = \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)$ Note that $f(\nu,-\mu) = 1/f(\nu,\mu)$

Now let's try to simplify a bit using Bateman p 140 (2) which says

$$f(\nu,\mu)\,e^{+i\pi\mu}Q_\nu^{-\mu}(z) = e^{-i\pi\mu}Q_\nu^\mu(z) \quad \text{or} \quad e^{+i2\pi\mu}Q_\nu^{-\mu}(z) = f(\nu,-\mu)\,Q_\nu^\mu(z)$$

which in turn says

$$f(-\nu-1,\mu)\,e^{+i\pi\mu}Q_{-\nu-1}^{-\mu}(z) = e^{-i\pi\mu}Q_{-\nu-1}^\mu(z) \quad \text{or} \quad e^{+i2\pi\mu}Q_{-\nu-1}^{-\mu}(z) = f(-\nu-1,-\mu)\,Q_{-\nu-1}^\mu(z)$$

Then we have

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L}$$

$$= \csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,[\; e^{-i2\pi\mu}\,\{\,\sin\pi(\nu+\mu)\,Q_\nu^\mu(z) \;-\; \sin\pi(\nu-\mu)\,Q_{-\nu-1}^\mu(z)\,\}$$

$$-\,f(\nu,\mu)\,\{\,\sin\pi(\nu-\mu)\,[f(\nu,-\mu)\,Q_\nu^\mu(z)\,] \;-\; \sin\pi(\nu+\mu)\,[f(-\nu-1,-\mu)\,Q_{-\nu-1}^\mu(z)]\,\}\;]$$

so we can now regroup to get

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L} = Q_\nu^\mu(z) \quad \csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,\{\; e^{-i2\pi\mu}\sin\pi(\nu+\mu) - \sin\pi(\nu-\mu)\,\}$$

$$+\; Q_{-\nu-1}^\mu(z)\,\csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,\{-e^{-i2\pi\mu}\sin\pi(\nu-\mu) + \sin\pi(\nu+\mu)\,f(\nu,\mu)\,f(-\nu-1,-\mu)\,\}$$

$$f(\nu,\mu) = \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)$$

This is not pretty, but the ugliness is just in the constants which could perhaps be simplified. The main point so far concerns large z behavior. We can see that

$$Q_\nu^\mu(z)_{0,1L} \rightarrow\; \alpha z^{-\nu-1} \;+ \beta\, z^\nu$$

and this then, finally, explains our observation that $Q_1(z)_{0,0R} \rightarrow z$ for $\mathrm{Im}(z) < 0$, and finally we are getting somewhere!

So let's try for some simplification of the coefficients.

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L} = E\,Q_\nu^\mu(z) + F\,Q_{-\nu-1}^\mu(z)$$

$$E = \csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,\{\; e^{-i2\pi\mu}\sin\pi(\nu+\mu) - \sin\pi(\nu-\mu)\,\}$$
$$F = \csc(\pi\mu)\,(1/2)\sec(\pi\nu)\,\{-e^{-i2\pi\mu}\sin\pi(\nu-\mu) + \sin\pi(\nu+\mu)\,f(\nu,\mu)\,f(-\nu-1,-\mu)\,\}$$

$$f(\nu,\mu) = \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)$$

$$f(-\nu-1,-\mu) = \Gamma(-\nu-\mu)/\Gamma(-\nu+\mu)$$

We quote Euler's Reflection rule

$$\Gamma(z+1)\Gamma(-z) = -\,\pi/\sin(\pi z)$$

to find that

$$\Gamma(\nu+\mu+1)\,\Gamma(-\nu-\mu) = -\,\pi\sin\pi(\nu+\mu)$$
$$\Gamma(-\nu+\mu+1)\,\Gamma(+\nu-\mu) = -\,\pi\sin\pi(-\nu+\mu) = \pi\sin\pi(\nu-\mu)$$

This says that

$$f(\nu,\mu)\,f(-\nu-1,-\mu) = [-\pi\sin\pi(\nu+\mu)]\,/\,[\pi\sin\pi(\nu-\mu)\,] = -\sin\pi(\nu+\mu)/\sin\pi(\nu-\mu)$$

