Reality of P(iz) and Q(iz) PhL 2,5.10

In this Sherlock Holmes monograph, we investigate a certain "problem" with certain Legendre functions.

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Background and the atomic forms. In oblate spheroidal coordinates, there are as usual three separated ODEs. Using Smythe notation and ordering, the coordinates are ξ,ζ,φ where ζ is the "radial" coordinate. The ODE for this coordinate is the Legendre ODE but with imaginary variable. In other words, here is the normal Legendre ODE:

$L_z = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2)$ // modified from Leg prop doc

$L_z u(z) = n(n+1)u(z)$

On the second line we have written "the eigenvalue problem" where $\lambda = n(n+1)$. The independent solutions of this equation can be taken as the usual $P_n^m(z)$ and $Q_n^m(z)$ Legendre functions. Now, the oblate ζ ODE is obtained by taking $z = i\zeta$ and then we have

$L_z = L_\zeta = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + m^2/(1+\zeta^2)$

$L_\zeta v(\zeta) = n(n+1)v(\zeta)$

The claim is that the solutions to the latter eigenvalue problem can be taken as $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$. Here is why:

$L_\zeta\, P_n^m(i\zeta) = L_z\, P_n^m(z) = n(n+1)\, P_n^m(z) = n(n+1)\, P_n^m(i\zeta)$

So for the ζ dimension of an oblate PDE, we can take as our "atoms" the functions $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$.

Smythe Scan. This fact is clearly stated in Smythe. Here is a clip from his page 160 confirming this idea:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \tag{1}$$

and

$$\frac{1}{\Xi}\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Xi}{d\xi}\right] + \frac{1}{Z}\frac{d}{d\zeta}\left[(1+\zeta^2)\frac{dZ}{d\zeta}\right] = \frac{m^2(\xi^2+\zeta^2)}{(1+\zeta^2)(1-\xi^2)} = \frac{m^2}{1-\xi^2} - \frac{m^2}{1+\zeta^2}$$

As before, this can be satisfied by taking

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Xi}{d\xi}\right] - \frac{m^2\Xi}{1-\xi^2} + n(n+1)\Xi = 0 \tag{2}$$

and

$$\frac{d}{d\zeta}\left[(1+\zeta^2)\frac{dZ}{d\zeta}\right] + \frac{m^2 Z}{1+\zeta^2} - n(n+1)Z = 0$$

Putting $\zeta' = j\zeta$ in the last equation gives

$$\frac{d}{d\zeta'}\left[(1-\zeta'^2)\frac{dZ}{d\zeta'}\right] - \frac{m^2 Z}{1-\zeta'^2} + n(n+1)Z = 0 \tag{3}$$

The solution of (1) is, from 5.23 (2),

$$\Phi_m = C\cos m\phi + D\sin m\phi \tag{4}$$

Since (2) and (3) are identical with 5.23 (3), the solution is given by 5.23 (5) to be

$$\Xi_{mn} = AP_n^m(\xi) + BQ_n^m(\xi) \tag{5}$$

$$Z_{mn} = A'P_n^m(\zeta') + B'Q_n^m(\zeta') = A'P_n^m(j\zeta) + B'Q_n^m(j\zeta) \tag{6}$$

The general solution is then

$$V = \sum_m \sum_n \Xi_{mn} Z_{mn} \Phi_m \tag{7}$$

In particular line (6) shows the atoms, Line 2 is the regular Legendre ODE in form [L+n(n+1)]u=0, and line (3) is our ζ version of that equation as noted above. So I think there is no doubt about the use of these P and Q "atoms" in the ζ dimension.

Smythe then goes on to do a few oblate spheroidal potential problems.

He does an iris with a distant planar E field on one side.

Then torque on a disk in a uniform E field.

Then he does the potential of a spheroid with a prescribed sticky surface charge distribution, which is really the Neumann problem for the spheroid. This problem is of particular interest to me because the potential is modeled using the atoms just mentioned, p 166

$$\zeta < \zeta_0 \qquad V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} M_{mn} P_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_m) \qquad (9)$$

$$\zeta > \zeta_0 \qquad V_o = \sum_{n=0}^{\infty} \sum_{m=0}^{n} N_{mn} Q_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_m) \qquad (10)$$

I review this problem in full detail in my doc " the Smythe method for Green's Functions, cone exampls.doc". The coefficients M and N involved integrals of the charge distribution σ. I shall return to this problem in a moment, but let's continue our "scan" of Smythe.

After solving the above prescribed charge distribution problem, he then treats a special case, where that charge distribution is a point charge. He then obtains the potential of a point charge in oblate coordinates, which is certainly interesting. It has exactly the form shown above, where he now does the integrals to get the M and N coefficients.

At this point, he switches to **prolate** spheroidal coordinates. It turns out that for these coordinates, the "third" coordinate like ζ has functions $P_n^m(\eta)$ and $Q_n^m(\eta)$ where η lies in (1,∞). So the prolate system does not involve imaginary arguments of P and Q functions. He treats the prolate spheroid in a uniform E field, and then he is DONE!. There are 6 problems at the end of Chapter 5 which involve "oblate".

Then that is really it. There are no other oblate problems treated in Smythe's electrostatics. On page 301 in his magnetic work we do have this problem stated,

21. There is a zonal distribution of current flowing in an oblate spheroidal shell $\zeta = \zeta_0$ such that current between $\xi = 1$ and $\xi = \xi_0$ is given by $\psi(\xi_0) = \Sigma C_n P_n(\xi_0)$. Evaluating the integral of 7.10 by the result in 5.274, show that the vector potential due to this current, when $\zeta < \zeta_0$, is

$$-\phi\mu j(1 + \xi_0^2)^{\frac{1}{2}} \sum_1^{\infty} \frac{C_n}{n(n+1)} Q_n^1(j\zeta_0) P_n^1(j\zeta) P_n^1(\xi)$$

which involves the atomic sum type I am today interested in.

M&F scan. Now the only other book I have that can even talk this language is M&F, so I will now do a little scan of its offerings in the oblate department. The action there would be in Section 10.3 which is on 3D solutions to the Laplace equation. On page 1293 we see the same atoms,

$$AP_1^{\eta}(\eta)Q_1^{\eta}(i\xi) + BP_1^1(\eta)Q_1^1(i\xi)$$

which is specific to some problem they are doing. Here they use (η,ξ,φ) instead of (ξ,ζ,φ) which is certainly unfortunate, but we deal with it. But sadly, MF don't do much else with oblates. They at least mention them, which Jackson does not. Also MF deal with T functions instead of P which is quite confusing as well. So this wonderful book really has nothing to say on my iζ atoms. Due diligence has been done.

Web Scan? A web scan on <"oblate spheroidal" electrostatics> gives about 4000 hits, and the leading ones are all proprietary, such as springerlink hits. As I download what I can from the top of the list (maybe 10 items) , I find some narrow specific applications (nukes!) One paper has the point charge thing in this manner

If a point charge q is placed at (η', θ', ϕ'), the potential in terms of the oblate spheroidal coordinates can be found as

$$\begin{aligned}\Phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{q}{\varepsilon_0 a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}\left\{\begin{array}{c} P_l^m(i\sinh\eta)\,Q_l^m(i\sinh\eta') \\ P_l^m(i\sinh\eta')\,Q_l^m(i\sinh\eta)\end{array}\right\} Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi'), \left\{\begin{array}{c}\eta<\eta' \\ \eta>\eta'\end{array}\right\} \qquad (2.136)\end{aligned}$$

Derivation of this expression is left for exercise. The Wronskian of the Legendre functions,

$$P_l^m(x)\frac{d}{dx}Q_l^m(x) - Q_l^m(x)\frac{d}{dx}P_l^m(x) = \frac{1}{x^2-1}\frac{(l+|m|)!}{(l-|m|)!}, \qquad (2.137)$$

which is a little different from the Smythe form because it has Y functions, but you see my "atoms" appearing. This author then claims a solution to the Green's function of a spheroid:

should be useful. Furthermore, the Green's function for an oblate spheroidal surface described by $\eta = \eta_0$ can readily be worked out to be:
for $\eta_0 < \eta < \eta'$,

$$\begin{aligned}G(\mathbf{r},\mathbf{r}') &= \frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}P_l^m(i\sinh\eta)\,Q_l^m(i\sinh\eta')\,Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi') \\ &\quad -\frac{1}{a}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-|m|)!}{(l+|m|)!}\frac{P_l^m(i\sinh\eta_0)}{Q_l^m(i\sinh\eta_0)}Q_l^m(i\sinh\eta)\,Q_l^m(i\sinh\eta')\,Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi') \qquad (2.138)\end{aligned}$$

which shows lots of atoms. Another PDF claims the following as a general atomic form:

$$\Phi(\zeta,\eta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left[A_l^m P_l^m(i\zeta) + B_l^m Q^m(i\zeta)\right]\left[C_l^m P_l^m(\eta) + D_l^m Q_l^m(\eta)\right]\left[E^m\cos m\varphi + F^m\sin m\varphi\right] \qquad (A.20)$$

Here the sum shows integral ℓ because author is thinking about problems with spheroidal conductors which have the full (-1,1) η range which forces ℓ to the integers.

Nobody so far is talking about Green's function for a hyperboloid! I can add that buzzword to the web search, we drop to 600 hits from 4000 hits. I add "Green's function" and down to 268. As I scan down, I find the following interesting claim:

computation of $\Phi(\vec{x})$ becomes straightforward. Green's functions for electrodes with various shapes (hyperboloids, oblate and prolate spheroids, toroids, flat-ring cyclide discs, etc) are well known in the literature [29, 30]. These kinds of electrodes can be put together in many different ways, so that a vast class of well-defined trapping volumes can theoretically be envisaged with them. Any such combination of differently shaped electrodes defining a closed trapping region is what we call a *hybrid Penning trap*.

So what are those references 29.30?

[29] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)
[30] Moon P and Spencer D E 1971 *Field Theory Handbook: Including Coordinate Systems, Differential Equations and Their Solutions* (Berlin: Springer)

Good old Moon and Spencer which I have not been able to get hold of! All you can get is the cover page. Marriott does not have it. But amazingly on scribd I just found a piece of this book, the piece on spheroidal coordinates, but just a few pages, no problems. It's always fun to search.

Now onto The Problem: $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$ are not real function

My problem is that the atoms $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$ are not real functions, even when m = integer and n = real. Therefore, when you try to assemble a Smythian form potential using these atoms, you are not assembling a real function. So how can this be a reasonable thing to do for an electrostatic potential?

In particular, when trying to compute the Green's Function for a Hyperboloid, I am able to get a result, but the potential comes out being complex! The n values are strange (but real) values which make the potential be zero on the bloid surface. This is determined by the cone angle function and then the ζ system just inherits these n values in the emitter follower sense.

So I wanted to see how the sources above dealt with this problem of the non-reality of the atoms, that is why I had such a long background section.

Non-reality of the atoms $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$: getting specific.

But first, why are these atoms not real? A good place to look is in the Bateman table where the F function has argument z^2 or z^{-2}, in which case the F function has argument $-\zeta^2$ or $-1/\zeta^2$ which is real, so the F function is real. The gamma functions are also real for real n,m (though if a denominator Γ has a pole, that term would vanish), so the non-reality comes from the following factors: [these are both Bateman off the cut functions]. [In each of the four cases, I show the potential non-reality factors involved for the two terms that make up the P and Q:]

for $P_n^m(i\zeta)$ and z^2: see p 126 (22)

$(z^2-1)^{-m/2}$

$(z^2-1)^{-m/2}$ z

for $P_n^m(i\zeta)$ and $1/z^2$ see p 126 (23)

$(z^2-1)^{-m/2}$ z^{-n+m-1}

$(z^2-1)^{-m/2}$ z^{n+m}

for $e^{-i\pi\mu}\ Q_n^m(i\zeta)$ and z^2: see p 135 (40)

$(z^2-1)^{-m/2}\quad (\pm i)^{m-n-1}$

$(z^2-1)^{-m2}\quad (\pm i)^{m-n}\qquad z$

for $e^{-i\pi\mu}\ Q_n^m(i\zeta)$ and $1/z^2$: see p 135 (41)

$(z^2-1)^{+m/2}\quad z^{-1-m-n}$

no second term

Example: $Q_3^2(z)$

Look at the Q cases and suppose m = 2 and n = 3.

