The Bateman Formulas for P and Q PhL 2.20.10

Confusing Issue #1 ..... **1**
Confusing Issue #2 ..... **2**
Comparison between Bateman Q (41) and Maple's LegendreQ **2**
Comparison between Bateman Q (43) and Maple's LegendreQ **3**

Confusing Issue #1

As an opener, consider Bateman page 122 (5) which is the famous (41) for Q. This formula includes the factor $(z^2\text{-}1)^{\mu/2}$. As I show in document "study of (1-z)^a and related.doc", the following two functions are *quite different*: (each being a function of a complex variable)

$f_1(z) \equiv (z^2\text{-}1)^{\alpha}$

$f_2(z) \equiv (z\text{-}1)^{\alpha}\,(z+1)^{\alpha}$

If you set α = .2 and enter into Maple, you get these results:

```
f1(-2+.000000001*I);
```

$$1.245730940 - .3321949172\,10^{-9}\,I$$

```
f2(-2+.000000001*I);
```

$$.3849520314 + 1.184760528\,I$$

In evaluating f_1 Maple evaluates things from inside to outside, so it sets z = -2, $z^2 = 4$ and $f_1 = 3^{\alpha}$ and that is what you get.

So, it is very important to understand the *meaning* of $(z^2\text{-}1)^{\mu/2}$ in Bateman p 122 (5). Which function is it? Both Bateman and A&S say that it is $f_2(z)$ that is implied, not $f_1(z)$, although it is *written* as $f_1(z)$. GR are completely silent on this subject! No one ever talks about the angle of z^2-1. So you can see that this is a huge pitfall that I didn't even realize until today, after "working with " P and Q functions for the last several months!!

Bateman on page 123 goes on to say that for ALL formulas in the table, the implication is this:

$(z^2\text{-}1)^{\alpha} \equiv (z\text{-}1)^{\alpha}(z+1)^{\alpha}$

so that is made pretty clear. Bateman goes on to verify my two little power rules from the doc just mentioned, to wit:

$(-z+1)^{\alpha}_{0R} = e^{\mp i\pi\alpha}\,(z\text{-}1)^{\alpha}_{0L}$
$(-z+1)^{\alpha} = e^{\mp i\pi\alpha}\,(z\text{-}1)^{\alpha}$
$(-z+1) = e^{\mp i\pi}\,(z\text{-}1)$ // symbolic, agrees with Bateman p 123 mid page

$(-z-1)^{\alpha}_{0R} = e^{\mp i\pi\alpha}(z+1)^{\alpha}_{0L}$

$(-z-1)^{\alpha} = e^{\mp i\pi\alpha}(z+1)^{\alpha}$

$(-z-1) = e^{\mp i\pi}(z+1)$ // symbolic form agrees with Bateman mid page 123 !!

And Bateman gives a third rule which I will now explain:

$(z^2-1)^{\alpha} \equiv (z-1)^{\alpha}(z+1)^{\alpha}$

$(1-z^2)^{\alpha} \equiv (1-z)^{\alpha}(1+z)^{\alpha}$ // he never writes this, but I claim it would be his definition

Then we have:

$(1-z^2)^{\alpha} \equiv (1-z)^{\alpha}(1+z)^{\alpha} = [e^{\mp i\pi\alpha}(z-1)^{\alpha}](1+z)^{\alpha} = e^{\mp i\pi\alpha}(z-1)^{\alpha}(z+1)^{\alpha} = e^{\mp i\pi\alpha}(z^2-1)^{\alpha}$

Which is to say:

$(1-z^2)^{\alpha} = e^{\mp i\pi\alpha}(z^2-1)^{\alpha}$

$(1-z^2) = e^{\mp i\pi}(z^2-1)$ // Bateman's third rule on page 123.

Confusing Issue #2

I think I have resolved this one now, but at least it deserves mention. When you use Maple's hypergeometric function, Maple really does know how to continue the thing to get the right answer for any z in the cut z-plane. Although Maple V says it just uses the series, the Maple 13 help makes it clear that it knows how to do the continuation to any z. I believe this now.

Comparison between Bateman Q (41) and Maple's LegendreQ

This is entered in file batemanpq.mws.

```
restart;
Q41 := (n,m,z)
->exp(+I*Pi*m)*2^(-1-n)*(Pi)^(1/2)*(GAMMA(1+n+m)/GAMMA(n+3/2))*z^(-1-n-m)*(z-1)
^(m/2)*(z+1)^(m/2)* hypergeom([1+n/2+m/2, 1/2+n/2+m/2],[n+3/2],1/z^2);
```