Then we have

$$F = \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ - e^{-i2\pi\mu}\sin\pi(\nu-\mu) + \sin\pi(\nu+\mu)\,f(\nu,\mu)\,f(-\nu-1,-\mu)\}$$

$$= \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ - e^{-i2\pi\mu}\sin\pi(\nu-\mu) - \sin\pi(\nu+\mu)^2/\sin\pi(\nu-\mu)\}$$

so we then have:

$$E = \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ e^{-i2\pi\mu}\sin\pi(\nu+\mu) - \sin\pi(\nu-\mu)\}$$
$$F = \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ - e^{-i2\pi\mu}\sin\pi(\nu-\mu) - \sin\pi(\nu+\mu)^2/\sin\pi(\nu-\mu)\}$$

Why is this looking so bad? Maybe time for an algebra recheck before grinding further? Well, let's plod on just a bit more. What is the limit of the above E and F as $\mu\to 0$?

$e^{-i2\pi\mu} \approx 1 - i2\pi\mu$
$\sin\pi(\nu+\mu) \approx \sin\pi\nu + \mu\,\pi\cos\pi(\nu)$
$\sin\pi(\nu-\mu) \approx \sin\pi\nu - \mu\,\pi\cos\pi(\nu)$

so in E we get

$$\{ (1 - i2\pi\mu)\,(\sin\pi\nu + \mu\,\pi\cos\pi(\nu)) - \sin\pi\nu - \mu\,\pi\cos\pi(\nu)\}$$

$$= \sin\pi\nu - \sin\pi\nu - i2\pi\mu\sin\pi\nu + \mu\,\pi\cos\pi(\nu) - \mu\,\pi\cos\pi(\nu) + O(\mu^2)$$

$$= - i2\pi\mu\sin\pi\nu$$

so then we have

$$E = \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ - i2\pi\mu\sin\pi\nu\} = (1/2)\,\sec(\pi\nu)\,\{ - i2\pi\mu\sin\pi\nu\}/\pi\mu$$

$$= (1/2)\,\sec(\pi\nu)\,\{ - i2\sin\pi\nu\} = -i\sin\pi\nu/\cos\pi\nu = -i\tan(\pi\nu)$$

Meanwhile, for F we don't have 0/0 so we can write it as follows:

$$F = \csc(\pi\mu)\,(1/2)\,\sec(\pi\nu)\,\{ - \sin\pi\nu - \sin\pi\nu\} = - \tan(\pi\nu)/\sin(\pi\mu).$$

This is infinite unless $\nu = 1$, say. But keep as is for the moment. We then seem to get

$$Q_\nu(z)_{0,1L} = -i\tan(\pi\nu)\,Q_\nu(z) - \tan(\pi\nu)/\sin(\pi\mu).\,Q_{-\nu-1}(z) \qquad \text{// } \mu \text{ missing?}$$

Backstep 1. But let's go back to an earlier form (*) where we had:

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L} = \csc(\pi\mu)\,(\pi/2)\,\{ e^{-i\pi\mu}\,P_\nu^\mu(z) - \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)\,e^{+i\pi\mu}\,P_\nu^{-\mu}(z)\}$$

Suppose now I want to know the large z behavior of $Q_\nu^\mu(z)_{0,1L}$. We can use Bateman p 126 (23) which has this form:

$$P_\nu^\mu(z) = Az^{-\nu-1} + Bz^\nu$$

The second term dominates assuming $\nu > -1/2$ (I think) so let's keep just this term:

$$P_\nu^\mu(z) \rightarrow (2^\nu/\sqrt{\pi})\,\Gamma(\nu+1/2)\,[1/\Gamma(1+\nu-\mu)]\,z^\nu * G(\nu,\mu; z^{-2})$$

where I write the F function as G, to reduce symbol count. Since F and G are series $1 + B/z^2 + C/z^4$ ..., we see that ALL powers from z^ν on down "see" the same $[1/\Gamma(1+\nu-\mu)]$ unless it is accidentally cancelled by something like B or C. In fact

$$G(\nu,\mu; z^{-2}) = F(-\nu/2-\mu/2,\ 1/2-\nu/2-\mu/2;\ 1/2-\nu;\ z^{-2})$$

For the time being, let's assume there is no such accidental cancellation, and we set F = G = 1 for limit.