Then for **large z**, we find that our function $Q_n^m(i\zeta)$ is **real**, since z^{-6} is real, and $e^{-i\pi 2} = 1$ = real and $(z^2-1)^1$ is real. We don't hit any bad spots in the F "c" or the gamma functions.

For **small z,** the factor $(\pm i)^{m-n} = (\pm i)^{-1} = \mp i$ while $(\pm i)^{m-n-1} = -1$ and $z = i\xi$, Thus, both terms are real.

It has $\Gamma(1/2+n/2 - m/2) = \Gamma(1/2+3/2-1) = \Gamma(1) = 1$ in the denominator, so this does not kill off the term. This says that as we pass in through the $|z| = 1$ disk, this function $Q_3^2(i\xi)$ suddenly changes from being real to being complex.

Maple agrees that this $Q_n^m(i\zeta)$ is real.

Now what is the actual function? Maxima tells us

```
assoc_legendre_q(3,2,z);expand(%);
```

$$\frac{15\log\left(-\frac{z+1}{z-1}\right)z^5-30z^4-30\log\left(-\frac{z+1}{z-1}\right)z^3+50z^2+15\log\left(-\frac{z+1}{z-1}\right)z-16}{2z^2-2}$$

So that

$$Q_3^2(z) = (1/2)\log[(z+1)/(z-1)]\ \{\ 15z^5 - 30z^3 + 15z\ \}/(z^2-1)\ \ + \{\ -30z^4 + 50z^2 - 16\ \}/(z^2-1)$$

where I have adjusted to make it be an off the cut function in the log. Simplify?

$$Q_3^2(z) = (1/2)\log[(z+1)/(z-1)]\ 15z\{\ z^4 - 2z^2 + 1\ \}/(z^2-1)\ \ + 2\{\ -15z^4 + 25z^2 - 8\ \}/(z^2-1)$$
$$= (1/2)\log[(z+1)/(z-1)]\ 15z\,(z^2-1)\ \ + 2\{\ -15z^4 + 25z^2 - 8\ \}/(z^2-1)$$

where the {} does not factor. Everything here is real except the "z" and the log. So how exactly does this thing manage to be real? Well, if $z = 1\zeta$, then $|z+1| = |z-1|$ so the log is pure imaginary, and that times z is pure real. Good.

The general form of a Q function is shown p 149 (11) and always involves this log. For $z = i\xi$, this log will always be pure imaginary.

The special problem of the hyperboloid Green's Function.

Here the n are set by $P_n(\xi_1) = 0$ where ξ_1 describes the bloid angle. In general, the n spectrum (discussed elsewhere) is real and positive and the n are not integers. This means that you never get a "kill-off" from a denominator Γ. That in turn means "both terms" are always present in the $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$ Bateman forms shown above. [As we try different ξ_1 values, the spectrum of $P_n(\xi_1) = 0$ will hit all possible positive n values, so when constructing examples, any n value will do, like n = 1.2 shown below.]

One candidate Smythian form for this problem involves

$$V_o(\zeta,\xi) = \Sigma_{m=0}^{\infty} \Sigma_n H_{nm} \{j Q_n^m(j\zeta) P_n^m(j\zeta_0)\} P_n^m(\xi) \cos(m\varphi) \qquad (***)$$

where the coefficients H_{nm} can be computed and are real, and ζ_0 is the location of the Green's point charge. Obviously the arguments of $Q_n^m(j\zeta)$ and $P_n^m(j\zeta_0)$ can be arbitrary points $\zeta > \zeta_0$ on the imaginary axis. It is easy to find values such that $\{j Q_n^m(j\zeta) P_n^m(j\zeta_0)\}$ is a complex number, for example,

```
I*LegendreQ(1.2,0,I*2)*LegendreP(1.2,0,I*1):evalf(%);
                    .05939268641 + .001646033692 I
```

Since the functions $P_n^m(\xi)\cos(m\varphi)$, which are sort of like the spherical harmonics, form a complete set for this problem, we end up with our "partial wave" coefficients being complex. That means V_o is complex, but that is impossible because this is a potential problem in electrostatics.

More specifically, since $P_n^m(\xi)\cos(m\varphi)$ are basis functions, we can invert to get

$$\int\int d\xi\, d\varphi\, V_o(\zeta,\xi) P_n^m(\xi) \cos(m\varphi) \sim H_{nm} [j Q_n^m(j\zeta) P_n^m(j\zeta_0)]$$

If V_o is real, as it must be, the LHS is real, but the RHS can be complex! Note that $P_n^m(\xi)$ can be taken as on-the-cut functions and these are real for any real n and integer m, and H_{nm} are real as noted above.

<u>This then is my Big Problem of the moment.</u> I did the big review above, because I wanted to see how authors dealt with this issue in their problem solving.

Smythe's spheroid Neumann problem . He has these forms,

$$\zeta < \zeta_0 \qquad V_i = \sum_{n=0}^{\infty} \sum_{m=0}^{n} M_{mn} P_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_m) \qquad (9)$$

$$\zeta > \zeta_0 \qquad V_o = \sum_{n=0}^{\infty} \sum_{m=0}^{n} N_{mn} Q_n^m(j\zeta) P_n^m(\xi) \cos m(\phi - \phi_m) \qquad (10)$$

Here the m and n are both integers, so we have some kill off possible. We know from

$P_\nu^m(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ m = 1,2,3....

$P_\nu^m(x) = (-1)^m (1-x^2)^{m/2}\partial_m P_\nu(x)$ m = 1,2,3.... // "CS phase"

$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....

that the $P_n(z)$ are simple polys even/odd, so the $P_n^m(z)$ are going to be the simple functions shown, which will then be either all imaginary or all real for a given set of n,m . The $Q_\nu^m(z)$ will have this same sense, with the log thrown in. Here are Smythe's M and N

$$M_{mn} = j(-1)^m \frac{(2-\delta_m^0)(2n+1)}{4\pi\epsilon c_1}\left[\frac{(n-m)!}{(n+m)!}\right]^2 Q_n^m(j\zeta_0) \int_{-1}^{+1}\int_0^{2\pi} \sigma P_n^m(\xi)\cos m(\phi-\phi_m) h_1 h_3\, d\xi\, d\phi \quad (11)$$

where $\delta_m^0 = 1$ if $m = 0$ and $\delta_m^0 = 0$ if $m \neq 0$. From (4) we have

$$N_{mn} = M_{mn}\frac{P_n^m(j\zeta_0)}{Q_n^m(j\zeta_0)} \quad (12)$$

So what really appears are

$V_i \sim \{ j\, P_n^m(j\xi)\, Q_n^m(j\xi_0)\}$

$V_0 \sim \{ j\, Q_n^m(j\xi)\, P_n^m(j\xi_0)\}$

so why doesn't he have my same problem? These are exactly my problematic combinations, except I have the extra complication of non-integer n. Is it something about these products that makes them real?

Let's go back to our reality rules from above:

for $P_n^m(i\zeta)$ and z^2: see p 126 (22)
$(z^2-1)^{-m/2}$
$(z^2-1)^{-m/2}$ z

for $P_n^m(i\zeta)$ and $1/z^2$ see p 126 (23)
$(z^2-1)^{-m/2}$ z^{-n+m-1}
$(z^2-1)^{-m/2}$ z^{n+m}

for $e^{-i\pi\mu}\, Q_n^m(i\zeta)$ and z^2: see p 135 (40)
$(z^2-1)^{-m/2}$ $(\pm i)^{m-n-1}$
$(z^2-1)^{-m2}$ $(\pm i)^{m-n}$ z

for $e^{-i\pi\mu}\, Q_n^m(i\zeta)$ and $1/z^2$: see p 135 (41)
$(z^2-1)^{+m/2}$ z^{-1-m-n}
no second term

Let's then consider large z only, so we have that single Q term. Then we have :

$j\, Q_n^m(j\xi)\, P_n^m(j\xi_0)$

$= j \times [e^{+i\pi m}(z^2-1)^{+m/2} z^{-1-m-n}] \times [\text{kill} * (z^2-1)^{-m/2} z^{-n+m-1} + (z^2-1)^{-m/2} z^{n+m}]$

$= j \;\; x \;\; [e^{+i\pi m} (z^2-1)^{+m/2} \, z^{-1-m-n}] \; x \; [(z^2-1)^{-m/2} \qquad z^{n+m} \;]$

$= j \;\; x \;\; [e^{+i\pi m} \; z^{-1}]$

$= j \; (1/j\zeta) \; (-1)^m = \;\; (-1)^m / \zeta \;$ = real !!!

So thanks to the kill-off of the second P term, these factors are pure real for Smythe, and of course this is in each partial wave. Probably the reality then tracks down to small z as well as large z. So we have at least bailed Smythe out of the fire!

Back to My Problem. It was my idea to try the $P_n(\xi_1) = 0$ idea for the bloid, stealing the idea from Smythe's cone job in sphericals. I have never seen anyone do this for a bloid. The resulting $P_n^m(\xi)$ seem to be a complete set since SL says they are. The Wronskian magically takes all the complex stuff as arguments and grinds it into something real, that is what makes the H_{nm} come our real.

Question: For any other ODE, we always have solutions which are *real*, so why do I have to accept complex solutions for this particular ODE? Is it just because people didn't want to have another named function to worry about? It must be possible to come up with some "real atoms" and they are the two independent solutions, and they have some simple Wronskian. They must then be linear combinations of the P and Q functions some how.

Attempt construction of a Q-like real atom.

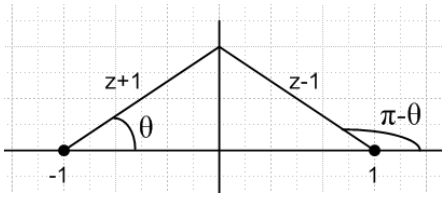
Recall from above,

for $e^{-i\pi\mu} \; Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(z^2-1)^{-m/2} \quad (\pm i)^{m-n-1}$

$(z^2-1)^{-m2} \quad (\pm i)^{m-n} \qquad z$

I have been putting this off, but time now to be precise about the first factor $(z^2-1)^{-m/2}$. Here is our famous picture on the 0,0L principle sheet:



So we have

$z+1 = |z+1|e^{i\theta} \qquad z-1 = |z-1|e^{i(\pi-\theta)} \qquad (z^2-1) \; = |z^2-1| \; e^{+i\pi}$

If we took z below decks, both angles simply negate, so result has opposite sign. Thus:

$(z^2-1) = |z^2-1|\, e^{\pm i\pi}$ // as a "symbolic function"

Then we have

$$(z^2-1)^{-m/2} = |z^2-1|^{-m/2}\, e^{\mp\, i\pi m/2} = |z^2-1|^{-m/2}\ (\mp i)^m = |z^2-1|^{-m/2}(\pm i)^{-m}$$

Then we can restate our Q rule from above as

for $e^{-i\pi\mu}\ Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(\pm i)^{-m}$ $(\pm i)^{m-n-1}$

$(\pm i)^{-m}$ $(\pm i)^{m-n}$ z

which simplifies to this rule

for $e^{-i\pi\mu}\ Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(\pm i)^{-n-1}$

$(\pm i)^{-n}$ z

But $e^{-i\pi m} = (-i)^{2m} = (i)^{-2m}$ so write the rule again as

for $Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(i)^{2m}\ (\pm i)^{-n-1}$

$(i)^{2m}\ (\pm i)^{-n}$ z

Now $z = i\zeta$ and this adds another i to the second term phase, so rewrite as

for $Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(i)^{2m}\ (\pm i)^{-n-1}$ term 1

