Custom Maple Legendre functions.doc PhL 2.8.11

I keep having problems with evaluation of LegendreP and LegendreQ failing, but I think the hypergeom function does not have any such problems. So I will try to just write my own functions customized to what I am looking for.

1. Mehler functions.

In my doc "doing Mehler functions" Appendix A I claim that

$$P_{-1/2+i\tau}(\text{ch}\alpha) = e^{-(1/2)\alpha} e^{i\alpha\tau} F(1/2-i\tau, 1/2; 1; 1 - e^{-2\alpha})$$

Let's first try this in Maple this way]

```
Pmehler := (alpha,tau) -> exp(-alpha/2 + I*alpha*tau)*hypergeom([1/2-I*tau, 1/2],[1], 1 - exp(-2*alpha));
```

```
Pmehler := (alpha,tau) -> exp(-alpha/2 + I*alpha*tau) * hypergeom([1/2-I*tau,
1/2],[1/2],[1], 1 - exp(-2*alpha));
```

$$Pmehler := (\alpha, \tau) \to e^{\left(-\frac{1}{2}\alpha + I\alpha\tau\right)} \text{hypergeom}\left(\left[\frac{1}{2} - I\tau, \frac{1}{2}\right], \left[\frac{1}{2}\right], [1], 1 - e^{(-2\alpha)}\right)$$

We always need to "do a test" to make sure things got entered correctly, so

```
Pmehler(1.5,3);
```

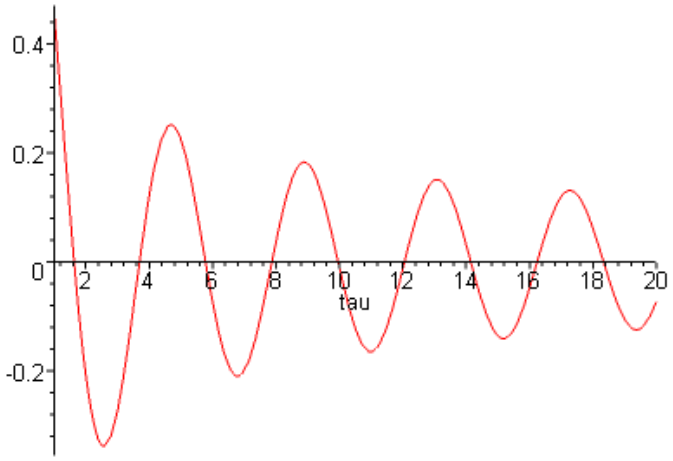
$$-.2724509302 - .2523324047\, 10^{-9} I$$

```
LegendreP(-1/2+I*3,cosh(1.5));
```

$$-.2724509301$$

and I am happy to have that tiny imaginary part which can be removed. Now we try a simple plot

```
plot(Re(Pmehler(1.5,tau)), tau = 1..20);
```



and this is what I was wanting to see. But the LegendreP plots works fine, I am unable to replicate the problem I was having!

2. Toroidal functions.

OK, I will put this doc on hold for a while until the problem arises again. // That didn't take long. Here is a problem:

```
U := evalf(LegendreQ(-1/2,1.5));
```

$$U := \text{LegendreQ}(-.5000000000, 0, 1.5)$$

Maple refuses to compute this function. So let's try to make our own version. Ptoroidal(nu,alpha). We now need a Q rep that converges on the positive real axis above 1. Looking at the converge doc, there are two choices: (42) and (46). They both have two terms, fine.

arg for (42) is $1-1/z^2$ and with $z = ch\xi$ this would be $1-sech^2\xi = th^2\xi$

so that seems pretty good. So here is the entry

Qtor := (xi,nu) -> stop. It has $\Gamma(\mu)$ sitting there, no good since $\mu = 0$.

How about (46)? Same problem. What does this mean?

How about (36) which should work at least down to 1.00001.

arg for (36) = $2/(1+z) = 2/(1+ch\xi) = 1/ ch^2(\xi/2)$

which is again fine. And (36) has only one term. So we try

```
f1 := 2^(nu);
f2 := (GAMMA(1+nu))^2;
```

```
f3 := (cosh(xi)+1)^(-nu-1)/GAMMA(2+2*nu);
f4 := hypergeom([1+nu,1+nu],[2+2*nu],1/(cosh(xi/2))^2;

    Qtor :=(xi,nu) -> 2^(nu)*(GAMMA(1+nu))^2*(cosh(xi)+1)
```