Then here is what we have:

$$e^{-i\pi\mu}Q_\nu^\mu(z)_{0,1L} \rightarrow \csc(\pi\mu)\,(\pi/2)\,(2^\nu/\sqrt{\pi})\,\Gamma(\nu+1/2)\,z^\nu *$$

$$\{ e^{-i\pi\mu}\,[1/\Gamma(1+\nu-\mu)] - e^{+i\pi\mu}\,(\Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1))\,[1/\Gamma(1+\nu+\mu)] \}$$

$$= \csc(\pi\mu)\,(\pi/2)\,(2^\nu/\sqrt{\pi})\,[\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\,z^\nu\,\{ e^{-i\pi\mu} - e^{+i\pi\mu} \}$$

$$= -\csc(\pi\mu)\,(\pi/2)\,(2^\nu/\sqrt{\pi})\,[\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\,z^\nu\,\{2i\sin(\pi\mu)\}$$

$$= -2i\,(\pi/2)\,(2^\nu/\sqrt{\pi})\,[\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\,z^\nu$$

$$= -i\sqrt{\pi}\ 2^\nu\ [\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\,z^\nu$$

which is a pretty simple result I would say. Let's test it for $\mu = 0$:

$$Q_\nu(z)_{0,1L} \rightarrow -i\sqrt{\pi}\ 2^\nu\ [\Gamma(\nu+1/2)/\Gamma(1+\nu)]\,z^\nu$$

$$Q_1(z)_{0,1L} \rightarrow -i\sqrt{\pi}\ 2\ [\Gamma(1+1/2)/\Gamma(1+1)]\,z^\nu = -i\sqrt{\pi}\ 2\,\Gamma(3/2)\,z^\nu$$

$$= -i\sqrt{\pi}\ 2(\sqrt{\pi}/2)\,z^\nu = -i\pi z$$

and, loud yelps, this does agree with our result above where we had

$$Q_1(z)_{0,1L} = Q_1(z)_{0,0L} - i\pi z = Q_1(z) - i\pi z$$

So here is My Big Result that I have been after for a long time:

$$\lim_{z\to\infty} e^{-i\pi\mu} Q_\nu^\mu(z)_{0,1L} = -i\sqrt{\pi}\ 2^\nu\ [\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\ z^\nu$$

so this is the limit that applies for a ray in the lower half plane for $e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0R}$.

You might kill off the z^ν divergence of this Q by choosing a specific spectrum for ν.

[At this point I am "pre cone method" so here is an idea that might say which "n" values are allowed such that the above Q decays rather than blows up. I no longer think this is a useful idea.]

One implication here is that you can *perhaps* kill off the divergence z^ν if you arrange to have $\Gamma(1+\nu-\mu)$ have a pole, which means if you select $1+\nu-\mu = 0,-1,-2...$ or $\nu-\mu = -1,-2...$ which says $\nu = \mu - 1, \mu-2$, etc. By our assumption just stated above, this would also kill off terms like $z^{\nu-2}$, $z^{\nu-4}$, etc which appear as we write out the F series, and which could potentially dominate. For such values of ν, our limit would then be $z^{-\nu-1}$.

Now consider this "spectral rule"

Theorem: For $\nu = \mu - 1, \mu-2, ...$ the function $e^{-i\pi\mu} Q_\nu^\mu(z)_{0,0R}$ will decay as $z^{-\nu-1}$ in all directions.

So what functions would be in this spectrally allowed set?

$Q_{\mu-J}^\mu(z)$ are allowed $J = 1,2,3...$

Examples:

$Q_{-1}^0(z)$, ...
$Q_0^1(z)$, $Q_{-1}^1(z)$, ...
$Q_1^2(z)$, $Q_0^2(z)$. ...
$Q_2^3(z)$, $Q_1^3(z)$, $Q_0^3(z)$...
etc.

What can be said about ν = - integer? Well, let's back step above to this point:

$$e^{-i\pi\mu} Q_\nu^\mu(z)_{0,1L} = \csc(\pi\mu)\ (\pi/2)\ \{\ e^{-i\pi\mu}\ P_\nu^\mu(z)\ -\ \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)\ e^{+i\pi\mu}\ P_\nu^{-\mu}(z)\ \}$$

$$P_\nu^\mu(z) \ \to\ (2^\nu/\sqrt{\pi})\ \Gamma(\nu+1/2)\ [1/\Gamma(1+\nu-\mu)]\ z^\nu\ +\ A_\nu^\mu\ z^{-\nu-1}$$
$$P_\nu^{-\mu}(z) \to\ (2^\nu/\sqrt{\pi})\ \Gamma(\nu+1/2)\ [1/\Gamma(1+\nu+\mu)]\ z^\nu\ +\ A_\nu^{-\mu}\ z^{-\nu-1}$$