$(i)^{2m+1}\ (\pm i)^{-n}$ term 2 // the +1 exponent is correct and has always been so

So try this idea

term A term B

$$q_n^m(\zeta) = [(-i)^{2m+1}\ (\mp i)^{-n} Q_n^m(i\zeta) + (-i)^{2m+1}\ (\pm i)^{-n} Q_n^m(-i\zeta)] \quad \text{// tentative}$$

The idea is this: for each Q function we unwind the phase of the second term in (40). Then due to the z factor, the second terms should cancel. As written above, the term A2 should have factor ζ, while the term B2 will be identical, but will have factor $-\zeta$, so we should have A2+B2 = 0. What is left is A1+B1. Notice that this is still a complex number, because we unwound the phase of the term 2's, not the term 1's. Now let's try to simply rewrite in a suggestive way:

$$q_n^m(\zeta) = [(-i)^{2m+1}\ (\mp i)^{-n} Q_n^m(i\zeta) + (-i)^{2m+1}\ (\pm i)^{-n} Q_n^m(-i\zeta)] \quad \text{// tentative}$$

$$= [(-i)^1(-i)^{2m}\ (\mp i)^{-n-1}\ (\mp i)^1 Q_n^m(i\zeta) + (-i)^1\ (-i)^{2m}\ (\pm i)^{-n-1}\ (\pm i)^1 Q_n^m(-i\zeta)]$$

$$= [(-i)^{1}(\mp i)^{1}\,(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (-i)^{1}\,(\pm i)^{1}\,(-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= [(-i)^{1}\,(\pm i)^{-1}\,(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (+i)^{-1}\,(\pm i)^{1}\,(-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= (-i)^{1}\,[(\pm i)^{-1}\,(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (\pm i)^{1}\,(-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= (-i)^{1}\,(\pm i)^{1}\,[(\pm i)^{-2}\,(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= (-i)^{1}\,(\pm i)^{1}\,[-\,(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= i\,(\pm i)^{1}\,[(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) - (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= i\,(\pm i)^{1}\,[(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) - (-i)^{2m}\,(i)^{2m}\,(-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= i\,(\pm i)^{1}\,[(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) - (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

$$= \mp\,[(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

term A term B

and we copy down our rule from above

for $Q_n^m(i\zeta)$ and z^2: see p 134 (40)

$(i)^{2m}\,(\pm i)^{-n-1}$	term 1
$(i)^{2m+1}\,(\pm i)^{-n}$	term 2

Notice that our term A $(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta)$ in its "first term" should be real, because we have exactly unwound its term 1 phase. And the same can be said for $(-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)$ in its "first term", so this too should be real in its first term. Our final result is then:

$$q_n^m(\zeta) = \mp\,[(-i)^{2m}\,(\mp i)^{-n-1}\,Q_n^m(i\zeta) + (-i)^{2m}\,(\pm i)^{-n-1}Q_n^m(-i\zeta)]$$

and we claim that this quantity is 100% real!

Observation: The function $(i)^{2m}\,q_n^m(\zeta)$ *as shown above* is antisymmetric in ζ.

The next step is to use this fact:

$$Q_\nu^\mu(-z) = -\,Q_\nu^\mu(z)\,e^{\pm i\pi\nu}$$

$$Q_n^m(-i\zeta) = -\,Q_n^m(i\zeta)\,e^{\pm i\pi n} = -\,Q_n^m(i\zeta)\,(\pm i)^{2n} \qquad e^{\pm i\pi n} = (\pm i)^{2n}$$

Then we get

$$q_n^m(\zeta) = \mp\ [(-i)^{2m}(\mp i)^{-n-1} Q_n^m(i\zeta) + (-i)^{2m}(\pm i)^{-n-1}\{- Q_n^m(i\zeta)(\pm i)^{2n}\}]$$

$$= \mp\ [(-i)^{2m}(\mp i)^{-n-1} Q_n^m(i\zeta) - (-i)^{2m}(\pm i)^{+n-1} Q_n^m(i\zeta)]$$

$$= \mp\ [(-i)^{2m}(\mp i)^{-n-1} - (-i)^{2m}(\pm i)^{+n-1}] Q_n^m(i\zeta)$$

$$= \mp (-i)^{2m}[(\mp i)^{-n-1} - (\pm i)^{+n-1}] Q_n^m(i\zeta)$$

$$= \mp (-i)^{2m}[(\pm i)^{n+1} - (\pm i)^{n-1}] Q_n^m(i\zeta)$$

$$= \mp (-i)^{2m}(\pm i)^n[(\pm i)^{+1} - (\pm i)^{-1}] Q_n^m(i\zeta)$$

$$= \mp (-i)^{2m}(\pm i)^n[(\pm i) - (\mp i)] Q_n^m(i\zeta)$$

$$= \mp (-i)^{2m}(\pm i)^n[(\pm i) + (\pm i)] Q_n^m(i\zeta)$$

$$= \mp\ 2\ (-i)^{2m}(\pm i)^{n+1} Q_n^m(i\zeta)$$ // tentative

In each line of algebra I could make a mistake, so it is extremely unlikely I did this all correctly since there have been about 10 sequential steps. So as usual, we are off to a Maple test.

But first: we just want something that is real. I like the idea of adding a factor 1/2 so we are really then averaging our two Q functions at the start, and I will keep the $\mp$ sign, so we then have

$$q_n^m(\zeta) = \mp (-i)^{2m}(\pm i)^{n+1} Q_n^m(i\zeta)$$

Now let's try some Maple testing on this thing. // Maple says we are good if we use the upper sign and if we use $\zeta > 0$ only. I varied n, m and ζ (n and m arbitrary real, no integer).

Now will try to verify the lower sign combination for negative ζ. // That looks good too!

This is the "magic phase" that makes a Q function be real!! This is what I have long been looking for. I have done lots of Maple testing with this thing, plus the entire development history is exposed. For the moment I will maintain the sign out front since it fell out naturally, but later I may dump it, I don't know yet.

Attempt construction of a P-like real atom.

What I really need here is any kind of "second solution" that is independent of q. Maybe it will turn out to be the P, maybe not. Let's just mimic the steps taken above. Start with the rule:

Recall from above,

for $P_n^m(i\zeta)$ and z^2: see p 126 (22)

$(z^2-1)^{-m/2}$

$(z^2-1)^{-m/2}$ z

Use the rule from above which says

$$(z^2-1)^{-m/2} = |z^2-1|^{-m/2}\, e^{\mp i\pi m/2} = |z^2-1|^{-m/2}\, (\mp i)^m = |z^2-1|^{-m/2}(\pm i)^{-m}$$

to get

for $P_n^m(i\zeta)$ and z^2: see p 126 (22)

$(\pm i)^{-m}$

$(\pm i)^{-m}$ z

Now $z = i\zeta$ and this adds another i to the second term phase, so rewrite as

for $P_n^m(i\zeta)$ and z^2: see p 126 (22)

$(\pm i)^{-m}$ term 1

$(i)^1\,(\pm i)^{-m}$ term 2

So try this idea

term A term B

$$p_n^m(\zeta) = [(-i)^1\,(\mp i)^{-m}\, P_n^m(i\zeta) + (-i)^1\,(\pm i)^{-m} P_n^m(-i\zeta)]$$

The idea is this: for each P function we unwind the phase of the second term in (22). Then due to the z factor, the second terms (the term-2 terms) should cancel. So what's left is A1 + B1. First rewrite as

$$p_n^m(\zeta) = (-i)\,[\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m}\, P_n^m(-i\zeta)]$$

But in this form, we have $(\mp i)^{-m}$ cancelling the phase of term A1 and $(\pm i)^{-m/2}$ cancelling the phase of term B1. So I think the [...] is real. So here is a candidate for a real atom:

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

I think it would be good to try a Maple test on this form and see if it really is real. // Yes, Maple says this is in fact real for lots of test cases I have done (n and m non integer but real for example).

Observation: the function $p_n^m(\zeta)$ is symmetric in ζ, from its definition above!

Now we can stop here, or we can replace the second term using

$$P_\nu^\mu(-z) = P_\nu^\mu(z)\, e^{\mp i\pi\nu} - (2/\pi)\sin\pi(\nu+\mu)\, Q_\nu^\mu(z)\, e^{-i\pi\mu}$$

$$P_n^m(-i\zeta) = P_\nu^m(i\zeta)\, e^{\mp i\pi n} - (2/\pi)\sin\pi(n+m)\, Q_n^m(i\zeta)\, e^{-i\pi m} \qquad \text{upper sign for } \zeta > 0$$

$$P_n^m(-i\zeta) = P_n^m(i\zeta)\,(\mp i)^{2n} - (2/\pi)\,\sin\pi(n+m)\,Q_n^m(i\zeta)\,(-i)^{2m} \qquad \text{upper sign for } \zeta > 0$$

so we then have:

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\,P_n^m(i\zeta) + (\pm i)^{-m}P_n^m(-i\zeta)]$$

$$= (\pm i)^{-m}\,[\,(\mp i)^{-m}\,(\pm i)^m\,P_n^m(i\zeta) + P_n^m(-i\zeta)]$$

$$= (\pm i)^{-m}\,[\,(\pm i)^{2m}\,P_n^m(i\zeta) + P_n^m(-i\zeta)]$$

$$= (\pm i)^{-m}\,[\,(\pm i)^{2m}\,P_n^m(i\zeta) + P_n^m(i\zeta)\,(\mp i)^{2n} - (2/\pi)\,\sin\pi(n+m)\,Q_n^m(i\zeta)\,(-i)^{2m}\,]$$

$$= (\pm i)^{-m}\,[\,\{\,(\pm i)^{2m} + (\mp i)^{2n}\,\}P_n^m(i\zeta) - (-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\,Q_n^m(i\zeta)\,]$$

This is no doubt correct, but I don't think it is a useful form to use, except I will use it to compute the Wronskian below!

Summary of our two real atoms:

$$q_n^m(\zeta) = \mp\,(-i)^{2m}\,(\pm i)^{n+1}\,Q_n^m(i\zeta)$$

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\,P_n^m(i\zeta) + (\pm i)^{-m}P_n^m(-i\zeta)]$$

$$= (\pm i)^{-m}\,[\,\{\,(\pm i)^{2m} + (\mp i)^{2n}\,\}P_n^m(i\zeta) - (-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\,Q_n^m(i\zeta)\,]$$

Both have been Maple tested! So these are the two real independent (?) solutions of the ζ dimension ODE that I have been looking for. The quantities m and n can be any complex numbers you want, but if they are real numbers, then $p_n^m(\zeta)$ and $q_n^m(\zeta)$ are real. I have NOT assumed m = integer anywhere.

The properties of these functions can be obtained from properties of the P and Q functions. It seems likely that p will blow up for large ζ, but q does not, which is what we need!

Now let's try to find a better way to write this awful factor above. Start with this approach:

$$\{\,(\pm i)^{2m} + (\mp i)^{2n}\,\} = (\pm i)^m\,(\pm i)^{-n}\,[\,(\pm i)^m(\pm i)^n + (\pm i)^{-m}(\pm i)^{-n}\,]$$

$$= (\pm i)^{-n+m}\,[\,(\pm i)^{n+m} + (\pm i)^{-n-m}\,]$$

$$= (\pm i)^{-n+m}\,2\cos[(n+m)\pi/2]$$

Therefore the full coefficient of $P_n^m(i\zeta)$ is this:

$$(\pm i)^{-m}\,[\,\{\,(\pm i)^{2m} + (\mp i)^{2n}\,\} = (\pm i)^{-m}(\pm i)^{-n+m}\,2\cos[(n+m)\pi/2]$$

$$= (\pm i)^{-n}\,2\cos[(n+m)\pi/2]$$

So at least we can now say:

$$p_n^m(\zeta) = (\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\, P_n^m(i\zeta) - (\pm i)^{-m/2}\,(-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\, Q_n^m(i\zeta)$$

New Summary of our two real atoms:

$$q_n^m(\zeta) = \mp\,(-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(i\zeta)$$

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$= (\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\, P_n^m(i\zeta) - (\pm i)^{-m/2}\,(-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\, Q_n^m(i\zeta)$$

It is true that there are values of n,m which will cause $p_n^m(\zeta) = 0$. But this was also true for $P_n^m(z)$, so maybe don't be worrying so much about that fact.

