The search for an asymmetric q PhL 2.21.10

Goal. Somehow I want to take a cut (-∞,+1) and rotate it down and to the right and thereby remove the cut in the region (-1,1) and by so doing, try to find a q function which solves the Legendre equation, but which is uncut in this region, and therefore generates functions like $Q_1(i\zeta)_{0R} = \zeta \cot^{-1}(\zeta) - 1$ which are asymmetric in ζ.

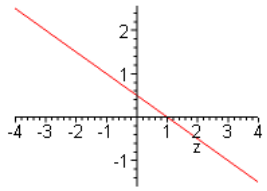
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Cut Structure for Q(32)

Let's start with Q (32). If has this form

$$Q = A\,(z-1)^{\mu/2}\;(z+1)^{-\mu/2}\;F(a,b,c,[1-z]/2) + B\,(z-1)^{-\mu/2}\;(z+1)^{\mu/2}\;F(a,b,c',[1-z]/2)$$

What is the cut structure of this animal? To find for F, we plot its arg



and we see that for z = (-∞,-1) this makes arg = (1,∞), so we conclude that (-∞,-1) is its cut. The factor $(z+1)^{\mu/2}$ has a branch point at z = -1, so we ignore it because we are interested in z = +1. Each term has a branch point at z = +1, and it is these guys that we want to "rotate" somehow.

How to rotate a branch cut.

Suppose we have

$$f(z) = (z-1)^{\alpha}{}_{0L}$$

The idea of rotating the branch cut here was studied in some detail in "study of (1-z)^a and related.doc", Section 3. The function with the cut rotated is called $(1-z)^{\alpha}_{0R}$ and we found this relation:

$$(z-1)^{\alpha}_{0R} = (z-1)^{\alpha}_{0L} \qquad \mathrm{Im}(z) \geq 0$$
$$(z-1)^{\alpha}_{0R} = e^{+i\alpha 2\pi}\,(z-1)^{\alpha}_{0L} \qquad \mathrm{Im}(z) \leq 0$$

or

$$(z-1)^{\alpha}_{0R} = \left(\frac{1}{e^{+2\pi i\alpha}}\right)(z-1)^{\alpha}_{0L} \qquad \mathrm{Im}(z) \gtrless 0$$

or

$$(z-1)^{\alpha}_{0R} = e^{i\alpha\pi}\,e^{\mp\, i\alpha\pi}\,(z-1)^{\alpha}_{0L} = e^{+i\pi[1\mp 1]\alpha}\,(z-1)^{\alpha}_{0L} \qquad \mathrm{Im}(z) \gtrless 0$$

Inverting we get

$$(z-1)^{\alpha}_{0L} = e^{-i\pi[1\mp 1]\alpha}\,(z-1)^{\alpha}_{0R}$$

Apply this rotation to our Q (32) formula.

From above we copy down and add the 0L notations

$$Q_{0L} = A\,(z-1)^{\mu/2}_{0L}\,(z+1)^{-\mu/2}_{0L}\,F(a,b,c,[1-z]/2) + B\,(z-1)^{-\mu/2}_{0L}\,(z+1)^{\mu/2}_{0L}\,F(a,b,c',[1-z]/2)$$

Remember that the F cut is $(-\infty,-1)$ which we leave put. We also leave put the z+1 factors. So

$$(z-1)^{\mu/2}_{0L} = e^{-i\pi\mu/2}\,e^{\pm i\pi\mu/2}\,(z-1)^{\mu/2}_{0R} = e^{-i\pi[1\mp 1]\mu/2}\,(z-1)^{\mu/2}_{0R}$$

$$(z-1)^{-\mu/2}_{0L} = e^{+i\pi\mu/2}\,e^{\mp\, i\pi\mu/2}\,(z-1)^{-\mu/2}_{0R} = e^{+i\pi[1\mp 1]\mu/2}\,(z-1)^{-\mu/2}_{0R}$$