If we allow $\nu < -1/2$, then we have to worry about the second terms like $A_\nu^\mu\ z^{-\nu-1}$ because they start blowing up then. Both these "second terms" contain $1/\Gamma(-\nu-\mu)$ and so die if $-\nu-\mu = 0,-1,-2..$ which means when $\nu = -\mu$, $-\mu+1$, etc. I should study all this, but for the moment, I am going to make this assumption: we shall not include the $\nu < 0$ Q functions in our list of spectrally allowed functions. Maybe this is wrong, but let's assume if for now.

Then here is are spectrally allowed set (ie, set that converge on (-i∞,i∞) :

$Q_{\mu-J}^{\mu}(z)$ are allowed J = 1,2,3...

Examples:

$Q_0^1(z)$
$Q_1^2(z)$, $Q_0^2(z)$
$Q_2^3(z)$, $Q_1^3(z)$, $Q_0^3(z)$
etc.

Notice that $Q_1(z)$ is not included, and we know why. All the functions above are meant to be 0,0R. You first write them out in their usual 0,0L form, they you "continue them" going south.

Technical Problem: If I do this in a Smythian form, if there is a corresponding $P_n^m(\xi)$, it gets killed off for every one of these spectrally allowed functions! So this kills off the entire Smythian form!!

So, my entire idea and plan has now completely collapsed!

But wait. Since we only use ξ in the range (1,0) with the new parameterization, anything concerning this function at ξ = -1 no longer applies. So maybe any n is now allowed on $P_n^m(\xi)$, not just integer n. But I do know when things are integers, this vanishes for n out of the normal range.

Ponder this just a bit. We do in fact have $P_n^m(\xi)$ as a factor in both lines of Smythian form. We know that m can be any integer from azimuth.

If we consider just negative m integers, then Bateman p 125 (14) is our rep for $P_n^m(\xi)$ very happy when ξ in (0,1). We have

$$P_n^m(\xi) = (\xi+1)^{m/2}\,(\xi-1)^{-m/2} / \Gamma(1-m) \;*\; F(-n,1+n;\, 1-m;\, (1-\xi)/2\,)$$

So let's take m = 0,-1,-2... . All factors are happy including the first since we never get near ξ = -1.
The F function converges for disk radius 2 around ξ = 1. and this includes all of (0,1), completely clean.
So it seems to me that there is no restriction at all on n, it can be any complex value and we converge here. So this is something very new to me.

Now can we extend this a bit using Bateman p 140 (7) ?

$P_\nu^m(\xi) = f(\nu,m)\, P_\nu^{-m}(\xi)$ m = 1,2,3... he claims

So take m → -m and this says

$P_\nu^{-m}(\xi) = f(\nu,-m)\, P_\nu^m(\xi)$ m = -1,-2,-3... he claims

So I think we then have, again for m = -1,-2,-3..,

$P_\nu^{-m}(\xi) = [\ \Gamma(\nu-m+1)/\ \Gamma(\nu+m+1)\ ]\ (\xi+1)^{m/2}\ (\xi-1)^{-m/2}\ /\ \Gamma(1-m)\ *\ F(-\nu,1+\nu;\ 1-m;\ (1-\xi)/2\ )$

So this gives us the P for positive upper indices. But vanishes when ν-m+1 = 0,-1.. or ν-m = -1,-2 or ν = m-1, m-2.... or ν = -|m|-1, -|m|-2, ... So these particular negative integer ν values would have killed off $P_\nu^{|m|}(\xi)$ functions. So here is my conclusion:

$P_\nu^{-|m|}(\xi) = (\xi+1)^{-|m|/2}\ (\xi-1)^{|m|/2}\ /\ \Gamma(1+|m|)\ *\ F(-\nu,1+\nu;\ 1+|m|;\ (1-\xi)/2\ )$
= exists, is non-singular, non-zero and convergent in (0,1) for all complex values of ν

$P_\nu^{+|m|}(\xi) = [\ \Gamma(\nu+|m|+1)/\ \Gamma(\nu-|m|+1)\ ]\ P_\nu^{-|m|}(\xi)$
= exists, but is singular for ν = -|m|-1, -|m|-2 , vanishes for ν = |m|-1, |m|-2,
and is otherwise convergent in (0,1). For ν to the left of -|m|, zero and pole cancel.
So all we know is that $P_n^{+|m|}(\xi)$ vanishes for n < |m| which is "the usual thing".
But for non-integer ν, $P_\nu^{+|m|}(\xi)$ is fine for all complex ν.