Cut structure in the ζ plane. If we now think of the complex ζ-plane, what is the cut structure of these functions, and where are they real, etc?? Remember that in any ODE analysis, this is something you always want to know. The ODE is this:

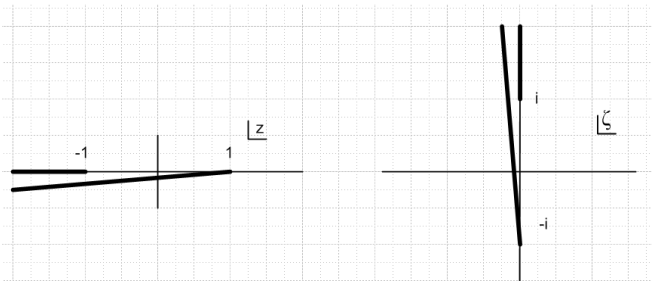
$$L_z = L_\zeta = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + m^2/(1+\zeta^2)$$
$$L_\zeta v(\zeta) = n(n+1)v(\zeta)$$

so we know the singular points are at $\xi = \pm i$ and ∞. So those are the branch points. How should we then draw the cuts? To the left? How does this relate to all our ± signs? There is some confusion going on here.

Let's consider this example: we know that $Q_1(z) = (z/2)\ln[(z+1)/(z-1)] - 1$ so

$$q_1^0(\zeta) \equiv \mp(\pm i)^2\, Q_1(i\zeta) = \pm Q_1(i\zeta) = \pm(i\zeta/2)\ln[(i\zeta+1)/(i\zeta-1)] - 1$$
$$= \pm(i/2)\;\zeta\;\ln[(\zeta-i)/(\zeta+i)] - 1$$

For real ζ, this looks imaginary, but in fact it is real since $|(\zeta-i)/(\zeta+i)| = 1$, not to worry. Now, when we have dual signs in an expression, such as we have here, it came from the idea of Imz > 0 for the upper sign. That now means Re(ζ) > 0. The transformation $z = i\xi$ is $\xi = -iz$ so we take a point in the z plane and rotate it CW by π/2. So here is our initial picture I think:



The picture on the right is that on the left rotated CW by 90 degrees. The cuts now run up, and the two signs then refer to the left and right side of the cut. Now ln [(z+1)/(z-1)] is real to the right of z=1 in the z plane because we have log of a positive real number in this case. And by the same argument, we would have that ln [(ζ-i) / (ζ+i)] was real for ζ chosen say at ζ = -5i.

So then what is the meaning of the two signs in ± in $q_1^0(\zeta)$? There can be only one value of the function $q_1(-5i)$ whether you approach from the left or the right. Defining the q function as I did is resulting in q being discontinuous at ζ = -5i. Imagine that we also rotated our angle defining scheme with the picture above. Then ζ = -5i ± ε => (ξ-[-i]) = ξ+i = -4i ± ε, and this vector pointing downward and a little to the left or right is basically at angle +δ and - δ, and so is continuous.

Let's try an example with m ≠ 0 and see what happens. It is hard to find the Q functions stated anywhere, but we can derive one here:

$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ $\qquad$ $Q_1(z) = (z/2)\ln[(z+1)/(z-1)] - 1$

$Q_1^1(z) = (z^2-1)^{1/2}\partial_1 Q_\nu(z) = (z^2-1)^{1/2}\,\partial_z[(z/2)\ln[(z+1)/(z-1)] - 1]$

$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] + (z/2)[(z+1)/(z-1)]^{-1}\{(z-1)*1 - (z+1)*1\}/(z-1)^2]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] + (z/2)[(z+1)/(z-1)]^{-1}\{-2\}/(z-1)^2]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - (z)[(z-1)/(z+1)]/(z-1)^2]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z[1/(z+1)]/(z-1)]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z[1/(z+1)(z-1)]]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z[1/(z^2-1)]]$
$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z/(z^2-1)]$

Smythe tells us this function for its on the cut version as

$$Q_1^1(\mu) = (1-\mu^2)^{\frac{1}{2}}\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)$$

so my result certainly looks reasonable for the off-cut version. Now then we have

$q_n^m(\zeta) = \mp(-i)^{2m}(\pm i)^{n+1} Q_n^m(i\zeta)$

$q_1^1(\zeta) = \mp(-i)^2(\pm i)^2 Q_1^1(i\zeta) = \mp Q_1^1(i\zeta)$ $\qquad$ $z^2 = (i\zeta)^2 = -\zeta^2$

$= \mp(z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z/(z^2-1)]$

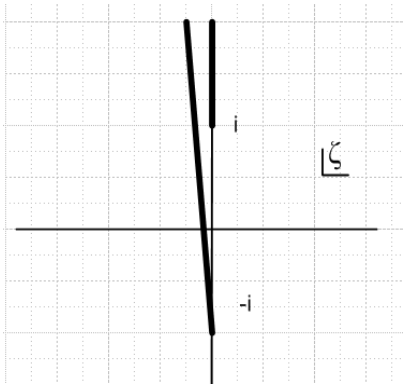
Rather than convert to ζ, just install z = 5 (ζ = -5i) and we see this thing is ∓ a real number. So we have exactly the same problem we had with $q_1^0(\zeta)$.

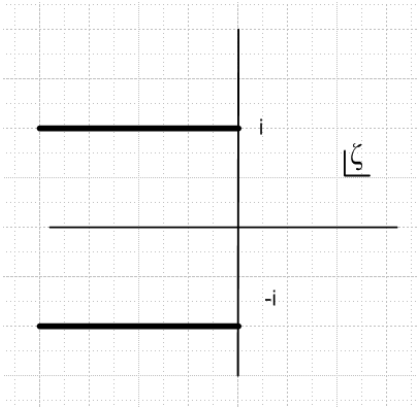
These two examples convince me to remove the sign ∓ in the q definition. This will then cause a sign to occur in the Wronskian which I guess we will have to live with. But at least the q function will be continuous where it should be!

Dialogo: It is a fact that if you want $q_n^m(\zeta)$ to be real for all real ζ, you *must* make the choice $q_n^m(\zeta) \equiv \mp (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$ where the upper sign is for $\mathrm{Re}(\zeta) > 0$ etc. If you try to just use the wrong phase, you will find that $q_n^m(\zeta)$ is not real for negative ζ .

Now consider the *regular* Legendre ODE. For that problem we want "real atoms" but we can only achieve realness on portions of the z axis where there is no cut! $Q_1(z)$ is not real to the left of the branch point at z = +1. It is complex and the imaginary part is different depending on whether you are above or below the cut. So the best we can do is "real analytic" which means the function is real on the real axis where there is no cut.

I think I am trying to thread a pathway to the picture on the right below:





In this case, the Maple or Bateman P and Q function will only be useful in the right half plane of my picture. Then for $\zeta = -2$, for example, Maple won't give me the right answer. But somehow I ought to be able to do *something* so Maple can give the results going with the above picture.

The other issue is how we parameterize things like $(\xi - [-i])$. In the left picture, if we just think of this as a rotation of the z picture, we get an answer to this question. For example, the point $\zeta = -i+1$ would be at angle $+\pi/2$ relative to an origin at the point -i. But in the picture on the right, the phase we want for this same vector is 0. Remember that the parameterization has everything to do with phases!

So let's try to somehow construct what we want. We start with this:

$q_n^m(\zeta) \equiv \mp (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$ valid only for $\mathrm{Re}(\zeta) > 0$ if using upper signs only

We have "full information" on q because we have full information on Q(z) for Im(z) > 0.

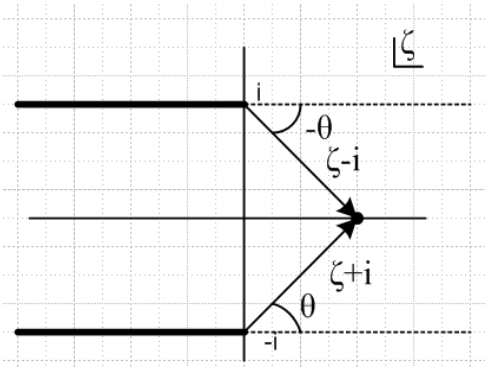
Warning: I think if we do as shown on the right, then q is going to blow up in the $-\zeta$ direction! I found this to be the case for Q(z) when I continued it down through the cut. But maybe somehow that will be different here. So again, let's "work with" our prototype function

$$Q_1(z) = (z/2) \ln [(z+1)/(z-1)] - 1$$

$$Q_1(i\zeta) = (i\zeta/2) \ln [(i\zeta+1)/(i\zeta-1)] - 1 = (i\zeta/2) \ln [(\zeta-i)/(\zeta+i)] - 1$$

$q_1(\zeta) \equiv +Q_1(i\zeta) = Q_1(i\zeta) = (i\zeta/2) \ln [(\zeta-i)/(\zeta+i)] - 1$ $\mathrm{Re}(\zeta) > 0$ only

Now let's parameterize things the "normal way you would" in the ζ plane, which means this picture:



Then we have

$(\zeta-i) = |\zeta-i|\, e^{-i\theta}$ $(\zeta+i) = |\zeta+i|\, e^{+i\theta}$ $(\zeta-i)/(\zeta+i) = e^{-2i\theta}$ $\ln[] = -2i\theta$

$q_1(\zeta) = (i\zeta/2) \ln [(\zeta-i)/(\zeta+i)] - 1 = (i\zeta/2)(-2i\theta) - 1 = \theta\zeta - 1$ = real

so we recover at least that $q_1(\zeta)$ is real on the positive ζ axis.

Now if we assume that this formula

$$q_1(\zeta) \equiv (-i)^{1-n} Q_1(i\zeta) = Q_1(i\zeta) = (i\zeta/2) \ln [(\zeta-i)/(\zeta+i)] - 1$$

is ALSO valid for $\zeta < 0$, we would find that $q_1(\zeta)$ is also real on the negative real axis and the value is given by the same result $q_1(\zeta) = \theta\zeta - 1$ but now angle θ is might be 0.7π instead of $.3\pi$.

What happens in our scenario here if we take $\zeta \to +\infty$? On scratch I show that

$\sin\theta = 1/\sqrt{\zeta^2+1}$ then

$\theta = \sin^{-1}[(\zeta^2 + 1)^{-1/2}] = 1/\zeta - (1/3)(1/\zeta^3) + ...$

Then our limit for large ζ is this:

$q_1(\zeta) = \theta\zeta - 1 = [1/\zeta - (1/3)(1/\zeta^3) + ...]\zeta - 1 = 1 - (1/3)(1/\zeta^3)\zeta + ... - 1$

$= - (1/3)(1/\zeta^3)\zeta + ... = (-1/3)\zeta^{-2} + ...$

and we obtain our "desired result" that $q_1(\zeta) \to \zeta^{-2} = \zeta^{-\nu-1}$.

Now what happens if we go off $\zeta \to -\infty$? In this case we have $\theta \to \pi$ and the limit is

$q_1(\zeta) = \pi\zeta - 1$

which diverges as ζ^1. This more or less matches my experience with $Q_1(z)$ in the z plane in the document "Legendre with z=1 cut taken to the right...". So we do NOT want to have such a thing appear in our oblate bloid solution form.

Why not then take the form $q_1(|\zeta|)$ as I think I did in my last attempt at the bloid problem? Then we only have to think about the right half plane where we have full information. In the neck then we are going to have $q_1(0)$ which we can compute coming in from the right.