We install these to get

$$Q_{0R} = A\,(z-1)^{\mu/2}_{0L}\,(z+1)^{-\mu/2}_{0L}\,F(a,b,c,[1-z]/2) + B\,(z-1)^{-\mu/2}_{0L}\,(z+1)^{\mu/2}_{0L}\,F(a,b,c',[1-z]/2)$$

$$= A\,e^{-i\pi\mu/2[1\mp 1]}\,(z-1)^{\mu/2}_{0R}\,(z+1)^{-\mu/2}_{0L}\,F(a,b,c,[1-z]/2) + B\,e^{+i\pi\mu/2[1\mp 1]}\,(z-1)^{-\mu/2}_{0R}\,(z+1)^{\mu/2}_{0L}\,F(a,b,c',[1-z]/2)$$

I need to maintain the 0R subscript because this is not the normal function Maple uses! But we can drop the 0L subscript since that is a standard function. Let's now write the above out in full for (32)

$$e^{-i\pi\mu}\,Q_{\nu}^{\mu}(z)_{0R} = (1/2)f(\nu,\mu)\Gamma(-\mu)\,e^{-i\pi\mu/2[1\mp 1]}\,(z-1)^{\mu/2}_{0R}\,(z+1)^{-\mu/2}\,F(-\nu,1+\nu,1+\mu,[1-z]/2) + (1/2)\,\Gamma(+\mu)\,e^{+i\pi\mu/2[1\mp 1]}\,(z-1)^{-\mu/2}_{0R}\,(z+1)^{\mu/2}\,F(-\nu,1+\nu,1-\mu,[1-z]/2)$$

$$e^{-i\pi m} Q_n^m(z)_{0R} = (1/2) f(n,m) \Gamma(-m)\, e^{-i\pi m/2[1\mp 1]}\, (z-1)^{m/2}{}_{0R}\ \ (z+1)^{-m/2}\, F(-n,1+n,1+m,[1-z]/2)$$
$$+ \ (1/2)\ \ \Gamma(+m)\, e^{+i\pi m/2[1\mp 1]}\, (z-1)^{-m/2}{}_{0R}\ \ (z+1)^{m/2}\ \ F(-n,1+n,1-m,[1-z]/2)$$

Comments on the new function Q_{0R}

(a) Since we have merely rotated a cut by 180 degrees, this function still solves the Legendre ODE.

(b) the angle arg(z-1) lies in the range (0,2π), whereas angle of arg(z+1) is the usual (-π,π).

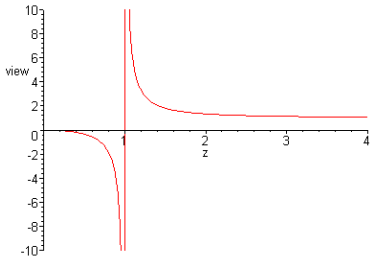
(c) this formula is no good for μ = 0 (or any integer), so we cannot test it for $Q_1(z)_{0R}$.

Cut Structure for Q(43)

Q (43) has this form: (see "zeros and poles..." for g_i)

$$e^{-i\pi m} Q_n^m(z) = \sqrt{\pi}\, 2^{m-1}\, g_1(n,m)\, e^{\mp i\pi(n+1/2)}\ \ (z+1)^{n/2}\ \ (z-1)^{n/2}\ \ F(a,b,c;\, z^2/(z^2-1))$$
$$+ \sqrt{\pi}\, 2^{m}\ \ g_2(n,m)\, e^{\mp i\pi(n-1/2)}\ \ z\ (z+1)^{(n-1)/2}\ \ (z-1)^{(n-1)/2}\ F(a',b',c';\, z^2/(z^2-1))$$

As z runs ∞ to 1+ε, the argument in the F function runs 1+ε to ∞. That is the right side red curve below in this plot of the arg versus z.



Therefore, the F function part of the functions has a cut (+1,+∞) and (-∞,-1).