Therefore, I conclude that $P_\nu^m(\xi)$, where m is any integer, is non-singular for any complex ν, assuming that our range of interest for ξ is (0,1) only. If we restrict to ν = integer, then we find that

$P_\nu^{-|m|}(\xi)$ is non-zero, non-singular, convergent for ν = any integer (either sign)

$P_\nu^{+|m|}(\xi)$ same thing except it vanishes for integers ν < |m|.

In general, we know $P_\nu = P_{-\nu-1}$ so we can show negative ν values if it becomes necessary, but let's not show these yet. So, in this integer ν case, here are some contributing $P_n^m(\xi)$ functions:

etc for higher m
$P_3^3(\xi)$, $P_4^3(\xi)$, $P_5^3(\xi)$....
$P_2^2(\xi)$, $P_3^2(\xi)$, $P_4^2(\xi)$....
$P_1^1(\xi)$, $P_2^1(\xi)$, $P_3^1(\xi)$,...
$P_0^0(\xi)$, $P_1^0(\xi)$, $P_2^0(\xi)$,...
$P_0^{-1}(\xi)$, $P_1^{-1}(\xi)$, $P_2^{-1}(\xi)$,...
$P_0^{-2}(\xi)$, $P_1^{-2}(\xi)$, $P_2^{-2}(\xi)$,...
etc for lower m.

Meanwhile, here was our allowed values for Q (quoted from above)

$Q_0^1(z)$
$Q_1^2(z)$, $Q_0^2(z)$
$Q_2^3(z)$, $Q_1^3(z)$, $Q_0^3(z)$
etc.

So our problem of "no overlap" seems to still exist. We don't show negative ν on the Q's because we think this diverge for large z. But what about negative m values on the Q's? My Q rule was this:

$Q_{\mu-J}^{\mu}(z)$ are allowed J = 1,2,3...

But for negative μ, you get negative ν which I think makes us diverge!

Conclusion: if I write my Smythian form like this:

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

then there exist no combinations $P_n^m(\xi)\, Q_n^m(j\zeta)$ where both factors are non-zero, such that the functions are always finite and decay at ∞. The mismatch is amazingly precise! The mystery continues for yet another day, and another, and another... [this is all before I switched to "the cone method".]

********** **scraps below this line** **********

****

I keep being led to this slightly frightening conclusion:

Conclusion #1: All the Bateman Legendre expressions and relations are only meaningful on the 0,0L principle sheet. There is no way to "continue them" down through the cut unless you are able to write out the specific function that $Q_\nu^\mu(z)$ represents.

Thus, for example, given the above result that

$$Q_1(z)_{0,0L} = 2^{-2}\sqrt{\pi}\,\Gamma(2)/\Gamma(5/2)\,(1/z^2)\,F(3/2,\,1;\,5/2;\;1/z^2) \qquad |z|>1 \qquad \text{// Bateman p 122 (5)}$$

we are helpless to continue this onto

because we get the wrong large-z behavior which is $Q_1(z)_{0,0R} \sim -i\pi z$ Im(z) < 0.

You have to wonder how this gets to be

$$Q_1(z) = (z/2)\ln[(z+1)/(z-1)] - 1$$

****

Example 1: $f(z) = Q_1(z)$ Consider,

$$Q_1(z)_{0,0L} = (z/2)\ln[(z+1)/(z-1)] - 1$$

Then we may calculate $Q_1(z)_{0,0R}$ as follows:

$$Q_1(z)_{0,0R} = (z/2)\ln[(z+1)/(z-1)] - 1 = Q_1(z)_{0,0L} \qquad Im(z) > 0$$

$$Q_1(z)_{0,0R} = (z/2)\ln[(z+1)/(z-1)] - 1 + (z/2)(-2i\pi) = Q_1(z)_{0,0L} - i\pi z \qquad Im(z) < 0$$

which agrees with our efforts above.