Could we try this kind of Smythian form for the bloid?

$$V_o(\zeta,\xi) = \Sigma_n \Sigma_{m=-\infty}^{\infty} F_{nm}\ q_n^m(|\zeta|)\ p_n^m(\zeta_0)\ P_n^m(\xi)\cos(m\varphi) \qquad (***)$$
$$V_i(\zeta,\xi) = \Sigma_n \Sigma_{m=-\infty}^{\infty} F_{nm}\ q_n^m(|\zeta_o|)\ p_n^m(\zeta)\ P_n^m(\xi)\cos(m\varphi)$$

That is, we could try this form in both the upper and lower halves. Remember that all $\zeta > 0$ in the upper region and $\zeta < 0$ in the lower region. So we get the desired large ζ behavior from V_0 in both upper and lower regions. It is unclear whether or not to use absolute values on the p argument. If we don't, then the above form allows for a distinction between upper ($\zeta>0$) and lower ($\zeta<0$). If we do, then we would have to use some G_{nm} maybe in the lower region.

Somehow I am thinking maybe the original parameterization of the bloid with $\zeta > 0$ always is better, even though ξ changes sign. [I tried this idea in "oblate bloid with p and q attempt 1.doc" and met with instant death. See that doc for reasons, but there are two bad things that happen. So forget this idea!

Wronskian

This is going to be needed if we try to use the above functions in our bloid solution. Here we go:

$$W[p_n^m(\zeta), q_n^m(\zeta)] \equiv p_n^m(\zeta)\,[\partial_\zeta\, q_n^m(\zeta)] - [\partial_\zeta\, p_n^m(\zeta)]\, q_n^m(\zeta)$$

Now set $\partial_\zeta = \partial/\partial\zeta = i\,\partial/\partial(i\zeta) = i\,\partial_{i\zeta}$ and write

$$W[p_n^m(\zeta), q_n^m(\zeta)] = i\,\{\ p_n^m(\zeta)\,[\partial_{i\zeta}\, q_n^m(\zeta)] - [\partial_{i\zeta}\, p_n^m(\zeta)]\, q_n^m(\zeta)\ \}$$

We want to have $\partial_{i\zeta} = \partial_z$ so that when we install the P and Q functions, the derivative will be the right derivative! So let's make up this notation

$$W[p_n^m(\zeta), q_n^m(\zeta)] = (i)\, W[p_n^m(\zeta), q_n^m(\zeta)]_{i\zeta}$$

where the label simply means that the derivative used is $\partial_{i\zeta}$. Then we can say

$$(-i)\, W[p_n^m(\zeta), q_n^m(\zeta)] = W[p_n^m(\zeta), q_n^m(\zeta)]_{i\zeta}$$

and NOW let's install our expressions for p and q,

$$W[p_n^m(\zeta), q_n^m(\zeta)]_{i\zeta} =$$

$W[(\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\, P_n^m(i\zeta) - (\pm i)^{-m/2}\,(-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\, Q_n^m(i\zeta),\ \mp (-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(i\zeta)]_{i\zeta}$

We can discard the W[Q,Q] part since it vanishes, and we are left with

$= W[(\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\, P_n^m(i\zeta),\ \mp (-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(i\zeta)]_{i\zeta}$

$= \mp\,(\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\,(-i)^{2m}\,(\pm i)^{n+1}\, W[P_n^m(i\zeta), Q_n^m(i\zeta)]_{i\zeta}$

$= \mp\,(\pm i) 2\cos[(n+m)\pi/2]\,(-i)^{2m}\, W[P_n^m(i\zeta), Q_n^m(i\zeta)]_{i\zeta}$

$= -i\, 2\cos[(n+m)\pi/2]\,(-i)^{2m}\, W[P_n^m(i\zeta), Q_n^m(i\zeta)]_{i\zeta}$

and of course we know that [Bateman p123 (13) with my mods done elsewhere]

$W[\, P_n^m(j\zeta), Q_n^m(j\zeta)]_{i\zeta} = (i)^{2m}\ [\Gamma(1+m+n)/\,\Gamma(1-m+n)] \ * \ 1/(1+\zeta^2)$

The $(i)^{2m}$ type factors cancel and we get

$W[p_n^m(\zeta), q_n^m(\zeta)]_{i\zeta} = -i\, 2\cos[(n+m)\pi/2]\ [\Gamma(1+m+n)/\,\Gamma(1-m+n)] \ * \ 1/(1+\zeta^2)$

so that

$W[p_n^m(\zeta), q_n^m(\zeta)] = (i)\, W[p_n^m(\zeta), q_n^m(\zeta)]_{i\zeta}$

$= -i^2\, 2\cos[(n+m)\pi/2]\ [\Gamma(1+m+n)/\,\Gamma(1-m+n)] \ * \ 1/(1+\zeta^2)$

$= 2\cos[(n+m)\pi/2] f(n,m) \ * \ 1/(1+\zeta^2)$

and as expected, our result is real. This result is valid for all m and n, nothing need be integer. Note: if I had not maintained the $\mp$ on the definition of q, the Wronskian would have two signs to worry about.

Summary of all results so far

$q_n^m(\zeta) = \mp\,(-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(i\zeta)$

$p_n^m(\zeta) = [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$

$= (\pm i)^{-n}\, 2\cos[(n+m)\pi/2]\, P_n^m(i\zeta) - (\pm i)^{-m/2}\,(-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\, Q_n^m(i\zeta)$

$W[p_n^m(\zeta), q_n^m(\zeta)] = 2\cos[(n+m)\pi/2] f(n,m) \ * \ 1/(1+\zeta^2)$

So I think these are the right functions to use in solving the Bloid Green's problem!

Negation of m property for integer m

For integer m we know from Leg prop doc that

$$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ P_\nu^{-m}(z) \ = \ P_\nu^{m}(z) \ = \ f(\nu,m)\ P_\nu^{-m}(z)$$
$$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ Q_\nu^{-m}(z) \ = \ Q_\nu^{m}(z) \ = \ f(\nu,m)\ Q_\nu^{-m}(z)$$

So here we go. First let's do the q:

$$q_n^m(\zeta) = \mp\ (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta)$$

$$\begin{aligned} q_n^{-m}(\zeta) &= \mp\ (-i)^{-2m}\ (\pm i)^{n+1}\ Q_n^{-m}(i\zeta) \\ &= \mp\ (-i)^{-2m}\ (\pm i)^{n+1}\ \{\ f(n,-m)\ Q_n^m(i\zeta)\} \\ &= (-i)^{-4m}\ \{\ f(n,-m)\ [\mp(-i)^{2m}\ (\pm i)^{n+1} Q_n^m(i\zeta)]\} \\ &= (-i)^{-4m}\ f(n,-m)\ q_n^m(\zeta) \\ &= (+i)^{4m}\ f(n,-m)\ q_n^m(\zeta) \\ &= (+i^4)^m\ f(n,-m)\ q_n^m(\zeta) \\ &= f(n,-m)\ q_n^m(\zeta) \end{aligned}$$

Now let's do p:

$$p_n^m(\zeta) \equiv\ [\ (\mp i)^{-m}\ P_n^m(i\zeta)\ + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$p_n^{-m}(\zeta) \equiv\ [\ (\mp i)^{m}\ P_n^{-m}(i\zeta)\ + (\pm i)^{m} P_n^{-m}(-i\zeta)]$$

$$= [\ (\mp i)^{m}\ \{\ f(n,-m)\ P_\nu^m(i\zeta)\}\ + (\pm i)^m\{\ f(n,-m)\ P_\nu^m(-i\zeta)\}]$$

$$= f(n,-m)\ [\ (\mp i)^{m}\ P_\nu^m(i\zeta)\ + (\pm i)^{m} P_\nu^m(-i\zeta)]$$

$$= f(n,-m)\ p_n^m(\zeta)$$

Summary of these two nice results:

$$q_n^{-m}(\zeta) = f(n,-m)\ q_n^m(\zeta) \qquad m = \text{integer, n = general, } \zeta = \text{anything}$$
$$p_n^{-m}(\zeta) = f(n,-m)\ p_n^m(\zeta)$$

Large ζ behavior:

We know these facts:

$$e^{-i\pi\mu} Q_\nu^\mu(z) \ = \ \sqrt{\pi}\ \ [\ \Gamma(1+\nu+\mu)/\ \Gamma(\nu+3/2)]\ \ (2z)^{-\nu-1} \qquad \nu \neq -3/2,-5/2..... \text{ // Bateman p 134 (41)}$$

$$P_\nu^\mu(z) \ = (2^\nu/\sqrt{\pi}\ )\ [\Gamma(\nu+1/2)/\ \Gamma(1+\nu-\mu)]\ z^\nu \qquad \mathrm{Re}(\nu) > -1/2 \qquad \text{// Bateman p 126 (23)}$$

which we now translate as follows

$(-i)^{2m}Q_n^m(z) = \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ (2z)^{-n-1}$ $\quad n \neq -3/2,-5/2.....$ // Bateman p 134 (41)

$P_n^m(z) = (2^n/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ z^n$ $\quad \mathrm{Re}(\nu) > -1/2$ // Bateman p 126 (23)

Then first we have:

$q_n^m(\zeta) \equiv \mp (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$

$= \mp (-i)^{2m} (\pm i)^{n+1} (+i)^{2m} \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ (2i\zeta)^{-n-1}$

$= \mp (\pm i)^{n+1} \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ (2i\zeta)^{-n-1}$

$= \mp (\pm i)^{n+1} \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ (\pm 2i|\zeta|)^{-n-1}$

$= \mp (\pm i)^{n+1} (\pm i)^{-n-1} \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ (2|\zeta|)^{-n-1}$

$= \mp \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ 2^{-n-1}|\zeta|^{-n-1}$ // which is real.

Next we have for p,

$p_n^m(\zeta) = (\pm i)^{-n}\ 2\cos[(n+m)\pi/2]\ P_n^m(i\zeta)$ + "$Q_n^m(i\zeta)$ term" which we ignore for large ζ

$= (\pm i)^{-n}\ 2\cos[(n+m)\pi/2]\ (2^n/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ (i\zeta)^n$

$= (\pm i)^{-n}\ 2\cos[(n+m)\pi/2]\ (2^n/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ (\pm i|\zeta|)^n$

$= (\pm i)^{-n}\ (\pm i)^n\ 2\cos[(n+m)\pi/2]\ (2^n/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ (|\zeta|)^n$

$= \cos[(n+m)\pi/2]\ (2^{n+1}/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ |\zeta|^n$ // which is real

So here are our large ζ results:

$q_n^m(\zeta) = \mp \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\ 2^{-n-1}|\zeta|^{-n-1}$

$p_n^m(\zeta) = \cos[(n+m)\pi/2]\ (2^{n+1}/\sqrt{\pi})\ [\Gamma(n+1/2)/\Gamma(1+n-m)]\ |\zeta|^n$

Summary of Everything So Far:

Results are true for general m and n except where indicated:

$q_n^m(\zeta) = \mp\ (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta)$ Re(ζ) > 0 upper sign, Re(ζ) < 0 lower sign

$p_n^m(\zeta) = [\ (\mp i)^{-m}\ P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$

$= (\pm i)^{-n}\ 2\cos[(n+m)\pi/2]\ P_n^m(i\zeta) - (\pm i)^{-m/2}\ (-i)^{2m}\ (2/\pi)\ \sin\pi(n+m)\ Q_n^m(i\zeta)$

$q_n^m(\zeta)$ and $p_n^m(\zeta)$ are real when n, m and ζ are real

$W[p_n^m(\zeta), q_n^m(\zeta)] = 2\cos[(n+m)\pi/2] f(n,m) * 1/(1+\zeta^2)$

$q_n^{-m}(\zeta) = f(n,-m)\ q_n^m(\zeta)$ m = integer, n = general, ζ = anything
$p_n^{-m}(\zeta) = f(n,-m)\ p_n^m(\zeta)$

$q_n^m(\zeta) \rightarrow \mp\sqrt{\pi}\ [\ \Gamma(1+n+m)/\ \Gamma(n+3/2)]\ 2^{-n-1}|\zeta|^{-n-1}$ large |ζ|

$p_n^m(\zeta) \rightarrow \cos[(n+m)\pi/2]\ (2^{n+1}/\sqrt{\pi}\ )\ [\Gamma(n+1/2)/\ \Gamma(1+n-m)]\ |\zeta|^n$ large |ζ|

Restatement of Everything So Far where we remove the ∓ sign from the q definition.

