Why are p and q symmetric ? PhL 2.22.10

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Appendix A: Power Series Solutions for regular Legendre **11**

Summary of this Appendix: 11

First, here is the ODE background (taken from reality of P and Q)

"Background and the atomic forms.

In oblate spheroidal coordinates, there are as usual three separated ODEs. Using Smythe notation and ordering, the coordinates are ξ,ζ,φ where ζ is the "radial" coordinate. The ODE for this coordinate is the Legendre ODE but with imaginary variable. In other words, here is the normal Legendre ODE:

$L_z = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2)$ // modified from Leg prop doc

$L_z u(z) = n(n+1)u(z)$

On the second line we have written "the eigenvalue problem" where $\lambda = n(n+1)$. The independent solutions of this equation can be taken as the usual $P_n^m(z)$ and $Q_n^m(z)$ Legendre functions. Now, the oblate ζ ODE is obtained by taking $z = i\zeta$ and then we have ("the flipped equation")

$L_z = L_\zeta = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + m^2/(1+\zeta^2)$

$L_\zeta v(\zeta) = n(n+1)v(\zeta)$

The claim is that the solutions to the latter eigenvalue problem can be taken as $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$. Here is why:

$$L_\zeta\, P_n^m(i\zeta) = L_z\, P_n^m(z) = n(n+1)\, P_n^m(z) = n(n+1)\, P_n^m(i\zeta)$$

So for the ζ dimension of an oblate PDE, we can take as our "atoms" the functions $P_n^m(i\zeta)$ and $Q_n^m(i\zeta)$"

Comment. The downside of using these functions is that they are in general complex-valued, so you can get into trouble trying to do make Smythian form solutions to electrostatic problems. The p and q functions are also solutions but are real, and they work well. These p and q are developed below, but also appeared in the "P and Q doc" where I discovered them for the first time.

Formulation of the Question.

For normal Legendre solutions P and Q, we have these rules

$$P_\nu^\mu(-z) = P_\nu^\mu(z)\, e^{\mp i\pi\nu} \qquad -(2/\pi)\sin\pi(\nu+\mu)\, Q_\nu^\mu(z)\, e^{-i\pi\mu}$$
$$Q_\nu^\mu(-z) = -Q_\nu^\mu(z)\, e^{\pm i\pi\nu}$$

So taken say on the interval (-1,1), neither of these functions is in general symmetric. Just to make the point:

```
LegendreP(1.2,2.3,0.4);
                                  -1.390113616 - .7082982652 I
LegendreP(1.2,2.3,-0.4);
                                  -1.139086236 - .5803934266 I
LegendreQ(1.2,2.3,0.4);
                                  -.6766801627 - 3.095265225 I
LegendreQ(1.2,2.3,-0.4);
                                  -1.271905500 - 2.901864789 I
```

So when you think of P(z) on the real axis, you don't have P being symmetric and the same for Q.

***** So why is it that when I construct real functions p(ζ) and q(ζ), both are symmetric?

Comment: Looking at the ODE

$$L_z = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2) \qquad \text{// modified from Leg prop doc}$$
$$L_z u(z) = n(n+1)u(z)$$

We can see that if u(z) is a solution, so is u(-z). But that does not imply u(z) = u(-z). If u(z) is a power series around z = 0, there will in general be even and odd terms, so it will have mixed symmetry. Those coefficients will be real.

So now look at the flipped ODE

$$L_z = L_\zeta = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + m^2/(1+\zeta^2)$$
$$L_\zeta v(\zeta) = n(n+1)v(\zeta)$$

It does seem again that if you made power series solutions, they would have even and odd terms. This is very mysterious.

Trying to construct Even and Odd (in z) functions from the Q function.

Start with Q (40) in z^2, appropriate for pondering symmetry near z = 0, and we find:

$$e^{-i\pi m} Q_n^m(z) = (z^2-1)^{-m/2}\{ (\pm i)^{m-n-1} A' F'(z^2) + (\pm i)^{m-n} B' z G'(z^2)\}$$

where we know the correct meaning of the outside function. We can use Bateman p 123 (12) to correctly flip around the outside factor:

$$(z^2-1) = (1-z^2)e^{\pm i\pi} = (1-z^2)(\pm i)^2$$

$$(z^2-1)^{-m/2} = (1-z^2)^{-m/2} (\pm i)^{2(-m/2)} = (1-z^2)^{-m/2} (\pm i)^{-m}$$

to get this form for the above

$$e^{-i\pi m} Q_n^m(z) = (1-z^2)^{-m/2} (\pm i)^{-m} \{ (\pm i)^{m-n-1} A' F'(z^2) + (\pm i)^{m-n} B' z G'(z^2)\}$$

