A study of $Q_1(z)$ using Bateman Q (40) PhL 2.23.10

It is interesting and useful to see how the various Bateman forms reproduce known simple functions such as $Q_1(z)$.

The Maple code is located in batemanpq.mws. The first order of business is to enter the expressions. I had Q (43) in this file and edited it to become Q (40).

```
> restart;
> g1 := (n,m) ->GAMMA(1/2+n/2+m/2)/GAMMA(1+n/2-m/2);
>
```

$$g1 := (n,m) \to \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m\right)}{\Gamma\left(1+\frac{1}{2}n-\frac{1}{2}m\right)}$$

```
> g2 := (n,m) ->GAMMA(1+n/2+m/2)/GAMMA(1/2+n/2-m/2);
```

$$g2 := (n,m) \to \frac{\Gamma\left(1+\frac{1}{2}n+\frac{1}{2}m\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}n-\frac{1}{2}m\right)}$$

```
> Q40_1 := (n,m,z)
  ->(Pi)^(1/2)*2^(m-1)*g1(n,m)*exp(signum(Im(z))*I*Pi*(m-n-1)/2)*(z-1)^(-m/2)
  *(z+1)^(-m/2)* hypergeom([-n/2-m/2, 1/2+n/2 -m/2],[1/2],z^2);
>
```

$$Q40_1 := (n,m,z) \to \sqrt{\pi}\, 2^{(m-1)}\, g1(n,m)\, \mathbf{e}^{\left(\frac{1}{2} I\, \mathrm{signum}(\Im(z))\, \pi\, (m-n-1)\right)} (z-1)^{\left(-\frac{1}{2}m\right)} (z+1)^{\left(-\frac{1}{2}m\right)} \mathrm{hypergeom}\left(\left[-\frac{1}{2}n-\frac{1}{2}m, \frac{1}{2}+\frac{1}{2}n-\frac{1}{2}m\right],\left[\frac{1}{2}\right], z^2\right)$$

```
> Q40_2 := (n,m,z) ->
  (Pi)^(1/2)*2^(m)*g2(n,m)*exp(+signum(Im(z))*I*Pi*(m-n)/2)*z*(z-1)^(-m/2)
  *(z+1)^(-m/2)* hypergeom([ 1/2-n/2 -m/2, 1+n/2 -m/2],[3/2],z^2);
```

$$Q40_2 := (n,m,z) \to \sqrt{\pi}\, 2^{m}\, g2(n,m)\, \mathbf{e}^{\left(\frac{1}{2} I\, \mathrm{signum}(\Im(z))\, \pi\, (m-n)\right)} z\, (z-1)^{\left(-\frac{1}{2}m\right)} (z+1)^{\left(-\frac{1}{2}m\right)} \mathrm{hypergeom}\left(\left[\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m, 1+\frac{1}{2}n-\frac{1}{2}m\right],\left[\frac{3}{2}\right], z^2\right)$$

```
>
> Q40 :=(n,m,z) -> exp(+I*Pi*m)*(Q40_1(n,m,z) + Q40_2(n,m,z));
```

Notice that the $(z^2\text{-}1)$ powers are written out the right way (otherwise we get in trouble!). The g_1 and g_2 factors appear in various places so I have them separated. I include the usual extra $e^{+i\pi m}$ phase of the Bateman Q function only in the last summing expression. I am going to evaluate the two terms separately here, and then the sum. Here is the first term,

```
t := Q40_1(1,0,z);
```

$$t := \frac{e^{(-I\,\mathrm{signum}(\Im(z))\,\pi)}\,\mathrm{hypergeom}\left(\left[\frac{-1}{2}, 1\right], \left[\frac{1}{2}\right], z^2\right)}{\mathrm{signum}(\Im(z))}$$

```
t1 := convert(t, StandardFunctions);
```

$$t1 := e^{(-I\,\mathrm{signum}(\Im(z))\,\pi)}\,(1 - z\,\mathrm{arctanh}(z))$$

```
t2 := convert(t1,ln);
```

$$t2 := e^{(-I\,\mathrm{signum}(\Im(z))\,\pi)}\left(1 - z\left(\frac{1}{2}\ln(z+1) - \frac{1}{2}\ln(1-z)\right)\right)$$

which I could write as (z/2) ln[(z+1)/(1-z)] - 1 . This is almost the Q_1 function, except for the (1-z) in the denominator. Now let's get the second term:

```
t := Q40_2(1,0,z);
```

$$t := \frac{1}{2}\pi\, e^{\left(-\frac{1}{2} I\,\mathrm{signum}(\Im(z))\,\pi\right)} z^{\mathrm{signum}(\Im(z))}$$

```
t1 := convert(t, StandardFunctions);
```

$$t1 := \frac{1}{2}\pi\, e^{\left(-\frac{1}{2} I\,\mathrm{signum}(\Im(z))\,\pi\right)} z$$

```
t2 := convert(t1,ln);
```

$$t2 := \frac{1}{2}\pi\, e^{\left(-\frac{1}{2} I\,\mathrm{signum}(\Im(z))\,\pi\right)} z$$

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which I could write as (∓i)πz/2 . So our total result is this:

(z/2) ln[(z+1)/(1-z)] - 1 (∓i)πz/2

Now we use Bateman p 123 (12) to write

$(1-z) = e^{\mp i\pi}(z-1)$

Then we resolve the result like this:

(z/2) ln[(z+1)/{ $e^{\mp i\pi}$(z-1)}] - 1 (∓i)πz/2

= (z/2) ln[(z+1)/{(z-1)}] + (±iπ)(z/2) - 1 (∓i)πz/2

= (z/2) ln[(z+1)/(z-1)] - 1

which at last is the correct result.

Comment #1: Notice how you have to use the two "convert" commands exactly in the sequence I show above. We get results first in terms of a $\tanh^{-1}$ function, then we convert that to log.

Comment #2: The "log part" is contained entirely in the first term.

Comment #3: We know that the function f(z) = z solves the ODE. So here we find that each of the two terms separately solves the Legendre ODE. This is something I have wondered about. I think in general that the "two terms" are how you add two independent solutions of the ODE to construct the official Q function, so I would **speculate** that it is a general fact that each term is separately a solution in all those Bateman forms.

// I show this in a general way for Q (40) in my recent doc "A Maple study of Bateman Q (40).doc" . Maple cannot do it for general arguments n,m,z but if you pick some random numbers, you can show it is true.