Results are true for general m and n except where indicated:

$q_n^m(\zeta) \equiv (-i)^{2m}\ (\pm i)^{n+1}\ Q_n^m(i\zeta)$ Re(ζ) > 0 upper sign, Re(ζ) < 0 lower sign

$p_n^m(\zeta) \equiv [\ (\mp i)^{-m}\ P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$

$= (\pm i)^{-n}\ 2\cos[(n+m)\pi/2]\ P_n^m(i\zeta) - (\pm i)^{-m/2}\ (-i)^{2m}\ (2/\pi)\ \sin\pi(n+m)\ Q_n^m(i\zeta)$

$q_n^m(\zeta)$ and $p_n^m(\zeta)$ are real when n, m and ζ are real

$W[p_n^m(\zeta), q_n^m(\zeta)] = \pm\ 2\cos[(n+m)\pi/2] f(n,m) * 1/(1+\zeta^2)$

$q_n^{-m}(\zeta) = f(n,-m)\ q_n^m(\zeta)$ m = integer, n = general, ζ = anything
$p_n^{-m}(\zeta) = f(n,-m)\ p_n^m(\zeta)$

$q_n^m(\zeta) \rightarrow \sqrt{\pi}\ [\ \Gamma(1+n+m)/\ \Gamma(n+3/2)]\ 2^{-n-1}|\zeta|^{-n-1}$ large |ζ|

$p_n^m(\zeta) \rightarrow \cos[(n+m)\pi/2]\ (2^{n+1}/\sqrt{\pi}\ )\ [\Gamma(n+1/2)/\ \Gamma(1+n-m)]\ |\zeta|^n$ large |ζ|

Negation of Argument:

<u>First do p:</u>

$$p_n^m(\zeta) \equiv [\ (\mp i)^{-m}\ P_n^m(i\zeta)\ + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$p_n^m(-\zeta) \equiv [(\pm i)^{-m}\ P_n^m(-i\zeta)\ + (\mp i)^{-m} P_n^m(i\zeta)]\ = p_n^m(\zeta)$$

Second do q:

$$q_n^m(\zeta) \equiv (-i)^{2m} (\pm i)^{n+1}\ Q_n^m(i\zeta) \qquad \mp$$

$$q_n^m(-\zeta) \equiv (-i)^{2m} (\mp i)^{n+1}\ Q_n^m(-i\zeta) \qquad \pm$$

But Leg prop doc says

$$Q_\nu^\mu(-z)\ = -\ Q_\nu^\mu(z)\ e^{\pm i\pi\nu}$$

$$Q_n^m(-i\zeta)\ = -\ Q_n^m(i\zeta)\ e^{\pm i\pi n}$$

so we get

$$q_n^m(-\zeta) \equiv (-i)^{2m} (\mp i)^{n+1}\ [-\ Q_n^m(i\zeta)\ e^{\pm i\pi n}] \qquad \pm$$

$$= (-i)^{2m} (\mp i)^{n+1}\ [-\ Q_n^m(i\zeta)\ (\pm i)^{2n}]$$

$$= (-i)^{2m} (\mp i)^{n+1}\ [-\ Q_n^m(i\zeta)\ (\mp i)^{-2n}]$$

$$= -\ (-i)^{2m} (\mp i)^{-n+1} Q_n^m(i\zeta) \qquad \pm$$

Now replace

$$Q_n^m(i\zeta) = q_n^m(\zeta)\ (-i)^{-2m}\ (\mp i)^{n+1} \qquad \pm$$

and we then get

$$q_n^m(-\zeta) = -\ (-i)^{2m} (\mp i)^{-n+1} Q_n^m(i\zeta)$$

$$= -\ (-i)^{2m} (\mp i)^{-n+1} [q_n^m(\zeta)\ (-i)^{-2m}\ (\mp i)^{n+1}] \qquad \pm\pm = 1$$

$$= -\ (\mp i)^2\ q_n^m(\zeta)$$

$$= q_n^m(\zeta)$$

Summary of results: [both functions are symmetric, very simple]

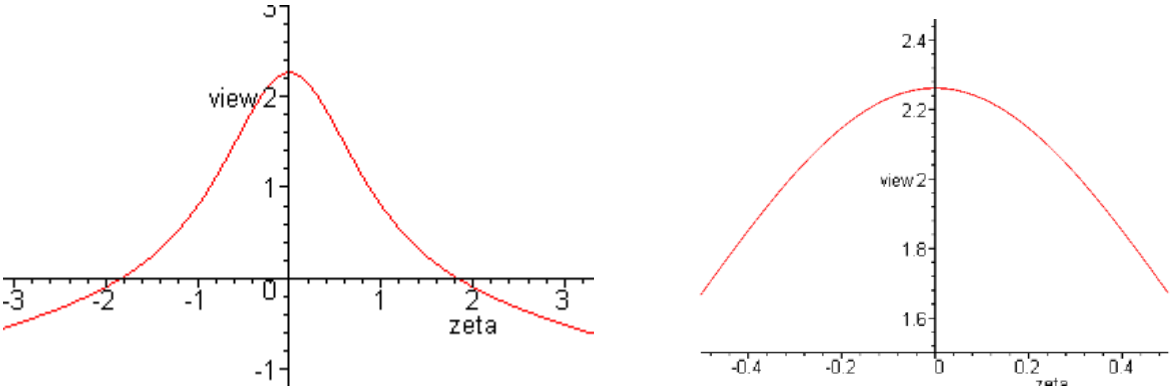
$p_n^m(-\zeta) = p_n^m(\zeta)$
$q_n^m(-\zeta) = q_n^m(\zeta)$

(My sign notes to the right show that q is symmetric if you add $\mp$ to its definition.)

Plots of the p function

$$p_n^m(\zeta) \equiv [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

```
p:= (n,m,zeta)-> (-signum(zeta)*I)^(-m)*LegendreP(n,m,+I*zeta)
                + (+signum(zeta)*I)^(-m)*LegendreP(n,m,-I*zeta);
plot(Re(p(1.2,2.3,zeta)), zeta = -0.5..0.5, view = 1.5..2.5);
```

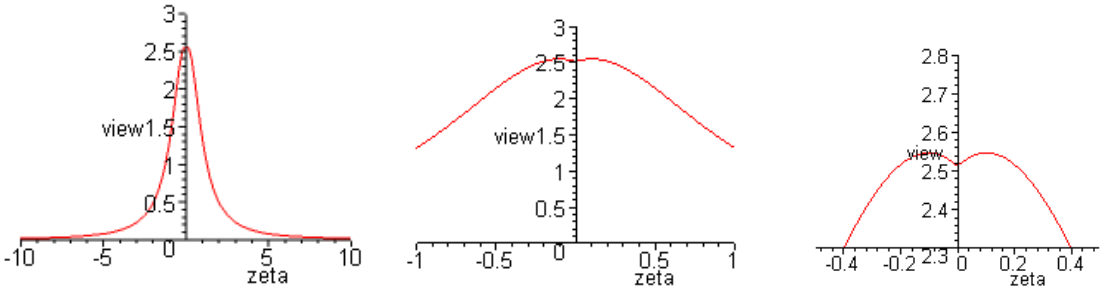


This function seems to have zero slope at $\zeta = 0$, but we will check on that below.

Plots of the q function

$$q_n^m(\zeta) \equiv (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta) \qquad \mathrm{Re}(\zeta) > 0 \text{ upper sign},\ \mathrm{Re}(\zeta) < 0 \text{ lower sign}$$

```
q := (n,m,zeta)-> (-I)^(2*m)*(signum(zeta)*I)^(n+1)*LegendreQ(n,m,I*zeta);
plot(Re(q(1.2,2.3,zeta)), zeta = -0.4..0.4, view = 2.3..2.7);
```



As we zoom in, we see that the slope is not 0 at $\zeta = 0$ (as I once thought it was)

Analytic and Symmetry Properties of the p and q functions

First, the p function. If we define

$$\rho_n^m(z) = [(\mp i)^{-m} P_n^m(z) + (\pm i)^{-m} P_n^m(-z)],$$

I found the following properties in "the meaning of f(-z)..." :

Fact 0: The function $\rho_n^m(z)$ has cuts $(-\infty,-1)$ and $(+1,+\infty)$ and is uncut on $(-1,1)$ where it is real. The function $\rho_n^m(z)$ is "real analytic" because it is real on a piece of the real axis. On the cuts, the value above and below are related by complex conjugation. [Of course we are saying $\rho_n^m(z)$ is an *analytic* function away from the branch points and their cuts.]

Fact 1:	$\rho_n^m(-z) = \rho_n^m(z)$	for arbitrary z
Fact 2:	$\rho_n^m(-z) = \rho_n^m(z)$ = real	for z in (-1,1)
Fact 3:	$\rho_n^m(-z) = \rho_n^m(z)$ = real	for z = imaginary

We can translate these "facts" into properties for $p_n^m(\zeta)$: // where $z = i\zeta$

Fact 0: The function $p_n^m(\zeta)$ is an analytic function of variable ζ . It has cuts $(+i,+i\infty)$ and $(-i,-i\infty)$.
The function $p_n^m(\zeta)$ is real on the real ζ axis (which is of course the imaginary axis of $\rho_n^m(z)$) and is therefore a real-analytic function (I think).

Fact 1:	$p_n^m(\zeta) = \rho_n^m(-\zeta)$	for arbitrary ζ
Fact 2:	$p_n^m(\zeta) = \rho_n^m(-\zeta)$ = real	for ζ in (-i,+i)
Fact 3:	$p_n^m(\zeta) = \rho_n^m(-\zeta)$ = real	for ζ on the real axis.

Comments: As an analytic function of ζ, the function $p_n^m(\zeta)$ has branch points which are off the real axis, being located at z = +i and z = -i. I am not aware of any other "special functions" that have branch points off the real axis like this.

Second, the q function

Analytic Structure of $\mathbf{s}_n^m(\mathbf{z}) \equiv (-\mathbf{i})^{2m} (\pm\mathbf{i})^{n+1} \mathbf{Q}_n^m(\mathbf{z})$? Just playing with Maple shows that this function has cuts everywhere on the real axis, including the positive side. I have not studied this in detail from a functional point of view. Here are the properties, obtained by Maple sample points:

Fact 1q:	$s_n^m(-z) = s_n^m(z)$	for arbitrary z (off the real axis)
Fact 3q:	$s_n^m(-z) = s_n^m(z)$ = real	for z = imaginary
Fact 4q:	$s_n^m(z^*) = (s_n^m(z))^*$	for arbitrary z

Fact 0q: The function $s_n^m(z)$ is cut on the entire real axis, having branch points at $z = \pm 1$. Some of the cut comes from Q, and some from the added phase factor, I think. You cannot say that $s_n^m(z)$ is real analytic since it is real nowhere on the real axis. It is analytic in z off the real axis, however.

Now we can translate these into properties of $q_n^m(\zeta) = s_n^m(z)$ where $z = i\zeta$.

Fact 0q: The function $q_n^m(\zeta)$ is cut on the entire imaginary axis, having branch points at $z = \pm i$. Some of the cut comes from Q, and some from the added phase factor, I think. $q_n^m(\zeta)$ is real for ζ on the real axis, so perhaps this means it is real analytic, I am not sure how the vertical cut affects things.