We can now treat this equation just as we did Q (32) above, using the following rule

$$(z-1)^{\alpha}{}_{0L} = e^{-i\pi[1\mp 1]\alpha}\ (z-1)^{\alpha}{}_{0R}$$

=>

$$(z-1)^{n/2}{}_{0L} = e^{-i\pi[1\mp 1]n/2}\ (z-1)^{n/2}{}_{0R}$$

$$(z-1)^{(n-1)/2}{}_{0L} = e^{-i\pi[1\mp 1](n-1)/2}\ (z-1)^{(n-1)/2}{}_{0R}$$

Our result is then:

$$e^{-i\pi m} Q_n^m(z) =$$

$$\sqrt{\pi}\, 2^{m-1}\, g_1(n,m)\, e^{\mp i\pi(n+1/2)} \quad (z+1)^{n/2} \quad e^{-i\pi[1\mp 1]n/2} \quad (z-1)^{n/2}{}_{0R} \quad F(a,b,c;\, z^2/(z^2-1))$$
$$+\ \sqrt{\pi}\, 2^{m} \quad g_2(n,m)\, e^{\mp i\pi(n-1/2)}\, z\, (z+1)^{(n-1)/2} \quad e^{-i\pi[1\mp 1](n-1)/2}\, (z-1)^{(n-1)/2}{}_{0R}\, F(a',b',c';\, z^2/(z^2-1))$$

Now let's try some phase combining:

first term:

$$\mp\, i\pi(n+1/2) - i\pi[1\mp 1]n/2 \;=\; \mp\, i\pi(2n+1)/2 - i\pi[1\mp 1]n/2 \;= i\pi/2\{\mp(2n+1) - 1 \pm n\} = i\pi/2\{\mp(n+1) - 1\}$$

second term:

$$\mp\, i\pi(n-1/2) - i\pi[1\mp 1](n-1)/2 = \mp\, i\pi(2n-1)/2 - i\pi[1\mp 1](n-1)/2 = i\pi/2\ \{\ \mp(2n-1) - [1\mp 1](n-1)\ \}$$

$$= i\pi/2\ \{\ \mp(2n-1) - (n-1) \pm (n-1)\ \} = i\pi/2\ \{\ \mp(2n) - (n-1) \pm (n)\ \} = i\pi/2\ \{\ \mp(n) - (n-1)\}$$

So summarize these two results as

first term phase $= i\pi/2\{\mp(n+1) - 1\} \qquad = i\pi/2\{\mp n \mp 1 - 1\}$

second term phase $= i\pi/2\{\ \mp(n) - (n-1)\} \qquad = i\pi/2\{\mp n - n + 1)\}$

Our result is then:

$$e^{-i\pi m} Q_n^m(z)_{0R} =$$

$$\sqrt{\pi}\, 2^{m-1}\, g_1(n,m)\, e^{i\pi/2\{\mp n \mp 1 - 1\}} \quad (z+1)^{n/2} \quad (z-1)^{n/2}{}_{0R} \quad F(a,b,c;\, z^2/(z^2-1))$$
$$+\ \sqrt{\pi}\, 2^{m} \quad g_2(n,m)\, e^{i\pi/2\{\mp n - n + 1)} \quad z\, (z+1)^{(n-1)/2}\, (z-1)^{(n-1)/2}{}_{0R}\, F(a',b',c';\, z^2/(z^2-1))$$

I was hoping to get something real here, but it is not working right! We still have a cut in (-1,1) which I am trying to get rid of.

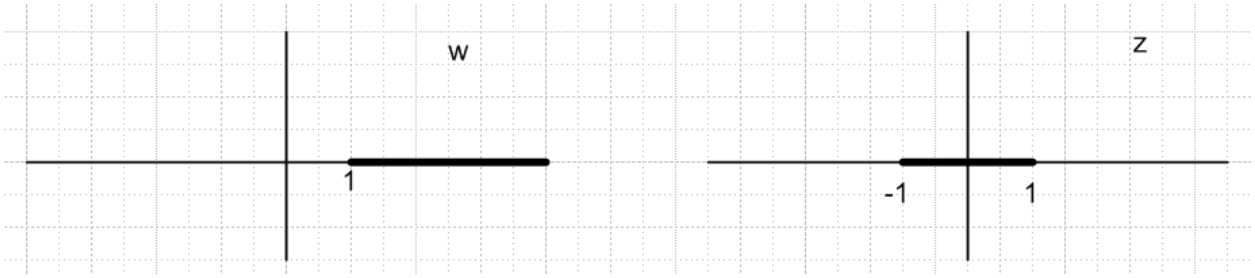
Comments

I have been looking for a Bateman form which can replicate my $Q_1(i\zeta)_{0R} = \zeta \cot^{-1}(\zeta) - 1$ asymmetric function. But when I came up with this expression starting with $Q_1(z) = (z/2)\ln[(z+1)/(z-1)] - 1$, the branch cut I "rotated" was a log one, not a power one. None of the Bateman forms "exposes" this log branch point, it is "hidden inside the F function" in all the forms.