$$= (1-z^2)^{-m/2}\{ (\pm i)^{-n-1} A' F'(z^2) + (\pm i)^{-n} B' z G'(z^2)\}$$

$$= (1-z^2)^{-m/2}(\pm i)^{-n} \{ (\pm i)^{-1} A' F'(z^2) + B' z G'(z^2)\}$$

$$= (1-z^2)^{-m/2}(\pm i)^{-n} \{ \mp i A' F'(z^2) + B' z G'(z^2)\} \qquad (*)$$

Now we can write this in z, and then again in -z, and the outside factor being $(1+z)^{-m/2}$ $(1-z)^{-m/2}$ is an *unambiguously* even function (whereas this is not clear for $(z^2-1)^{-m/2}$). Thus:

$$e^{-i\pi m} Q_n^m(z) = (1-z^2)^{-m/2}(\pm i)^{-n} \{ \mp i A' F'(z^2) + B' z G'(z^2)\}$$
$$e^{-i\pi m} Q_n^m(-z) = (1-z^2)^{-m/2}(\mp i)^{-n} \{ \pm i A' F'(z^2) - B' z G'(z^2)\}$$

and I take a - sign out of the second factor to get

$$e^{-i\pi m} Q_n^m(z) = (1-z^2)^{-m/2}(\pm i)^{-n}\{ \mp i A' F'(z^2) + B' z G'(z^2)\}$$
$$e^{-i\pi m} Q_n^m(-z) = -(1-z^2)^{-m/2}(\mp i)^{-n} \{ \mp i A' F'(z^2) + B' z G'(z^2)\}$$

This shows that we cannot "isolate" either of the two terms in the {} bracket by doing a linear combination of the above two functions. In fact we can take the ratio to get

$$Q_n^m(z)/ Q_n^m(-z) = (\pm i)^{-n}/ [-^2(\mp i)^{-n}] = - (\pm i)^{-2n} = - e^{\mp i\pi n}$$

which says that $Q_n^m(-z) = - e^{\pm i\pi n} Q_n^m(z)$ which is exactly the well-known property of Q.

So all we have shown at this point is that you cannot isolate either of A' or B' terms by doing linear combinations of Q(z) and Q(-z). But consider this function with some unknown phase function δ(z) :

$$f_n^m(z) \equiv e^{i\delta(z)} Q_n^m(z)$$
$$f_n^m(-z) \equiv e^{i\delta(-z)} Q_n^m(-z) = - e^{\pm i\pi n} Q_n^m(z) e^{i\delta(-z)} = - e^{\pm i\pi n} Q_n^m(z) e^{i\delta(z)} e^{i2\delta(-z)}$$

$$= -\, e^{\pm i n \pi}\, e^{i 2\delta(-z)}\, f_n^m(z)$$

We could cause f to be an even function in z if we could find a phase function such that

$$e^{\pm i n \pi}\, e^{i 2\delta(-z)} = -1 \qquad \Rightarrow \qquad e^{i 2\delta(-z)} = -\, e^{\mp i n \pi} = (\mp i)^2\, e^{\mp i n \pi} = (\mp i)^2\, (\mp i)^{2n} = (\mp i)^{2n+2}$$

$$\Rightarrow\ e^{i\delta(-z)} = (\mp i)^{n+1} \qquad \Rightarrow \qquad e^{i\delta(z)} = (\pm i)^{n+1}$$

Therefore, the following function will be even in z:

$$f_n^m(z) = (\pm i)^{n+1}\, Q_n^m(z)$$

which we can write out using the above:

$$f_n^m(z) = (\pm i)^{n+1}\, e^{+i n \pi}\, (1-z^2)^{-m/2} (\pm i)^{-n} \{ \mp i\ \ A'\, F'(z^2) + B'\, z\, G'(z^2) \}$$

$$= (\pm i)^{1}\, e^{+i n \pi}\, (1-z^2)^{-m/2} \{ \mp i\ \ A'\, F'(z^2) + B'\, z\, G'(z^2) \}$$

$$= e^{+i n \pi}\, (1-z^2)^{-m/2} \{ A'\, F'(z^2) \pm i\, B'\, z\, G'(z^2) \}$$

We can now explicitly demonstrate that this function is even in z:

$$f_n^m(z) = e^{+i n \pi}\, (1-z^2)^{-m/2} \{ A'\, F'(z^2) \pm i\, B' \quad z\, G'(z^2) \}$$
$$f_n^m(-z) = e^{+i n \pi}\, (1-z^2)^{-m/2} \{ A'\, F'(z^2) \mp i\, B'\, (-z)\, G'(z^2) \}$$

Now having done this, let's define $g_n^m(z) \equiv f_n^m(z)\, e^{-i n \pi}$ so we then have a new even function g:

$$g_n^m(z) = (1-z^2)^{-m/2} \{ A'\, F'(z^2) \pm i\, B' \quad z\ \ G'(z^2) \} \qquad g_n^m(z) = e^{-i n \pi}\ \ (\pm i)^{n+1}\, Q_n^m(z)$$
$$g_n^m(-z) = (1-z^2)^{-m/2} \{ A'\, F'(z^2) \mp i\, B'\, (-z)\, G'(z^2) \}$$