Fact 1q:	$q_n^m(-\zeta) = q_n^m(\zeta)$	for arbitrary ζ (off the imaginary axis)
Fact 3q:	$q_n^m(-\zeta) = q_n^m(\zeta)$ = real	for ζ = real
Fact 4q:	$q_n^m(\zeta^*) = (q_n^m(-\zeta))^*$	for arbitrary ζ

Our main interest is that the p and q functions are real for real ζ, and are independent according to the Wronskian. Therefore we can use them as atoms in constructing oblate Laplace solutions.

Value of p and q at zero

First we do p:

$$p_n^m(\zeta) = [\,(\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)] \qquad \mathrm{Re}(\zeta) > 0 \text{ upper sign, } \mathrm{Re}(\zeta) < 0 \text{ lower sign}$$
$$p_n^m(0\pm) = [\,(\mp i)^{-m}\, P_n^m(i0\pm) + (\pm i)^{-m} P_n^m(i0\mp)]$$

and we call upon Leg prop doc

$$P_\nu^\mu(z=i0\pm) = 2^\mu\sqrt{\pi}\; e^{\mp i\pi\mu/2} / [\,\Gamma(1/2 - \nu/2 - \mu/2)\,\Gamma(1 + \nu/2 - \mu/2)\,] \qquad \text{// p 126 (22)}$$
$$P_n^m(i0\pm) = 2^m\sqrt{\pi}\; (\mp i)^m / [\,\Gamma(1/2 - n/2 - m/2)\,\Gamma(1 + n/2 - m/2)\,] \qquad \text{// p 126 (22)}$$

We can factor out common factors as we insert these expressions into the above:

$$p_n^m(0\pm) = (\,2^m\sqrt{\pi}\; / [\,\Gamma(1/2 - n/2 - m/2)\,\Gamma(1 + n/2 - m/2)\,]\,)\,[(\mp i)^{-m}(\mp i)^m + (\pm i)^{-m}(\pm i)^m\,]$$

$$= (\,2^{m+1}\sqrt{\pi}\; / [\,\Gamma(1/2 - n/2 - m/2)\,\Gamma(1 + n/2 - m/2)\,]\,)$$

As expected from the symmetry property mentioned above, namely that $p_n^m(\zeta) = \rho_n^m(-\zeta)$ = real for ζ on the real axis, and function is uncut on (-i,i), we are (now) *not* surprised to find that we get the same real value approaching from positive or negative ζ.

Second we do q:

$$q_n^m(\zeta) \equiv (-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(i\zeta)$$

$$q_n^m(0\pm) \equiv (-i)^{2m}\,(\pm i)^{n+1}\, Q_n^m(0\pm)$$

From Leg prop doc we have,

$$Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\, 2^{\mu-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)] \quad // \text{p 134 (40)}$$

$$Q_n^m(z=0\pm) = \sqrt{\pi}\, 2^{m-1}\, e^{\pm i\pi(-n-1)/2}\, [\, \Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)] \quad // \text{p 134 (40)}$$

$$Q_n^m(z=0\pm) = \sqrt{\pi}\, 2^{m-1}(\pm i)^{-n-1}\, [\, \Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)] \quad // \text{p 134 (40)}$$

Therefore

$$q_n^m(0\pm) \equiv (-i)^{2m}\, (\pm i)^{n+1}\, Q_n^m(0\pm)$$

$$= (-i)^{2m}\, (\pm i)^{n+1}\, \sqrt{\pi}\, 2^{m-1}(\pm i)^{-n-1}\, [\, \Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$$

$$= (-1)^m \sqrt{\pi}\, 2^{m-1}\, [\, \Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$$

As expected from the symmetry property mentioned above, namely that $q_n^m(-\zeta) = q_n^m(\zeta)$ = real for ζ on the real axis, we get the same result coming in from either direction. Unlike the p case, the function q has a cut from (-i,i) as Maple shows, but it happens that in the middle of this cut on the real ζ axis, you get the same number coming in from either side. Maple confirms this explicitly, as do the plots above.

Summary of our results:

$$p_n^m(0\pm) = 2^{m+1}\sqrt{\pi}\ / [\, \Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)\,]$$

$$q_n^m(0\pm) = (-1)^m \sqrt{\pi}\, 2^{m-1}\, [\, \Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$$

Derivatives of p and q evaluated at $\zeta = 0$ (for general n and m)

First, we do p:

$$p_n^m(\zeta) = [\, (\mp i)^{-m}\, P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

$$\partial_\zeta\, p_n^m(\zeta) = i\, [\, (\mp i)^{-m}\, P_n^{m\,\prime}(i\zeta) - (\pm i)^{-m} P_n^{m\,\prime}(-i\zeta)]$$

$$[\partial_\zeta\, p_n^m(\zeta)](0\pm\varepsilon) = i\, [\, (\mp i)^{-m}\, P_n^{m\,\prime}(0\pm i\varepsilon) - (\pm i)^{-m} P_n^{m\,\prime}(0\mp i\varepsilon)]$$

From Leg prop we have (I have very high confidence in this since two derivations agreed)

$$P_n^{m\,\prime}(0\pm i\varepsilon) = -(\mp i)^m\ \sqrt{\pi}\, 2^{m+1} / [\, \Gamma(1/2+n/2-m/2)\, \Gamma(-n/2-m/2)]$$
$$P_n^{m\,\prime}(0\mp i\varepsilon) = -(\pm i)^m\ \sqrt{\pi}\, 2^{m+1} / [\, \Gamma(1/2+n/2-m/2)\, \Gamma(-n/2-m/2)]$$

Thus we insert to get

$$[\partial_\zeta\, p_n^m(\zeta)](0\pm\varepsilon) = -\ i\sqrt{\pi}\ 2^{m+1}/\ [\ \Gamma(1/2+n/2-m/2)\ \Gamma(-n/2-m/2)] * [(\mp i)^{-m}(\mp i)^m - (\pm i)^{-m}(\pm i)^m]$$

$$= -\ i\sqrt{\pi}\ 2^{m+1}/\ [\ \Gamma(1/2+n/2-m/2)\ \Gamma(-n/2-m/2)] * [1 - 1]$$

$$= 0$$

"But in retrospect, since $p_n^m(\zeta)$ is symmetric (and finite), this has to be the case, duh! " [Think again, blasto breath. Symmetric and finite does not imply zero slope. But in this case the slope really is zero, and that correlates with the fact that there is not cut.]

<u>Second, we do q:</u>

$$q_n^m(\zeta) \equiv (-i)^{2m}(\pm i)^{n+1}\, Q_n^m(i\zeta) \qquad \mathrm{Re}(\zeta) > 0 \text{ upper sign},\ \mathrm{Re}(\zeta) < 0 \text{ lower sign}$$

$$\partial_\zeta\, q_n^m(\zeta) = i\,(-i)^{2m}(\pm i)^{n+1}\, Q_n^m{}'(i\zeta)$$

Our Leg prop doc says (high confidence since two derivations)

$$Q_n^m{}'(0\pm i\varepsilon) = e^{+inm}\ e^{\mp inn/2}\sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

So we find

$$q_n^m{}'(0\pm) = i\,(-i)^{2m}(\pm i)^{n+1}\, e^{+inm}\ e^{\mp inn/2}\sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

$$= i\,(-i)^{2m}(\pm i)^{n+1} \qquad (i)^{2m} \qquad (\mp i)^n\sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

$$= i\,(-i)^{2m}(\pm i)^{1} \qquad (i)^{2m} \qquad \sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

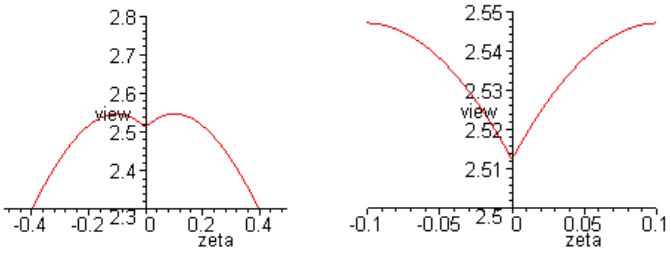
$$= i \quad (\pm i)^{1} \qquad \sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

$$= \pm(\pm i) \quad (\pm i)^{1} \qquad \sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

$$= \pm(\pm i)^2 \qquad \sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

$$= \mp\sqrt{\pi}\ 2^m\ [\ \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

Let's check this against my plots: Picture on the right suggests pos side slope = (2.525 -2.513)/.02 = 0.6



I had n = 1.2 and m = 2.3 so my results above are

$$\mp\sqrt{\pi}\, 2^{2.3}\ [\,\Gamma(1+1.2/2+2.3/2) / \Gamma(1/2+1.2/2-2.3/2)]$$

$$\mp\sqrt{\pi}\, 2^{2.3}\ [\,\Gamma(1+0.6+1.15) / \Gamma(0.5+0.6-1.15)]$$

$$\mp\sqrt{\pi}\, 2^{2.3}\ [\,\Gamma(1.6+1.15) / \Gamma(1.1-1.15)]$$

$$\mp\sqrt{\pi}\, 2^{2.3}\ [\,\Gamma(2.75) / \Gamma(-.05)]$$

Now Maple tells me that

$$\sqrt{\pi}\, 2^{2.3}\ [\,\Gamma(2.75) / \Gamma(-.05)] \quad = -\ 0.68$$

```
sqrt(Pi)*2^(2.3)*GAMMA(2.75)/GAMMA(-0.05):evalf(%);
```

```
                                -.6805302920
```

So the positive side slope is about + 0.7, good enough! I think I believe these results:

<u>Summary:</u>

$p_n^m\,'(0\pm) = 0$

$q_n^m\,'(0\pm) = \mp\sqrt{\pi}\, 2^m\ [\,\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$

New summary of Everything So Far

Results are true for general m and n except where indicated:

Definitions:

$q_n^m(\zeta) \equiv (-i)^{2m}\,(\pm i)^{n+1}\,Q_n^m(i\zeta)$ Re(ζ) > 0 upper sign, Re(ζ) < 0 lower sign

$p_n^m(\zeta) \equiv [\,(\mp i)^{-m}\,P_n^m(i\zeta) + (\pm i)^{-m}P_n^m(-i\zeta)]$

$= (\pm i)^{-n}\,2\cos[(n+m)\pi/2]\,P_n^m(i\zeta) - (\pm i)^{-m/2}\,(-i)^{2m}\,(2/\pi)\,\sin\pi(n+m)\,Q_n^m(i\zeta)$

Reality on real axis:

$q_n^m(\zeta)$ and $p_n^m(\zeta)$ are real when n, m and ζ are real

Wronskian:

$W[p_n^m(\zeta), q_n^m(\zeta)] = \pm 2\cos[(n+m)\pi/2] f(n,m) * 1/(1+\zeta^2)$

Negation of the order:

$q_n^{-m}(\zeta) = f(n,-m)\, q_n^m(\zeta)$ m = integer, n = general, ζ = anything
$p_n^{-m}(\zeta) = f(n,-m)\, p_n^m(\zeta)$

Large ζ behavior:

$q_n^m(\zeta) \rightarrow \sqrt{\pi}\ [\Gamma(1+n+m)/\Gamma(n+3/2)]\, 2^{-n-1} |\zeta|^{-n-1}$ large $|\zeta|$
$p_n^m(\zeta) \rightarrow \cos[(n+m)\pi/2]\, (2^{n+1}/\sqrt{\pi})\, [\Gamma(n+1/2)/\Gamma(1+n-m)]\, |\zeta|^n$ large $|\zeta|$

Negation of the argument:

$p_n^m(-\zeta) = p_n^m(\zeta)$
$q_n^m(-\zeta) = q_n^m(\zeta)$

Values at zero: [notation here means $0 \pm \varepsilon$]

$p_n^m(0\pm) = 2^{m+1}\sqrt{\pi} / [\Gamma(1/2 - n/2 - m/2)\, \Gamma(1 + n/2 - m/2)]$
$q_n^m(0\pm) = (-1)^m \sqrt{\pi}\, 2^{m-1} [\Gamma(1/2 + n/2 + m/2) / \Gamma(1 + n/2 - m/2)]$

$p_n^m{}'(0\pm) = 0$
$q_n^m{}'(0\pm) = \mp\sqrt{\pi}\, 2^m [\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$

Analytic and Symmetry properties with ζ as a complex variable:

Fact 0p: The function $p_n^m(\zeta)$ is an analytic function of variable ζ . It has cuts $(+i,+i\infty)$ and $(-i,-i\infty)$. The function $p_n^m(\zeta)$ is real on the real ζ axis and is therefore a real-analytic function (I think).