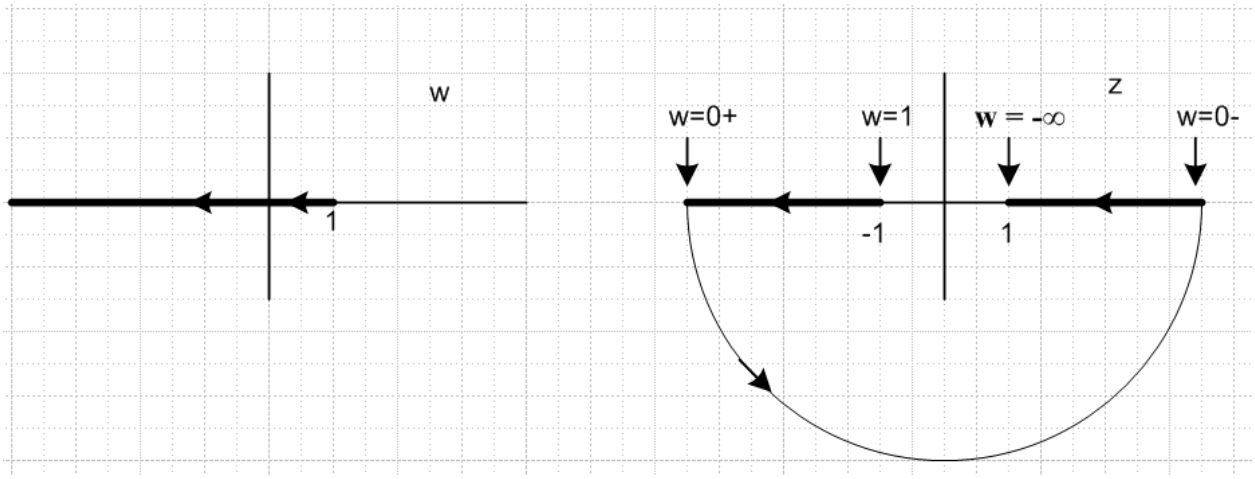
So I think this says that I can NEVER arrive at a form like $\zeta \cot^{-1}(\zeta) - 1$ by trying to do cut rotations in the Bateman forms. Somehow we have to rotate a cut inside the F function.

Cut Structure of Q (37)

The F functions both have argument $w = 2/(1-z)$ and therefore $z = 1 - 2/w$. Here are some pictures showing the cut of F(a,b,c,w) in w space and in z space. Here is the usual starting picture:



Now if we could rotate the w-space cut to the left, here is what happens:



So this would then clean up the (-1,1) region in Q(37).

How do you rotate the cut of F(a,b,c,z) ?

Consider the integral representation p 114 (2)

$F(a,b,c,z)_{0R} = K \int dt\ t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a}{}_{0R}$ $\qquad (1-zt) = t\ (1/t - z)$

$= K \int dt\ t^{b-1} (1-t)^{c-b-1} t^{-a}(1/t-z)^{-a}{}_{0R}$

Now use the rule which says:

$(z-1)^{\alpha}{}_{0L} = e^{\pm i\pi\alpha} (1-z)^{\alpha}{}_{0R}$

$(z-1/t)^{\alpha}{}_{0L} = e^{\pm i\pi\alpha} (1/t-z)^{\alpha}{}_{0R} \qquad \Rightarrow \qquad (1/t-z)^{\alpha}{}_{0R} = e^{\mp i\pi\alpha}(z-1/t)^{\alpha}{}_{0L}$