If we choose z = real, the function $g_n^m(z)$ is complex and cannot be "rephased" to be real. The complexness is intrinsic in the bracket factor {...}. But suppose we set $z = i\zeta$ with ζ real. Then the $\pm$ sign refers to the sign of ζ instead of the sign of Im(z) and we have

$$g_n^m(i\zeta) = (1+\zeta^2)^{-m/2} \{ A'\, F'(-\zeta^2) \pm i\, B' \quad i\zeta\ \ G'(-\zeta^2) \}$$
$$g_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \{ A'\, F'(-\zeta^2) \mp i\, B'\, (-i\zeta)\, G'(-\zeta^2) \}$$

or

$$g_n^m(i\zeta) = (1+\zeta^2)^{-m/2} \{ A'\, F'(-\zeta^2) \mp B'\ \ \zeta\ \ G'(-\zeta^2) \}$$
$$g_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \{ A'\, F'(-\zeta^2) \mp\ \ B'\, \zeta\, G'(-\zeta^2)\ \}$$

Now as a function of ζ, our g function is <u>both even and real</u>! So let's define

$$h_n^m(\zeta) \equiv g_n^m(i\zeta) = f_n^m(i\zeta)\, e^{-i\pi m} = (\pm i)^{n+1} Q_n^m(i\zeta)\, e^{-i\pi m} = (\pm i)^{n+1} Q_n^m(i\zeta)\, (-i)^{2m}$$

and we are very happy so see that this exactly aligns with our function from the P and Q doc:

$$q_n^m(\zeta) \equiv (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta) \qquad \mathrm{Re}(\zeta) > 0 \text{ upper sign, } \mathrm{Re}(\zeta) < 0 \text{ lower sign}$$

It is pretty trivial to now construct a function which is odd in z, and which is real and odd in ζ :

$$gg_n^m(z) \equiv \pm k\, g_n^m(z) \qquad k =$$

$$qq_n^m(\zeta) \equiv \pm k\, q_n^m(\zeta)$$

but this is just a technical point, there is no significant difference between the even and odd functions, they are not independent for example. They are the same up to a phase.

Summary: We notice that the two terms in Q(40) ,

$$e^{-i\pi m} Q_n^m(z) = (1-z^2)^{-m/2} (\pm i)^{-n} \{ \mp i\ A' F'(z^2) + B' z G'(z^2)\}$$

appear at first glance to consist of an even A' and an odd B' term. But when the $\mp i$ is considered, we realize that in fact the bracket {...} as a whole is really purely odd in z;

$$b(z) = \{ \mp i\ A' F'(z^2) + B' z G'(z^2)\}$$
$$b(-z) = \{ \pm i\ A' F'(z^2) - B' z G'(z^2)\} = -\{ \mp i\ A' F'(z^2) + B' z G'(z^2)\} = - b(z)$$

Therefore the function

$$e^{-i\pi m} Q_n^m(z) (\pm i)^n = (1-z^2)^{-m/2}\, b(z)$$

is odd in z. This function may be written in terms of the even function g defined above,

$$= \mp i\, (-i)^{2m} Q_n^m(z) (\pm i)^{n+1} = \mp i\, g_n^m(z)$$

and this really explains everything we have found in this section.

Another way to look at it. The functions Q(z) and Q(-z) are not linearly independent, due to the famous relation between them. So there is no way you can linearly combine them to get two independent functions. We constructed $g_n^m(z)$ as an (in general complex) linear combination of Q(z) and Q(-z) which is designed to be even in z. This is "all there is", just a rephasing of Q. There is no independent $G_n^m(z)$ which is odd in z.

Our original question was this:

"So <u>why is it</u> that when I construct real functions $p(\zeta)$ and $q(\zeta)$, both are symmetric? "

We have thus answered the question regarding q(ζ). We could have rephased it to be antisymmetric. The answer is that "this is the nature of Q". Phasing the function to be real is not the cause of the q(ζ) being symmetric.

My history on this subject. When I first did this study, I thought I was somehow isolating just the A' term in {...} and that was causing my $q_n^m(\zeta)$ function to be even in ζ. That made me think there must be some interesting and different other function that would be odd in ζ and which would be associated with the B' term. In this regard, I gained a great interest in seeing whether the A' and B' terms in Q (40) are separately solutions of the Legendre equation. Although I think they are (see " A Maple study of Bateman Q (40).doc") , the jury is still out. But whether or not true, the above work is unaffected.

Preview. In the above section, we did not produce any definite-parity Q-generated functions that were real for real z. We will do that in a section below!

Trying to construct Even and Odd (in z) functions from the P function.

