

SPHEROIDAL FUNCTIONS OF THE SECOND KIND

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In a recent note in these PROCEEDINGS,¹ henceforth to be referred to as I, certain solutions of the equation

$$(1 - z^2)w'' - 2(a + 1)zw' + (b - c^2z^2)w = 0 \quad (1)$$

were defined. Corresponding to a discrete set of characteristic values of the parameter b_l there exist solutions of the type

$$Re_{a,l}^1(c, z) = (cz)^{-a-1/2} \sum_n' a_n^l J_{n+a+1/2}(cz), \quad (I-10)$$

convergent throughout the entire z -plane and behaving as $(cz)^{-a-1} \sin \left(cz - \frac{l+a}{2} \pi \right)$ as $z \rightarrow \infty$. In the neighborhood of $z = 0$ a formal solution may be obtained in terms of an expansion in hypergeometric functions,

$$Se_{a,l}^1(c, z) = \sum_n' d_n^l T_n^a(z), \quad (I-8)$$

which for the characteristic values b_l is also bounded at the singularities $z = \pm 1$. With the aid of the integral representation (I-18) it was shown that (I-8) and (I-10) are proportional,

$$Se_{a,l}^1(c, z) = \lambda_l k_l Re_{a,l}^1(c, z), \quad (I-26)$$

and that the expansion coefficients are related by the simple expression (I-22).

An independent particular solution of (1) behaving at infinity as $(cz)^{-a-1} \cos \left(cz - \frac{l+a}{2} \pi \right)$ may be defined formally by

$$R_{a,l}^2(c, z) = (cz)^{-a-1/2} \sum_n' a_n^l N_{n+a+1/2}(cz). \quad (I-11)$$

This expansion, however, diverges in general at the branch points $z = \pm 1$ and consequently is usable only for large values of the independent variable. It is the object of the present note to define a second solution valid in the neighborhood of $z = 0$, and to determine its analytic connection with the functions (I-10), (I-11).

By analogy with the definition (I-4) of the functions T_n^a , let

$$\mathbf{T}_n^a(z) = (1 - z^2)^{-a/2} Q_{n+a}^a(z), \quad (2)$$

where the Q_{n+a}^a are Associated Legendre functions of the second kind as defined by Hobson.² If $R(n+a+1) > 0$,

$$\mathbf{T}_n^a = \frac{e^{a\pi i} \Gamma(n+2a+1)}{2^{n+a+1} \Gamma(n+a+1)} \int_{-1}^1 (1-t^2)^{n+a} (z-t)^{-n-2a-1} dt. \quad (3)$$

Since both the T_n^a and the $\mathbf{T}_n^a$ thus defined satisfy the same recursion formulas, a second formal solution of (1) may be defined,

$$Se_{a,l}^2(c, z) = \sum_n' d_n^l \mathbf{T}_n^a(z), \quad (4)$$

which may be expected to converge in the neighborhood of $z = 0$, since $\sum_n' d_n^l$ is bounded when b assumes one of the characteristic values b_l .

It is known that (1) is satisfied by the Laplace transform

$$w(z) = \int_C e^{icz} (1-t^2)^a u(t) dt, \quad (\text{I-18})$$

where $u(t)$ itself satisfies (1) and the contour C is to be determined by the bilinear concomitant

$$\left| e^{icz} (1-t^2)^{a+1} \left[iczu - \frac{du}{dt} \right] \right|_C \equiv 0. \quad (\text{I-19})$$

As in I let $u(t) = Se_{a,l}^1(t) = \lambda_l k_l Re_{a,l}^1(ct)$. If $t \rightarrow \infty$ and $|arg ct| < \pi$,

$$Se_{l,a}^1(t) \rightarrow \lambda_l k_l (ct)^{-a-1} \sin \left(ct - \frac{l+a}{2} \pi \right).$$