$$Q41 := (n, m, z) \to e^{(I\pi m)}\, 2^{(-1-n)}\, \sqrt{\pi}\, \Gamma(1+n+m)\, z^{(-1-n-m)}\, (z-1)^{\left(\frac{1}{2}m\right)}\, (z+1)^{\left(\frac{1}{2}m\right)}$$
$$\text{hypergeom}\left(\left[1+\frac{1}{2}n+\frac{1}{2}m, \frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m\right], \left[n+\frac{3}{2}\right], \frac{1}{z^2}\right) \Big/ \Gamma\left(n+\frac{3}{2}\right)$$

```
n := 1.2:
m := 2.3:
z := -.4 -.2*I;
```

$$z := -.4 - .2\, I$$

```
Q41(n,m,z):evalf(%);
```

$$-.5542721416 - 3.005133026\, I$$

```
LegendreQ(n,m,z):evalf(%);
```

$$-.5542721424 - 3.005133026\, I$$

I verified this for all four quadrants and all agree. Notice that this z value has |z| < 1 which is not in the range of the series shown, but we do get the right answer. If I use $(z^2\text{-}1)^{m/2}$ in the definition of Q41 above, then Q41 gives the wrong answer for Re(z) < 0! This relates to Confusing Issue #1 above. So you cannot enter the thing "as it appears". You have to write out the separate factors.

Comparison between Bateman Q (43) and Maple's LegendreQ

This (43) has my interest since it converges for all imaginary z inside an hourglass shape. This one requires considerably more code entry, in the same file quoted above,

```
restart;
g1 := (n,m) ->GAMMA(1/2+n/2+m/2)/GAMMA(1+n/2-m/2);
```

$$g1 := (n, m) \to \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m\right)}{\Gamma\left(1+\frac{1}{2}n-\frac{1}{2}m\right)}$$

```
g2 := (n,m) ->GAMMA(1+n/2+m/2)/GAMMA(1/2+n/2-m/2);
```

$$g2 := (n, m) \to \frac{\Gamma\left(1+\frac{1}{2}n+\frac{1}{2}m\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}n-\frac{1}{2}m\right)}$$

```
> Q43_1z := (n,m,z)
  ->(Pi)^(1/2)*2^(m-1)*g1(n,m)*exp(-signum(Im(z))*I*Pi*(n+1/2))*(z-1)^(n/2)
  *(z+1)^(n/2)* hypergeom([-n/2-m/2, -n/2 +m/2],[1/2],z^2/(z^2-1));
>
```

$$Q43_1z := (n,m,z) \to \sqrt{\pi}\, 2^{(m-1)}\, g1(n,m)\, \mathbf{e}^{\left(-I\,\mathrm{signum}(\Im(z))\,\pi\left(n+\frac{1}{2}\right)\right)} (z-1)^{\left(\frac{1}{2}n\right)} (z+1)^{\left(\frac{1}{2}n\right)} \mathrm{hypergeom}\left(\left[-\frac{1}{2}n-\frac{1}{2}m, -\frac{1}{2}n+\frac{1}{2}m\right], \left[\frac{1}{2}\right], \frac{z^2}{z^2-1}\right)$$

```
> Q43_2z := (n,m,z) ->
  (Pi)^(1/2)*2^(m)*g2(n,m)*exp(-signum(Im(z))*I*Pi*(n-1/2))*z*(z-1)^(n/2-1/2)
  *(z+1)^(n/2-1/2)* hypergeom([ 1/2-n/2 +m/2, 1/2-n/2 -m/2],[3/2],z^2/(z^2-1));
```

$$Q43_2z := (n,m,z) \to \sqrt{\pi}\, 2^{m}\, g2(n,m)\, \mathbf{e}^{\left(-I\,\mathrm{signum}(\Im(z))\,\pi\left(n-\frac{1}{2}\right)\right)} z\,(z-1)^{\left(\frac{1}{2}n-\frac{1}{2}\right)} (z+1)^{\left(\frac{1}{2}n-\frac{1}{2}\right)} \mathrm{hypergeom}\left(\left[\frac{1}{2}-\frac{1}{2}n+\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m\right], \left[\frac{3}{2}\right], \frac{z^2}{z^2-1}\right)$$

```
> Q43z :=(n,m,z) -> exp(+I*Pi*m)*(Q43_1z(n,m,z) + Q43_2z(n,m,z));

n := 1.2;
```

$$n := 1.2$$

```
m := 2.3;
```

$$m := 2.3$$

```
z := -.4 -.2*I;
```

$$z := -.4 - .2\,I$$

```
Q43z(n,m,z):evalf(%);
```

$$-.5542721391 - 3.005133023\,I$$

```
LegendreQ(n,m,z):evalf(%);
```

$$-.5542721424 - 3.005133026\,I$$

Again, I check all four quadrants and it agrees in each. I have therefore verified my typo correction in Bateman (43) where he has 2μ which should be 2^{μ}. I have shown that he has no other errors in this formula (43), that is something I wanted to know.