The table entry for P is (22) and this entry has exactly the same "structure" as Q (40) except for one very important fact: there are no unequal ± phase factors as there is in Q (40). This has big implications. We start off as above for Q, but we now have

$$P_n^m(z) = (z^2-1)^{-m/2}\{ A' F'(z^2) + B' z G'(z^2)\}$$

$$= (1-z^2)^{-m/2} (\pm i)^{-m} \{ A' F'(z^2) + B' z G'(z^2)\}$$

or

$$(\pm i)^m P_n^m(z) = (1-z^2)^{-m/2}\{ A' F'(z^2) + B' z G'(z^2)\}$$
$$(\mp i)^m P_n^m(-z) = (1-z^2)^{-m/2}\{ A' F'(z^2) - B' z G'(z^2)\}$$

In this case we CAN isolate the A' and the B' terms! We then find that

$$f_{even}(z) = (\pm i)^m P_n^m(z) + (\mp i)^m P_n^m(-z) = (1-z^2)^{-m/2} 2A' F'(z^2) \quad = \text{even}$$
$$f_{odd}(z) = (\pm i)^m P_n^m(z) - (\mp i)^m P_n^m(-z) = (1-z^2)^{-m/2} B' z G'(z^2) \quad = \text{odd}$$

So in this case, we DO have two interesting functions which are independent of each other and have definite parity. This of course follows from the fact that P(z) and P(-z) are, in general, independent functions, as Bateman p 140 (10) reminds us where it says P(-z) = mixture of P(z) and Q(z). If we put z on the real axis, both functions above are real. If we set z = iζ, the first function is real and the second can be made real by multiplying by -i.

In my P and Q doc, I stumbled onto this function

$$p_n^m(\zeta) = [(\mp i)^{-m} P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

which is exactly $f_{even}(i\zeta)$ above. I never constructed the odd function.

Since our even and odd functions above are linear combinations of Legendre ODE solutions, there is no question but that the A' and B' terms as shown above separately solve the Legendre ODE. So for Bateman table entry (22) , we have our proof of this fact. Whether it is true for other table entries is, as I said above, still not determined though I think it is true.

Implication for the flipped Legendre equation in ζ.

If one treats this as a standalone real ODE in variable ζ, one could expend lots of effort, do the power series recursion relations, and come up with two independent and real solutions of the ODE. I am able to avoid doing all that work, because the following two functions fill that bill:

$$q_n^m(\zeta) \equiv (-i)^{2m} (\pm i)^{n+1} Q_n^m(i\zeta)$$

$$p_n^m(\zeta) \equiv [(\mp i)^{-m} P_n^m(i\zeta) + (\pm i)^{-m} P_n^m(-i\zeta)]$$

The first function is intrinsically even, and we "like it" because it has the decaying large ζ behavior of the Q(z) function which is so important in doing Smythian forms for electrostatic problems where the potential decays to 0 at infinity. This intrinsic evenness of q could be intrinsic oddness if we just added a ± sign to the definition.

The second function *happens* to be even because we selected the even linear combination as our second independent solution. We could have selected the very different odd function, $-if_{odd}(i\zeta)$, but we did not. Obviously we could write down two real independent solutions of our flipped Legendre equation which had no specific parity. One could do this by linearly combining the q and p function with the $if_{odd}(i\zeta)$ function, for example.

When you do a power-series solution $f = \Sigma_n a_n \zeta^2$ about the point ζ = 0, probably the recursion relations are spaced by 2, and you would naturally come up with solutions of even and odd parity. You would probably come up with the $f_{even}(i\zeta)$ and $if_{odd}(i\zeta)$ above. Then it would take you a while to construct the linear combination of these which has large ζ decay, although we know that MF would use "the trick" formula to construct the "second kind" solution.

So having selected the p and q as above, I did the work to derive various "properties" of these functions in my P and Q doc, and then I as able -- I think -- to use these functions to solve the problem of the Green's function of the oblate hyperboloid.

Trying to construct Even and Odd (in *real* z = x) functions from the Q function.

First, a little preliminary stuff:

Theorem 1 about ODE solutions.

Suppose we have some ODE Lu(z) = 0 with a solution u(z) = 0. We usually think of z as being on the principal sheet of u(z), but in fact u(z) is a solution of Lu(z) = 0 for z on ANY sheet. So imagine we have two functions $u(z)_1$ and $u(z)_2$ where the second function is the analytic continuation of the first onto a neighboring sheet. If we know that $Lu(z)_1 = 0$ then it will also be true that $Lu(z)_2 = 0$. This is so because the ODE does not know which sheet z is on! In general, we will have $u(z)_1 \neq u(z)_2$. As we recall from

ancient times, when you continue a function around a branch point, you tend to pick up a linear combination of the two independent solutions and this causes $u(z)_1 \neq u(z)_2$. More on this elsewhere.