Let it be assumed that $I[c(z+1)] > 0$, $I[c(z-1)] > 0$. This is in accord with the conditions of most physical problems wherein z is real and $I(c) > 0$. The contour may now start and terminate at $t = +\infty$. In particular a solution may be represented by

$$w(z) = \int_{\infty}^{(-1+1+)} e^{icz} (1-t^2)^a Se_{a,l}^1(t) dt. \quad (5)$$

$$\text{Let } I_n = \int_{\infty}^{(-1+1+)} e^{icz} (1-t^2)^a T_n^a(t) dt.$$

Then on replacing T_n^a by the differential expression

$$T_n^a(t) = \frac{(-1)^n \Gamma(n+2a+1)}{2^{n+a} \Gamma(n+1) \Gamma(n+a+1)} (1-t^2)^{-a} \frac{d^n}{dt^n} (1-t^2)^{n+a}, \quad (6)$$

and integrating partially one finds

$$I_n = \frac{\Gamma(n+2a+1)}{2^{n+a} \Gamma(n+1) \Gamma(n+a+1)} i^n e^{(n+a)\pi i} (cz)^n \int_{\infty}^{(-1+1+)} e^{icz} (t^2-1)^{n+a} dt. \quad (7)$$

It has been shown by Watson³ that

$$J_{-n-a-1/2}(cz) = \frac{\Gamma(-n-a)e^{(n+a)\pi i}(cz)^{n+a+1/2}}{2^{n+a+1}\pi\sqrt{2\pi}} \int_{\infty}^{(-1+,1+)} e^{icst}(t^2-1)^{n+a}dt, \quad (8)$$

and hence

$$I_n = \frac{i^n \Gamma(n+2a+1)}{\Gamma(n+1)} (-1)^{l+1} 2\sqrt{2\pi} \sin a\pi (cz)^{-a-1/2} J_{-n-a-1/2}(cz). \quad (9)$$

Multiply both sides of (9) by d_n^l , replace d_n^l on the right side by a_n^l according to (I-22), and sum over odd or even values of n as l is odd or even. There results

$$w(z) = \int_{\infty}^{(-1+,1+)} e^{icst}(1-t^2)^a Se_{a,l}^1(t)dt = \frac{2k_l(-1)^{l+1} \sin a\pi (cz)^{-a-1/2}}{\sum_n'} a_n^l J_{-n-a-1/2}(cz). \quad (10)$$

The integral thus evaluated may be readily expressed in terms of the functions $Re_{a,l}^1$ and $Re_{a,l}^2$. In virtue of the definition of the Neumann function $N_{n+a+1/2}$ when n is an integer,

$$N_{n+a+1/2} = -\tan a\pi J_{n+a+1/2} - \frac{(-1)^n}{\cos a\pi} J_{-n-a-1/2}, \quad (11)$$

it follows that

$$(-1)^{l+1}(cz)^{-a-1/2} \sum_n' a_n^l J_{-n-a-1/2}(cz) = \cos a\pi [\tan a\pi Re_{a,l}^1 + Re_{a,l}^2]. \quad (12)$$

In the domain $1 < z \leq \infty$ one has therefore

$$\int_{\infty}^{(-1+,1+)} e^{icst}(1-t^2)^a Se_{a,l}^1(t)dt = k_l \sin 2a\pi [\tan a\pi Re_{a,l}^1 + Re_{a,l}^2]. \quad (13)$$

The contour integral must now be evaluated for values of z lying in domain $-1 < z < 1$. To accomplish this, let $Se_{a,l}^1(t)$ be replaced by $\lambda_l k_l Re_{a,l}^1(ct)$ in the integrand of (13) and consider integrals of the type

$$K_n = \int_{\infty}^{(-1+,1+)} e^{icst}(t^2-1)^a (ct)^{-a-1/2} J_{n+a+1/2}(ct)dt. \quad (14)$$

The Bessel function may be replaced by its representation as a definite integral,

$$J_{n+a+1/2}(ct) = \frac{(ct)^{n+a+1/2}}{2^{n+a}\sqrt{2\pi}\Gamma(n+a+1)} \int_{-1}^1 e^{ias}(1-s^2)^{n+a}ds, \quad (15)$$

and on reversing the order of integration there results for (14),

$$K_n = \frac{1}{2^{n+a}\sqrt{2\pi}\Gamma(n+a+1)} \int_{-1}^1 ds (1-s^2)^{n+a} \int_{\infty}^{(-1+,1+)} e^{ict(s-s)} (t^2-1)^a (ct)^n dt. \quad (16)$$