$(1/t-z)^{-a}{}_{0R} = e^{\pm i\pi a}(z-1/t)^{-a}{}_{0L}$

Then we have

$$F(a,b,c,z)_{0R} = K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,t^{-a}(1/t-z)^{-a}{}_{0R}$$

$$= K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,t^{-a}\,e^{\pm i\pi a}(z-1/t)^{-a}{}_{0L}$$

$$= e^{\pm i\pi a}\,K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,t^{-a}\,(z-1/t)^{-a}{}_{0L}$$

$$= e^{\pm i\pi a}\,K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,t^{-a}\,(z-1/t)^{-a}{}_{0L} \qquad t\,(z-1/t) = (tz-1) \qquad (*)$$

$$= e^{\pm i\pi a}\,K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,(zt-1)^{-a}{}_{0L}$$

$$= e^{\pm i\pi a}\;K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,(zt-1)^{-a}$$

We might then define this integral as $F(a,b,c,z)_{0L}$ and from the (*) form you see that it has a branch cut going from +1 to the left, so $(-\infty,1)$, which is what I was after:

$$F(a,b,c,z)_{0L} \equiv \quad K \int dt\, t^{b-1}\,(1-t)^{c-b-1}\,(zt-1)^{-a}$$

$$= \; e^{\mp i\pi a}\,F(a,b,c,z)_{0R}$$

In retrospect, this seems pretty simple. We are just using this phase to flip remove the cut from the right and add it to the left. The question was to find out what phase does the trick. Since $F(a,b,c,z)_{0R}$ is symmetric in a and b, I guess $F(a,b,c,z)_{0L}$ no longer has this symmetry.

Apply to Q (37)

We showed above that for w = 2/(1-z) we could clear the (-1,1) region in z space if we could take the F cut off to the left. This is Bateman (37) for Q, so write that as

$$e^{-i\pi\mu}\,Q_\nu{}^\mu(z)_{0R} = 2^\nu\,[\,\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)/\Gamma(2+2\nu)]\,)\,(z+1)^{\mu/2}\,(z-1)^{-\nu-1-\mu/2}F(1+\nu+\mu,1+\nu,2+2\nu,\,2/(1-z)\,)_{0R}$$

$$= K\,(z+1)^{\mu/2}\,(z-1)^{-\nu-1-\mu/2}F(1+\nu+\mu,1+\nu,2+2\nu,\,2/(1-z)\,)_{0R}$$

Now if we want to "clear out (-1,1)" we have to do two things. First we have to rotate the power cut

$$(z-1)^\alpha{}_{0R} = = \; e^{+i\pi[1\mp1]\alpha}\;(z-1)^\alpha{}_{0L} \qquad \mathrm{Im}(z) \gtrless 0$$

$$(z\text{-}1)^{-\nu-1-\mu/2}{}_{0R} = e^{+i\pi[1\mp1](-\nu-1-\mu/2)}\ (z\text{-}1)^{-\nu-1-\mu/2}{}_{0L} \qquad \mathrm{Im}(z) \gtrless 0$$

$$(z\text{-}1)^{-\nu-1-\mu/2}{}_{0L} = e^{+i\pi[1\mp1](\nu+1+\mu/2)}\ (z\text{-}1)^{-\nu-1-\mu/2}{}_{0R}$$

Second, we have to flip the F function cut:

$$F(1+\nu+\mu,1+\nu,2+2\nu,\ 2/(1\text{-}z))_{0R} = e^{\pm i\pi(1+\nu+\mu)}\ F(1+\nu+\mu,1+\nu,2+2\nu,\ 2/(1\text{-}z))_{0L}$$

Our result is then:

$$= K\,(z+1)^{\mu/2}\,(z\text{-}1)^{-\nu-1-\mu/2}{}_{0R}\,F(1+\nu+\mu,1+\nu,2+2\nu,\ 2/(1\text{-}z))_{0L}\ \ e^{+i\pi[1\mp1](\nu+1+\mu/2)}\ e^{\pm i\pi(1+\nu+\mu)}$$