Corollary

Imagine a function that is cut along the real axis somewhere, and we have $u(x)_1 = f(x+i\varepsilon)$ and $u(x)_2 = f(x-i\varepsilon)$ at least for x on the real axis. The difference of the two cut sides is that the points are on two adjacent sheets. Then if ODE $Lu(x)_1 = 0$, we know at once that ODE $Lu(x)_2 = 0$. So then we know that any linear combination of the form $g(x) = A\, f(x+i\varepsilon) + B\, f(x-i\varepsilon)$ is also a solution of the ODE *at the point x*.

Note added 2.24.10. I have firmed up the above claims in "complex ODEs.doc". Solid now.

And now onto the show:

In the "Q section" above, we found the following even function in z:

$$g_n^m(z) = (1-z^2)^{-m/2}\{ A' F'(z^2) \pm i B' \; z \; G'(z^2)\} \qquad g_n^m(z) = e^{-i\pi m} (\pm i)^{n+1} Q_n^m(z)$$

and the only "odd" partner is the same function with a ± prefix. This function is even for all z including real z, but it is not *real* for real z. We shall now construct from the Q function alone functions which are real for real z and have definite parity. I have no particular use for these functions, but since I did the work, I will put it into the record.

Our starting point is equation (*) from the Q section above,

$$e^{-i\pi m} Q_n^m(z) = (1-z^2)^{-m/2}(\pm i)^{-n} \{ \mp i \; A' F'(z^2) + B' z G'(z^2)\} \qquad (*)$$

which we write as the first line below for just above and below the cut. The second line is the same as the first but below and above the cut instead, so all the signs change since Im(z) changes (the second line is not the complex conjugation of the first line)

$$e^{-i\pi m} Q_n^m(x\pm i\varepsilon) = (1-x^2)^{-m/2}(\pm i)^{-n} \{ \mp i \; A' F'(x^2) + B' x G'(x^2)\}$$
$$e^{-i\pi m} Q_n^m(x\mp i\varepsilon) = (1-x^2)^{-m/2}(\mp i)^{-n} \{ \pm i \; A' F'(x^2) + B' x G'(x^2)\} \qquad (*)$$

We are now ready to create our even and odd functions. We try out:

$$(\pm i)^n e^{-i\pi m} Q_n^m(x\pm i\varepsilon) + (\mp i)^n e^{-i\pi m} Q_n^m(x\mp i\varepsilon)$$

$$= (1-x^2)^{-m/2} [\{\mp i \; A' F'(x^2) + B' x G'(x^2)\} + \{\pm i \; A' F'(x^2) + B' x G'(x^2) \}]$$

$$= (1-x^2)^{-m/2} \, 2 B' x G'(x^2) \qquad = \text{odd and real}$$

$$(\pm i)^n e^{-i\pi m} Q_n^m(x\pm i\varepsilon) - (\mp i)^n e^{-i\pi m} Q_n^m(x\mp i\varepsilon)$$

$= (1-x^2)^{-m/2}\,[\,\{\mp i\;\; A'\,F'(x^2)\;\; + B'\,x\,G'(x^2)\} - \{\pm i\;\; A'\,F'(x^2)\;\; + B'\,x\,G'(x^2)\}\,]$

$= (1-x^2)^{-m/2}\,\{(\mp i)\;\; 2A'\,F'(x^2)$ = even and imaginary

Since we would like to have a real function here, we do this:

$(\pm i)\,[\,(\pm i)^n\, e^{-i\pi m}\; Q_n^m(x\pm i\varepsilon)\; - (\mp i)^n\, e^{-i\pi m}\; Q_n^m(x\mp i\varepsilon)\,] = (1-x^2)^{-m/2} 2A'\,F'(x^2)$ //real even

Then here is our little summary: For $|x| < 1$, we construct:

$(\pm i)\,[\,(\pm i)^n\, e^{-i\pi m}\; Q_n^m(x\pm i\varepsilon)\; - (\mp i)^n\, e^{-i\pi m}\; Q_n^m(x\mp i\varepsilon)\,] = (1-x^2)^{-m/2}\;\; 2A'\,F'(x^2)$ //real even

$[(\pm i)^n\, e^{-i\pi m}\; Q_n^m(x\pm i\varepsilon)\; + (\mp i)^n\, e^{-i\pi m}\; Q_n^m(x\mp i\varepsilon)\,] = (1-x^2)^{-m/2}\; 2\;B'\,x\,G'(x^2)$ //real odd

According to our claim above (from "complex ODEs'doc"), both the LHS Q functions are solutions of the Legendre ODE at the same point x (just on different sheets). Therefore, the following two functions are solutions of the Legendre equation:

$q_n^m(x)_{even} = (1-x^2)^{-m/2}\;\; 2A'\,F'(x^2)$ m and n real, $|x| < 1$

$q_n^m(x)_{odd} = (1-x^2)^{-m/2}\;\; 2\;B'\,x\,G'(x^2)$

The functions are real (for real x) and have definite parity. I don't think these functions have any name.