Let the contour of integration be taken around the points $(-1, +1)$ such that $t > 1$ over the entire path. Then $(t^2-1)^a$ may be expanded in a convergent series of inverse powers of t , and it may be shown⁴ that

$$\int_{\infty}^{(-1+,1+)} e^{ict(s-s)} (t^2-1)^a (ct)^n dt = \frac{2\pi e^{-a\pi i - \frac{n\pi i}{2}}}{c^{2a+1}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a)}{m! \Gamma(-a) \Gamma(2m-n-2a)} c^{2m} (z-s)^{2m-2a-n+1}. \quad (17)$$

$$K_n = \frac{1}{2^{n+a}\sqrt{2\pi}\Gamma(n+a+1)} \frac{2e^{-a\pi i - \frac{n\pi i}{2}} (-1)^{l+1} \sin 2a\pi}{c^{2a+1}} \times \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a) \Gamma(n+2a+1-2m)}{m! (-a)} c^{2m} \int_{-1}^1 (1-s^2)^{n+a} (z-s)^{2m-2a-n-1} ds \quad (18)$$

Now by (3) it appears that

$$\int_{-1}^1 (1-s^2)^{n+a} (z-s)^{2m-2a-n-1} ds = \frac{2^{n+a+1} \Gamma(n+a+1)}{\Gamma(n+2a-2m+1)} e^{-a\pi i} \mathbf{T}_{n+2m}^{a-2m} \quad (19)$$

and on substituting in (18) and reducing, one obtains

$$K_n = 2\sqrt{\frac{2}{\pi}} \frac{e^{-2a\pi i} \sin 2a\pi (-1)^{l+1}}{c^{2a+1}} e^{-\frac{n\pi i}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a)}{m! \Gamma(-a)} c^{2m} \mathbf{T}_{n+2m}^{a-2m}. \quad (20)$$

Finally, in virtue of (14) and (13), there results

$$\sum_n' a_n^l K_n = \frac{e^{-a\pi i}}{\lambda_l k_l} \int_{\infty}^{(-1+,1+)} e^{ict} (1-t^2)^a \text{Se}_{a,l}^1(t) dt, \quad (21)$$

or

$$\tan a\pi \text{Re}_{a,l}^1 + \text{Re}_{a,l}^2 = \lambda_l 2\sqrt{\frac{2}{\pi}} \frac{e^{-a\pi i} (-1)^{l+1}}{c^{2a+1}} \sum_n' a_n^l e^{-\frac{n\pi i}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a)}{m! \Gamma(-a)} c^{2m} \mathbf{T}_{n+2m}^{a-2m} \quad (22)$$

The right-hand side of (22) is next to be expressed as a multiple of the function $\text{Se}_{a,l}^2$ defined in (4). In the simple case $a = 0$ corresponding to axially symmetric spheroidal functions, the summation over m breaks

off with the first term. Replacing a_n^l by d_n^l according to (I-22) yields at once

$$Re_{0,l}^2 = -\frac{2\lambda_l}{c} e^{\frac{i\pi l}{2}} Se_{0,l}^2. \quad (23)$$

In order to determine the proportionality factor, let us write

$$\sum_n' d_n^l e^{-\frac{\pi i l}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m-a)}{m! \Gamma(-a)} c^{2m} T_{n+\frac{1}{2}m}^{a-\frac{1}{2}m} = \mu_l(c) \sum_n' d_n^l T_n^a, \quad (24)$$

and evaluate at the ordinary point $z = 0$. By an obvious modification of Hobson's expressions for the functions Q_{n+a}^a on the cross cut between $\pm 1^5$ we have

$$T_n^a(0) = -2^{a-1} \sin \frac{n+2a}{2} \pi \frac{\Gamma\left(\frac{n+2a+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \sqrt{\pi}, \quad (25)$$

and a corresponding expression for $T_{n+\frac{1}{2}m}^{a-\frac{1}{2}m}(0)$. Introducing these values into (24) one obtains an expression for the joining factor μ_l in terms of the known expansion coefficients.