Now examine the phase:

$$e^{+i\pi[1\mp1](\nu+1+\mu/2)}\ e^{\pm i\pi(1+\nu+\mu)} = e^{i\pi(\nu+1+\mu/2)}\ e^{i\pi\ [\mp(\nu+1+\mu/2)\ \pm(1+\nu+\mu)}$$

$$= e^{i\pi(\nu+1+\mu/2)}\ e^{i\pi\ [\mp(\mu/2)\ \pm(\mu)]} = e^{i\pi(\nu+1+\mu/2)}\ e^{\pm i\pi\mu/2)}$$

I was hoping to get no ± phase so I could claim we are clear on (-1,1). But taking as it comes, we have

$$e^{-i\pi\mu}\,Q_\nu^\mu(z)_{0R} = e^{i\pi(\nu+1+\mu/2)}\ e^{\pm i\pi\mu/2)}\ K\,(z+1)^{\mu/2}\,(z\text{-}1)^{-\nu-1-\mu/2}{}_{0R}\,F(1+\nu+\mu,1+\nu,2+2\nu,\ 2/(1\text{-}z))_{0L}$$

Let's just try this for $\mu = 0$ and $\nu = 1$

$$Q_1(z)_{0R} = e^{i\pi(\nu+1)}\ K(z\text{-}1)^{-\nu-1}\ F(2,2,4,\ 2/(1\text{-}z))_{0L}$$

This does seem to have no cut. And we can say:

$$F(2,2,4,\ 2/(1\text{-}z))_{0L} = F(2,2,4,\ 2/(1\text{-}z))_{0R}\ e^{\mp i\pi(1+\nu)} = F(2,2,4,\ 2/(1\text{-}z))_{0R}\ e^{\mp i\pi 2} = F(2,2,4,\ 2/(1\text{-}z))_{0R}$$

so we then have

$$Q_1(z)_{0R} = 2[\,\Gamma(2)\,\Gamma(2)/\Gamma(4)]\,(z\text{-}1)^{-2}\,F(2,2,4,\ 2/(1\text{-}z))_{0R}$$

$$= (1/3)\,(z\text{-}1)^{-2}\,F(2,2,4,\ 2/(1\text{-}z))$$

Maple tells us that:

```
restart;
f := (1/3)*(z-1)^(-2)*hypergeom([2,2],[4],2/(1-z));
```

$$f := \frac{1}{3}\frac{\text{hypergeom}\left([2,2],[4],2\frac{1}{1-z}\right)}{(z-1)^2}$$

```
g := convert(f,StandardFunctions);
```

$$g := \frac{1}{3}\frac{-3(1-z)^2+\frac{1}{8}\left(-12+12\frac{1}{1-z}\right)(1-z)^3\ln\left(1-2\frac{1}{1-z}\right)}{(z-1)^2}$$

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simplify(g);
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$$-1+\frac{1}{2}z\ln\left(\frac{z+1}{z-1}\right)$$

Time To Review

1. I wanted to solve the Green's function for an oblate hyperboloid.

2. I constructed a Smythian solution using P(iζ) and Q(iζ) and got a solution, but the solution was complex because these functions are complex.

3. I looked for two independent real solutions of the flipped Legendre equation in ζ. I found some and called them p(ζ) and q(ζ). Both were symmetrical in ζ which I was not happy about. I felt I could not construct an asymmetric solution for the bloid using symmetric functions. The solution has to be asymmetric since the Green's charge is one the upper half and not on the bottom half.

4. I then kept looking for some more general solutions that had asymmetric properties like $Q_1(i\zeta)_{OR} = \zeta\cot^{-1}\zeta - 1$. I thought I might find some by "rotating cuts" to the right to "clear out" the (-1,1) region. I then realized that you have to somehow rotate cuts of powers and in addition rotate cuts of F functions. Despite a lot of effort, I came up with nothing new. By considering Q (43), I merely reproduced my symmetric q function. So having done due diligence, I think I will now "give up" on trying to "find" symmetric q-like functions hidden in all those Bateman forms.

It is not clear to me how I would have used an asymmetric q function, since it would blow up at least in one direction.