An accidental proof. I think the above constitutes a proof of this claim: the two terms in Q (40) each separately are solutions of the Legendre ODE. We just showed it was true for real argument. It therefore has to be true for any complex argument z. The phases from flipping the leading factor around and the phases shown in Q (40) have no bearing on this conclusion. I reached the same conclusion with Maple in doc "a maple study of Bateman Q (40).doc".

Example of the above accidental proof:

Let's try to compute these even and odd functions for a simple case n = 1 and m = 0 :

$q_1^0(x)_{even} = 2\sqrt{\pi}\,(1/2)\,[\Gamma(1)/\Gamma(3/2)]\;F(-1/2,1,1/2;\,z^2) = 2\;F(-1/2,1,1/2;\,z^2)$

= 2 times function shown below:

```
f := hypergeom([-1/2,1],[1/2],z^2);
```

$$f := \text{hypergeom}\left(\left[1, \frac{-1}{2}\right], \left[\frac{1}{2}\right], z^2\right)$$

```
f1 := convert(f,StandardFunctions);
```

$$f1 := 1 - z\,\text{arctanh}(z)$$

```
f2 :=convert(f1,ln);
```

$$f2 := 1 - z\left(\frac{1}{2}\ln(z+1) - \frac{1}{2}\ln(1-z)\right)$$

Notice that this f2 function is manifestly even because the (...) quantity is odd. The odd function is

$$q_1{}^0(x)_{odd} = \; 2\,x\,\sqrt{\pi}\;\; F(0,3/2,3/2;\,x^2) \;= 2\sqrt{\pi}\;\; x \qquad\qquad\qquad // \; F = 1$$

We know that $P_1(x) = x$, so x is a good solution. And it is easy to show that the log term f2 is a solution:

```
n :=1; m := 0;
```

$$n := 1$$
$$m := 0$$

```
g := -(1-z^2)*diff(f2,z,z) + 2*z*diff(f2,z) + ( m^2/(1-z^2) - n*(n+1))*f2:simplify(%);
```

$$0$$

The Bateman on-the-cut Q function

Neither of my above functions is even close to the on-the-cut function of Bateman p 143 (2). Here is that Bateman function:

$$Q_{on_cut}(x) = \; [(-i)^m\, e^{-i\pi m}\; Q_n^m\,(x+i\varepsilon) \; + (+i)^m\, e^{-i\pi m}\; Q_n^m\,(x-i\varepsilon)\;]/2$$

We can evaluate this using our first of the pair above

$$e^{-i\pi m}\; Q_n^m\,(x\pm i\varepsilon) \;=\; (1-x^2)^{-m/2}(\pm i)^{-n}\;\{\mp i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$

so that

$$e^{-i\pi m}\; Q_n^m\,(x+i\varepsilon) \;=\; (1-x^2)^{-m/2}(+i)^{-n}\;\{-i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$
$$e^{-i\pi m}\; Q_n^m\,(x-i\varepsilon) \;=\; (1-x^2)^{-m/2}(-i)^{-n}\;\{+i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$

$$(-i)^m\, e^{-i\pi m}\; Q_n^m\,(x+i\varepsilon) \;=\; (-i)^m\,(1-x^2)^{-m/2}(+i)^{-n}\;\{-i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$
$$(+i)^m\, e^{-i\pi m}\; Q_n^m\,(x-i\varepsilon) \;=\; (+i)^m\,(1-x^2)^{-m/2}(-i)^{-n}\;\{+i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$

$$(-i)^m\, e^{-i\pi m}\; Q_n^m\,(x+i\varepsilon) \;=\; (1-x^2)^{-m/2}(+i)^{-n-m}\;\{-i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$
$$(+i)^m\, e^{-i\pi m}\; Q_n^m\,(x-i\varepsilon) \;=\; (1-x^2)^{-m/2}(-i)^{-n-m}\;\{+i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}$$

Therefore we find that

$$Q_{on_cut}(x) = \;\; = \;\; (1-x^2)^{-m/2}\;[(+i)^{-n-m}\{-i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\} \;\;\; + c.c\;]\,/2$$

$$= \; (1-x^2)^{-m/2}\;\mathrm{Re}\;[(+i)^{-n-m}\{-i\;\; A'\,F'(x^2) + B'\,x\,G'(x^2)\}\;]$$

So this function is definitely real for |x| < 1, but it is neither even nor odd. Since it is a sum of Q functions at the same point (but on different sheets), it is a solution of the Legendre equation. The motivation of this function is more easily seen using the Q (32) table entry and not the Q (40) we have used here, as I show in "legendre on the cut.doc".