Fact 1p: $p_n^m(\zeta) = \rho_n^m(-\zeta)$ for arbitrary ζ
Fact 2p: $p_n^m(\zeta) = \rho_n^m(-\zeta)$ = real for ζ in $(-i,+i)$
Fact 3p: $p_n^m(\zeta) = \rho_n^m(-\zeta)$ = real for ζ on the real axis.

Fact 0q: The function $q_n^m(\zeta)$ is cut on the *entire* imaginary axis, having branch points at $z = \pm i$. Some of the cut comes from Q, and some from the added phase factor, I think. $q_n^m(\zeta)$ is real for ζ on the real axis, so perhaps this means it is real analytic, I am not sure how the vertical cut affects things.

Fact 1q:	$q_n^m(-\zeta) = q_n^m(\zeta)$	for arbitrary ζ (off the imaginary axis)
Fact 3q:	$q_n^m(-\zeta) = q_n^m(\zeta)$ = real	for ζ = real
Fact 4q:	$q_n^m(\zeta^*) = (q_n^m(-\zeta))^*$	for arbitrary ζ

Comments on the p and q functions developed here. [Feb 14, 2010]

I was surprised when I "discovered" that the p and q functions are perfectly symmetric in ζ . I have now seen that this fact implies that p' and q' are both zero at $\zeta = 0$, and this has the sad implication that if you try to form an oblate bloid Smythian form for V_i out of these functions, you find that for anything you can put together with p and q as your two ζ independent functions, you get $\partial_\zeta V = 0$ at the "neck", and this means the form is not acceptable. More generally, it now seems pretty obvious that you cannot construct any kind of bloid solution from symmetric functions, since you know the bloid solution will NOT be symmetric! RETHINK THESE STATEMENTS! NO LONGER TRUE.

My original goal was just to find two independent real solutions of the ζ ODE and I would call them p and q. But my "method of construction" is what forced them to be even in ζ. I constructed each of these functions to cancel the F function term having the linear z factor. Once this second term was cancelled, then of course what is left is even in ζ, it is now a complete no-brainer. But at the time, I did not realize the significance this was going to have.

So it seems clear that you cannot allow cancellation of the second terms! So is there some other way to construct p and q functions that are real, but are more general than the ones I found above? If there is no way, then why is that so? Is it because the branch points of f(ζ) are not on the real axis?

What happens if I apply the "Frobenius theory" directly to the ζ ODE (see "essay") , which is this:

$$L = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + [\,m^2/(1+\zeta^2) - n(n+1)] \qquad Lu = 0 \quad \text{is the ODE}$$

General Frobenius form

$$L = D^2 + pD + q$$

so divide through by $(1+\zeta^2)$ to get (don't confuse with the p and q used earlier in this doc!)

$$L_1 = \partial_\zeta^2 - 2[\zeta/(1+\zeta^2)]\,\partial_\zeta + [\,m^2/(1+\zeta^2)^2 - n(n+1)/(1+\zeta^2)]$$

$$p = -2[\zeta/(1+\zeta^2)] \qquad q = [\,m^2/(1+\zeta^2)^2 - n(n+1)/(1+\zeta^2)]$$

Both these functions are analytic at $\zeta = 0$ so we could do a power series there. We expect the Frobenius exponents to be r = 0,1. But let's go ahead and derive this fact:

$$p(0) = 0 \qquad q(0) = m^2 - n(n+1)$$

Try $u(\zeta) = \Sigma_n a_n \zeta^n$.

Let's change everything from ζ to x so can type faster. So we haev

$L_1 = \partial_x^2 - 2[x/(1+x^2)]\,\partial_x + [\,m^2/(1+x^2)^2 - n(n+1)/(1+x^2)]$ $\qquad L_1 u = 0$

$p = -2[x/(1+x^2)] \qquad q = [\,m^2/(1+x^2)^2 - \eta(\eta+1)/(1+x^2)]$

$p(0) = 0 \qquad q(0) = m^2 - \eta(\eta+1)$ // since n is a summation index below

Try $u(x) = x^r \Sigma_{n=0}^{\infty} a_n x^n$. Get

$\partial_x^2[\Sigma_{n=0}^{\infty} a_n x^{n+r}] + p(x)\,\partial_x[\Sigma_{n=0}^{\infty} a_n x^{n+r}] + q(x)\,[\Sigma_{n=0}^{\infty} a_n x^{n+r}] = 0$

$[\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}] + p(x)\,[\Sigma_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}] + q(x)\,[\Sigma_{n=0}^{\infty} a_n x^{n+r}] = 0$

$[\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}] + p(x)\,[\Sigma_{n=1}^{\infty}(n-1+r)\,a_n x^{n+r-2}] + q(x)\,[\Sigma_{n=2}^{\infty} a_{n-2} x^{n+r-2}] = 0$

The lowest power term is n = 0 and we get

$(n+r)(n+r-1)a_n x^{n+r-2} = 0 \quad \Rightarrow \quad (r)(r-1)a_0 x^{r-2} = 0 \quad \Rightarrow \quad r = 0,\ \text{and}\ r = 1$

So, we should then find two independent and real solutions of the form

$u_1(x) = \Sigma_{n=0}^{\infty} a_n x^n$

$u_2(x) = x\,\Sigma_{n=0}^{\infty} b_n x^n$

This solution should converge in a disk of radius 1 surrounding the origin in the $\zeta = x$ complex plane. We expect the solutions to be real because all the recursion relations are real, p and q are real, and so on.

Question: What are these two real solutions, and how are they related to $P_\eta^m(ix)$ and $Q_\eta^m(ix)$?

Digression on Simple Case Study: [Feb 15, 2010]

$L = -\partial_z^2 + k^2$. Then we have $Lu=0$ saying $-\partial_z^2 u = k^2 u$. Our solutions are $r_1(z) = \sin(kz)$ and $r_2(z) = \cos(kz)$, but real, both power series, one even, one odd. We could also write the solutions as $c_1(z) = e^{ikz}$ and $c_2(z) = e^{-ikz}$ and each of these solutions is complex and neither even nor odd.

Now the "flipped" ODE is obtained from $z = i\zeta$ so we then have $+\partial_\zeta^2 v(\zeta) = k^2 v(\zeta)$. We would assume that the solutions could be taken to be

$v_1(\zeta) = -i\,r_1(i\zeta) = -i\sin(ki\zeta) = \sinh(k\xi)$ real and odd in ζ
$v_2(\zeta) = r_2(i\zeta) = \cos(ki\zeta) = \cosh(k\xi)$ real and even in ζ

where I have rephased the first solution to make it real. So we obtain two independent functions which serve as good solutions to problems using the new equations. Nothing is "missing". Another solution set:

$v_1(\zeta) = c_1(i\zeta) = e^{+k\zeta}$ = real, neither even nor odd

$v_2(\zeta) = c_2(i\zeta) = e^{-k\zeta}$ = real, neither even nor odd

If we were doing a schematic "bloid" solution with these functions, we might do this and hope for the best. We know that nothing is "missing".

1	$e^{-k\zeta}$	$v_2(\zeta)$
2	$A\, e^{+k\zeta} + B\, e^{-k\zeta}$	
3	$C\, e^{+k\zeta} + D\, e^{-k\zeta}$	
4	$e^{+k\zeta}$	$v_1(\zeta)$

This case study is simple in that there are no branch points to think about! Another simple feature is that the function $e^{-k\zeta}$ blows up in one direction and blows down in another direction. In contrast, our function $q_n^m(\zeta)$ blows down in both directions because there is a cut present at z = (-1,1) which allows this to happen. If we went through the cut, we would get behavior similar to $e^{-k\zeta}$. Here are the forms I used in my Smythe section bloid attempt #3:

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

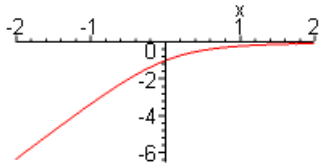
$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(\zeta)\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

Suppose I had used the "through the cut" version of $q_n^m(\zeta)$, call it $q_n^m(\zeta)_{0,0R}$. I keep coming back to our prototype example,

$$Q_1(j\zeta) = \zeta \cot^{-1}(\zeta) - 1 \qquad Q_1(z) = (z/2) \ln [\,(z+1)/(z-1)] - 1$$

which is blatantly NOT symmetric under $\zeta \rightarrow -\zeta$

```
plot(x*arccot(x)-1, x=-2..2);
```



We could define

$$q_n^m(\zeta)_{0,0R} = (-i)^{2m} (\pm i)^{n+1}\ Q_n^m(i\zeta)_{0,0R}$$

$$q_1(\zeta)_{0,0R} = (\pm i)^{1+1}\ Q_1(i\zeta)_{0,0R} = -\,Q_1(i\zeta)_{0,0R} = 1 - \zeta \cot^{-1}(\zeta)$$

This q function would NOT be symmetric, it would blow down for + ζ, and blow up for - ζ . If all the q's did this, we might rewrite our Smythian form as

$$V_o(\zeta,\xi) = \Sigma_{nm} [B_{nm}\ p_n^m(\zeta_0) + C_{nm}\ q_n^m(\zeta_0)]\ q_n^m(\zeta)_{0,0R}\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//upper} \quad \zeta \geq 0$$

$$V_o(\zeta,\xi) = \Sigma_{nm} [B'_{nm}\ p_n^m(-\zeta_0) + C'_{nm}\ q_n^m(-\zeta_0)]\ q_n^m(-\zeta)_{0,0R}\ P_n^m(\xi) \cos(m\varphi) \qquad \text{//lower} \quad \zeta \leq 0$$

The function $q_n^m(\zeta)_{0,0R}$ is not symmetric, and does not have to therefore have zero slope at the bloid neck.

So why did I reject these 0,0R functions? I study this Q_1 case in *great* detail in " legendre with z=1 cut taken to the right" and find this important result

$$Q_1(z)_{0,1L} = Q_1(z)_{0,0L} - i\pi z$$

Then with the cut to the right, I define

$Q_1(z)_{0,0R} = Q_1(z)_{0,0L}$	$\mathrm{Im}(z) > 0$	blows down z^{-1-1}
$Q_1(z)_{0,0R} = Q_1(z)_{0,1L} = Q_1(z)_{0,0L} - i\pi z$	$\mathrm{Im}(z) < 0$	blows up z^1 (second term)

and I then show that

$$Q_1(z)_{0,0R} = -1 + \zeta \cot^{-1}(\zeta)$$

My rejection was based on my pre-cone-method approach where n was an integer, and arrived at this conclusion:

"Conclusion: if I write my Smythian form like this:

$$V_o(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi_0) - Q_n^m(\xi_1) P_n^m(\xi_0)] P_n^m(\xi) Q_n^m(j\zeta) e^{im\varphi}$$
$$V_i(\zeta,\xi,\varphi) = \Sigma_{nm} E_{nm} [P_n^m(\xi_1) Q_n^m(\xi) - Q_n^m(\xi_1) P_n^m(\xi)] P_n^m(\xi_0) Q_n^m(j\zeta) e^{im\varphi}$$

then there exist no combinations $P_n^m(\xi)\, Q_n^m(j\zeta)$ where both factors are non-zero, such that the functions are always finite and decay at ∞. The mismatch is amazingly precise! The mystery continues for yet another day, and another, and another... [this is all before I switched to "the cone method".] "

I don't know why I didn't think to fix this problem just by putting $Q_n^m(-j\zeta)$ into the lower bloid V_o as noted above, then we don't care that it blows up for negative ζ .