$$\begin{aligned} \mu_l(c) \sum_n' d_n^l e^{\frac{i\pi l}{2}} \frac{\Gamma\left(\frac{n+2a+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \\ \sum_n' a_n^l \sum_{m=0}^{\infty} \frac{\Gamma(m-a) c^{2m}}{m! \Gamma(-a) 2^{2m}} \frac{\Gamma\left(\frac{n+2a-2m+1}{2}\right)}{\Gamma\left(\frac{n+2m+2}{2}\right)}. \end{aligned} \quad (26)$$

The analytic connection between the functions of the second kind is therefore

$$\tan a\pi Re_{a,l}^1 + Re_{a,l}^2 = 2\sqrt{\frac{2}{\pi}} \frac{(-1)^{l+1} e^{-a\pi i}}{c^{2a+1}} \lambda_l \mu_l Se_{a,l}^2. \quad (27)$$

In case a is an integer, as in the spheroidal wave functions, $Se_{a,l}^2$ represents the analytic continuation of $Re_{a,l}^2$ into the origin. In particular

$$\mu_l = \sqrt{\frac{\pi}{2}} e^{-\frac{\pi i l}{2}}, \quad (a = 0), \quad (28)$$

and

$$\mu_l = \sqrt{\frac{\pi}{2}} e^{-\frac{\pi i l}{2}} \left(\frac{b_l}{2} - \frac{c^2}{2} + 1 \right), \quad (a = 1). \quad (29)$$

If a is half an odd integer, as in the case of the Mathieu functions ($a = -1/2$), $\tan a\pi$ in (27) becomes infinite. The function $(cz)^{-a-1/2} \sum_n' a_n^l J_{-n-a-1/2}(cz)$, however, is then no longer independent of $Re_{a,l}^1$ and, as might be expected, it is easy to show that $Se_{a,l}^2$ is now proportional to $Se_{a,l}^1$. An independent expression for $Se_{a,l}^2$ applicable in this case may be obtained by a limiting process, the details of which will be published later.

¹ J. A. Stratton, "Spheroidal Functions," *Proc. Nat. Acad. Sci.*, **21**, 51-56 (1935).

² E. W. Hobson, *Theory of Spherical and Ellipsoidal Harmonics*, p. 193.

³ G. N. Watson, *Theory of Bessel Functions*, p. 164.

⁴ Whittaker and Watson, *Modern Analysis*, 4th Ed., p. 245.

⁵ Hobson, loc. cit., p. 229.

THERMAL EQUILIBRIUM IN A GENERAL GRAVITATIONAL FIELD

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1. *Introduction.*—In the absence of any appreciable gravitational field the conditions satisfied by the temperature of a medium having no thermal flow from one portion to another, are well known to be those of uniformity throughout the system. In the presence of a gravitational field, however, the conditions imposed at thermal equilibrium on the proper temperature—as measured at different points of the medium by local observers—are considerably more complicated, owing to the association of inertia and weight with all forms of energy and the consequent tendency for heat to flow from regions of higher to those of lower gravitational potential. Nevertheless, for the special case of static gravitational fields, these more complicated conditions have already been obtained,^{1,2} and it is the purpose of the present article to investigate the conditions for thermal equilibrium in the general case of any kind of gravitational field.

The investigation shows that a reasonably straightforward treatment can be found which leads to an apparently satisfactory, covariant expression of the general conditions for thermal equilibrium. The possible value of such a result can be two-fold. In the first place it satisfies our desire for coherent theory, since our previous expression of the conditions for thermal equilibrium, which applied only to static systems and was made in the non-covariant language of a kind of coördinate system then appropriate, can now be regarded as a special case of the generally covariant expression applying to any kind of system. In the second place, the expression ob-