Constructing Even and Odd Real versions of the Q function: Version 2

Although I am pretty convinced that *all* the two-term table entries for the P and Q function are each Legendre solutions, I have accidentally proven above that this is for sure true for the Q (40) two term entry. I also have support on this from " A Maple study of Bateman Q (40).doc". So this means that if you want even and odd real functions for expanding around z = 0, you just look at the Q (40) entry and you conclude that

$$F_n^m(z)_{even} = (z^2-1)^{-m/2}\ 2A'\ F'(z^2)$$
$$F_n^m(z)_{odd} = (z^2-1)^{-m/2}\ 2\ B'\ z\ G'(z^2)$$

are even and odd solutions of the Legendre equation for general complex z. For real |x|<1, we write these with a different phase as

$$q_n^m(x)_{even} = (1-x^2)^{-m/2}\ 2A'\ F'(x^2) \qquad \text{m and n real, } |x| < 1$$
$$q_n^m(x)_{odd} = (1-x^2)^{-m/2}\ 2\ B'\ x\ G'(x^2)$$

and then the functions have definite parity and are real.

The down side of these functions is that neither of them has the desirable asymptotic decay in z that our even q function found earlier does have.

Appendix A: Power Series Solutions for regular Legendre

Summary of this Appendix: This appendix is not very relevant, but I did the work so will leave it here. I show the Frobenius form for the Legendre ODE, exponents are ±m/2. But the commonly used solutions don't have the simple Frobenius form $(z-1)^{\pm m/2}\ \Sigma_{k=0}\ a_k(z-1)^k$ that you would expect. Instead they are

$$w_{\pm}(z) = P_n^{\pm m}(z) = (z+1)^{\pm m/2}\ (z-1)^{\mp m/2}\ F(-n,n+1;\ 1\mp m;\ [1-z]/2)$$

where we have the extra factor $(z+1)^{\pm m/2}$. I think this is a historical situation where the F function was there first, so when you transform the solutions $w_{\pm}(z) = (z+1)^{\pm m/2}\ W_{\pm}(z)$, the Legendre ODE for w must become the hypergeometric ODE for W which has those same exponents. Then the solution comes out as shown above. I started on a proof of this below, but decided not to pursue it.

The ODE here is in "standard Frobenius form" as stated, only if you are expanding around z = ± 1. I show details of this in " Frobenius Method Applied to the Associated Legendre Equation.doc". For example, at z = +1 we find that the Frobenius exponents are r = ±m/2 and we end up with power series solutions of this form

$$w_{\pm}(z) = (z-1)^{\pm m/2} \sum_{k=0} a_k (z-1)^k$$

which is valid as long as m ≠ integer so the difference in the exponents is not an integer. The Frobenius functions are these

$$(z-z_0)^2 u''(z) + (z-z_0)\, P(z) u'(z) + Q(z) u(z) \qquad \text{for } z_0 = 1$$

$$P(z) = \{2z/(1+z)\} \qquad Q(z) = \{ -\lambda(z-1)/(1+z) - m^2/(1+z)^2 \}$$

$$P(1) = 1 \qquad Q(1) = -m^2/4$$

To get the full power series solutions, you have to expand P and Q about z=1. I don't have any books which do all this detail (I checked M&M, Schiff, Saxon). Of course we know exactly what these series are without doing all this work. For m ≠ integer, we use Bateman p 124 (14) where the (1-z) series is expressed by the F function. So you can regard $w_{\pm}(z) = P_n^{\pm m}(z)$ from this formula.

$$w_{\pm}(z) = P_n^{\pm m}(z) = (z+1)^{\pm m/2} \; (z-1)^{\mp m/2} \, F(-n, n+1; 1 \mp m; [1-z]/2)$$

I have commented elsewhere (but cannot now find it) about that $(z+1)^{\pm m/2}$ factor: for the convergence disk of this F, we never hit the cut of $(z+1)^{\pm m/2}$ so it is an analytic multiplying function for this disk. In reality we have *transformed* our ODE into another one (never mentioned) where $w_{\pm}(z) = (z+1)^{\pm m/2} W_{\pm}(z)$, and then we have found a power series for $W_{\pm}(z)$. I don't think I have ever seen this stated.

Aside: Let's try it here for $w_+(z)$ only so don't have all those signs to mess with:

$$L_z = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2) \qquad \text{// modified from Leg prop doc}$$

$$L_z u(z) = n(n+1)u(z) = \lambda\, u(z)$$

$$-(1-z^2)u'' + 2zu' + [\, m^2(1-z^2)^{-1} - \lambda]\, u = 0$$

$$u = (z+1)^{m/2} U$$

$$u' = (z+1)^{m/2} U' + (m/2)\,(z+1)^{m/2-1}\, U$$

$$u'' = (z+1)^{m/2} U'' + 2\,(m/2)\,(z+1)^{m/2-1}\, U' + (m/2)(m/2-1)\,(z+1)^{m/2-2}\, U$$