Example 2: $f(z) = (z-1)^{\alpha}$

For Imz> 0, we just write $f(z)_{0,0L} = f(z)_{0,0R} = (z-1)^{\alpha}$

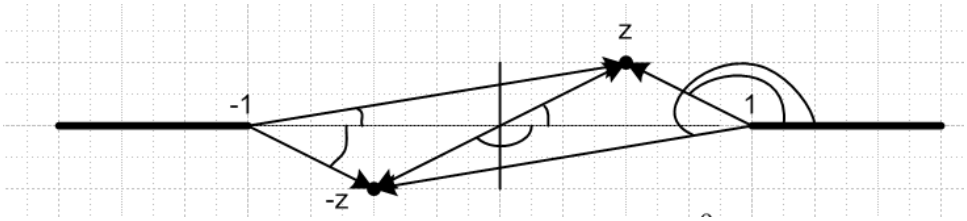
Now as a convention, let's use this notation

$$(z-1)_{0,0L} = (z-1)$$
$$(z-1)_{0,0R} = (z-1)_R$$

It is not immediately obvious how to even "find" the P and Q functions on the 0,0R sheet. The entire Bateman Legendre chapter concerns the 0,0L functions. Somehow, the secret lies in the F functions, because the other z dependence can only create a phase. You could never, for example, develop a new factor of "z" as we say in the Q_1 case above where we found that:

****

We of course define the principle sheet for P and Q as the sheet where the angles are as shown here:



Our main work is interpreting the factor $(z-1)^{\alpha}$ where it appears in the various Bateman expressions. The branch point at z = -1 is unchanged. So we think this way

$(z-1) = |z-1| e^{i\theta}$ // angle shown above as inner smaller angle top right.

$(z-1)^{\alpha} = |z-1|^{\alpha} e^{i\alpha\theta}$ θ in range $(0,2\pi)$

Obviously this function and the P and Q functions have no discontinuity on (-1,1) since θ is continuous there. We now want to consider ALL of our "rules" and see what happens to them. This could take me a very long time. That is because for most of the rules, I just quoted from Bateman, but now I would have to actually derive each one.

We have a **few useful guidelines**:

(1) for things only in the upper half plane, everything is the same as in my Legendre doc. The reason is this. With the cut pulled left, the upper half plane was θ in $(0,\pi)$ for angle of (z-1), our of range $(-\pi,\pi)$. With cut pulled right, the upper half plane has θ in $(0,\pi)$ for angle of (z-1), our of range $(0,2\pi)$. So if we only look at the range $(0,\pi)$, the two functions are identical!!!

Value of $Q_{\nu}^{\mu}(0)$

For example, for cut to the left we had (L means cut to the left)

$$Q_{\nu}^{\mu}(z=0\pm)_{0,0L} = \sqrt{\pi}\, 2^{\mu-1} e^{\pm i\pi(-\nu-1)/2} [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)] \quad // \text{p 134 (40)}$$

Therefore we may conclude that

$$Q_{\nu}^{\mu}(z=0+)_{0,0R} = \sqrt{\pi}\, 2^{\mu-1} e^{+i\pi(-\nu-1)/2} [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)] \quad // \text{p 134 (40)}$$

But for R versions, this limit is the same approached from either direction, so we say

$$Q_{\nu}^{\mu}(0)_{0,0R} = \sqrt{\pi}\, 2^{\mu-1} e^{+i\pi(-\nu-1)/2} [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)]$$

Large z behavior of $Q_\nu^\mu(z)$

How do we figure this out for the lower half plane of the 0,0R principle sheet. Here is what we know:

$$Q_1(z)_{0,1L} = 2^{-2}\sqrt{\pi}\,\Gamma(2)/\Gamma(5/2)\;\; z^{-2}\,F(3/2,\,1;\,5/2;\;\,1/z^2) \qquad \text{// Bateman p 122 (5)}$$

You have to wonder how this gets to be

$$Q_1(z) = (z/2)\,\ln\,[\,(z+1)/(z-1)\,] - 1$$

I have shown above that

$$Q_1(-z)_{0,1L} = Q_1(z)_{0,1L} + 2i\pi z$$

from which we get a hint that "something is fishy" with the idea that $Q_1 \rightarrow z^{-2}$ for large $|z|$.