Stuff these in and see what happens:

$$-(1-z^2)\{\,(z+1)^{m/2} U'' + 2\,(m/2)\,(z+1)^{m/2-1}\, U' + (m/2)(m/2-1)\,(z+1)^{m/2-2}\, U\,\} +$$
$$2z\{(z+1)^{m/2} U' + (m/2)\,(z+1)^{m/2-1}\, U\,\} +$$
$$[\, m^2(1-z^2)^{-1} - \lambda]\{\,(z+1)^{m/2} U\,\} = 0$$

$$-(1-z^2)\{\,(z+1)^{m/2} U'' + 2\,(m/2)\,(z+1)^{m/2-1}\, U' + (m/2)(m/2-1)\,(z+1)^{m/2-2}\, U\,\} +$$
$$2z\{(z+1)^{m/2} U' + (m/2)\,(z+1)^{m/2-1}\, U\,\} +$$
$$[\, m^2(1-z^2)^{-1} - \lambda]\{\,(z+1)^{m/2} U\,\} = 0$$

$$-(1-z^2)(z+1)^{m/2}U'' + [2\,(m/2)\,(z+1)^{m/2-1} + 2z(z+1)^{m/2}\,]\,U'$$
$$+ \{\,(m/2)(m/2-1)\,(z+1)^{m/2-2} + (m/2)\,(z+1)^{m/2-1} + (z+1)^{m/2}[\,m^2(1-z^2)^{-1} - \lambda]\,\}\,U = 0$$

$$-(z+1)^{m/2}\,(1-z^2)U'' + (z+1)^{m/2}\,[2\,(m/2)\,(z+1)^{-1} + 2z\,]\,U'$$
$$+ (z+1)^{m/2}\{\,(m/2)(m/2-1)\,(z+1)^{-2} + (m/2)\,(z+1)^{-1} + [\,m^2(1-z^2)^{-1} - \lambda]\,\}\,U = 0$$

So divide out our factor $(z+1)^{m/2}$ to get

$$-(1-z^2)U'' + [2\,(m/2)\,(z+1)^{-1} + 2z\,]\,U'$$
$$+ \{\,(m/2)(m/2-1)\,(z+1)^{-2} + (m/2)\,(z+1)^{-1} + [\,m^2(1-z^2)^{-1} - \lambda]\,\}\,U = 0$$

Now maybe mult thru by $(z+1)^2$:

$$-(1-z^2)\,(z+1)^2U'' + [2\,(m/2)\,(z+1) + 2z\,(z+1)^2\,]\,U'$$
$$+ \{\,(m/2)(m/2-1) + (m/2)\,(z+1) + (z+1)^2\,[\,m^2(1-z^2)^{-1} - \lambda]\,\}\,U = 0$$

$$-(1-z^2)\,(z+1)^2U'' + [2\,(m/2)\,(z+1) + 2z\,(z+1)^2\,]\,U'$$
$$+ \{\,(m/2)(m/2) + (m/2)\,(z) + (z+1)^2\,[\,m^2(1-z^2)^{-1} - \lambda]\,\}\,U = 0$$

$$-(1-z^2)\,(z+1)^2U'' + [2\,(m/2)\,(z+1) + 2z\,(z+1)^2\,]\,U'$$
$$+ \{\,(m/2)(m/2) + (m/2)\,(z) + (z+1)\,[(z+1)m^2(1-z^2)^{-1} - (z+1)\lambda]\,\}\,U = 0$$

$$-(1-z^2)\,(z+1)^2U'' + 2(z+1)\,[(m/2) + z(z+1)\,]\,U'$$
$$+ \{\,(m/2)(m/2) + (m/2)\,(z) + (z+1)\,[m^2(1-z)^{-1} - (z+1)\lambda]\,\}\,U = 0$$

This certainly doesn't look simpler that the starting equation, so I will terminate this thread.

Back now to

$$w_{\pm}(z) = P_n^{\pm m}(z) = (z+1)^{\pm m/2}\,(z-1)^{\mp m/2}\,F(-n,n+1;\,1\mp m;\,[1-z]/2)$$

I guess we could imagine expanding

$$(z+1)^{\alpha} = [2 + (z-1)]^{\alpha} = 2^{\alpha}\,[\,1 + (z-1)/2\,]^{\alpha} = 2^{\alpha}\,(1-x)^{\alpha} \qquad x = [1-z]/2$$
$$= 2^{\alpha}\,\Sigma_{k=0}\binom{\alpha}{k}(-1)^k x^k = 2^{\alpha}\,\Sigma_{k=0}\binom{\alpha}{k}\,(-1)^k\,\{\,[1-z]/2\}^k = \Sigma_{k=0}\,K_{\alpha k}\,\{\,[1-z]/2\}^k$$

Then it would be the product $(z+1)^{\pm m/2}\,F(-n,n+1;\,1\mp m;\,[1-z]/2)$ that is our $w_{\pm}(z)$ Frobenius series. So somehow that mess above must by the hypergeometric ODE.