OK, here now is a specific example of the problem:

```
xi := 2;
```

$$\xi := 2$$

```
a := evalf(arccosh(xi));
```

$$a := 1.316957897$$

```
cosh(a);
```

$$2.000000000$$

```
evalf(Qtor(-1/2,a));
```

$$1.656638169$$

```
f := evalf(LegendreQ(-1/2,xi));
```

$$f := \text{LegendreQ}(-.5000000000, 0, 2.)$$

My call to Qtor(-1/2,a) gives a result, but the same LegendreQ *refuses to evaluate*. It is not a cut problem because this also fails

```
f := evalf(LegendreQ(-1/2,2.000+.00001*I));
```

$$f := \text{LegendreQ}(-.5000000000, 0, 2.000 + .00001\, I)$$

Now having shown up a problem, let's show that we agree at some other value:

```
xi := 3;
```

$$\xi := 3$$

```
a := evalf(arccosh(xi));
```

$$a := 1.762747174$$

```
cosh(a);
```

$$3.000000000$$

```
evalf(Qtor(-1/2,a));
```

$$1.311028777$$

```
f := evalf(LegendreQ(-1/2,xi));
```

$$f := 1.311028777$$

Here LegendreQ seems able to evaluate, and we get the same result.

So we seem pretty successful with (36) for Q. The "partner": to (36) is (18) but that has two terms.

Can we use the same P as used above, which was this

$$P_{-1/2+i\tau}(\text{ch}\alpha) = e^{-(1/2)\alpha} e^{i\alpha\tau} F(1/2\text{-}i\tau,\ 1/2;\ 1;\ 1 - e^{-2\alpha})$$

$P_\nu(z) = z^\nu\ F(-\nu/2,\ 1/2-\nu/2;\ 1;\ 1-1/z^2)$

So we could define

Ptor(nu,xi)

$P_\nu(z) = z^\nu\ F(-\nu/2,\ 1/2-\nu/2;\ 1;\ 1-1/z^2)$
$P(\nu,\xi) \equiv\ P_\nu(ch\xi) = (ch\xi)^\nu\ F(-\nu/2,\ 1/2-\nu/2;\ 1;\ th^2\xi)$

$Q_\nu(z) = 2^\nu\ [\Gamma(1+\nu)]^2\ (z+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+z)$
$Q(\nu,\xi) = Q_\nu(ch\xi) = 2^\nu\ [\Gamma(1+\nu)]^2\ (ch\xi+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+ch\xi)$

I have now entered the above stuff into PandQ.mws and it looks very good

P and Q functions for z >= 1

For P, we select Bateman (24) which says this

$P_\nu(z) = z^\nu\ F(-\nu/2,\ 1/2-\nu/2;\ 1;\ 1-1/z^2)$
$P(\nu,\xi) \equiv\ P_\nu(ch\xi) = (ch\xi)^\nu\ F(-\nu/2,\ 1/2-\nu/2;\ 1;\ th^2\xi)$

```
> restart;
> P := (nu,xi) ->
  evalf(cosh(xi)^nu*hypergeom([-nu/2,1/2-nu/2],[1],tanh(xi)^2));
```

$$P := (\nu, \xi) \rightarrow \mathrm{evalf}\left(\cosh(\xi)^{\nu}\ \mathrm{hypergeom}\left(\left[-\frac{1}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu\right], [1], \tanh(\xi)^2\right)\right)$$

Now show that these two functions agree at least somewhere!

```
> evalf(P(3,0.5));
```

1.893115746

```
> evalf(LegendreP(3,cosh(0.5)));
```

1.893115745

For Q, we select Bateman (36) which says this

$Q_\nu(z) = 2^\nu\ [\Gamma(1+\nu)]^2\ (z+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+z)$
$Q(\nu,\xi) = Q_\nu(ch\xi) = 2^\nu\ [\Gamma(1+\nu)]^2\ (ch\xi+1)^{-\nu-1}/\Gamma(2+2\nu) * F(1+\nu,1+\nu,2+2\nu,\ 2/(1+ch\xi)$

```
> Q := (nu,xi) ->
  evalf(2^(nu)*GAMMA(1+nu)^2*(cosh(xi)+1)^(-nu-1)*(1/GAMMA(2+2*nu))*hypergeom
  ([1+nu,1+nu],[2+2*nu],2/(1+cosh(xi))));
>
```

$$Q := (\nu, \xi) \rightarrow \mathrm{evalf}\left(\frac{2^{\nu}\,\Gamma(\nu+1)^2\,(\cosh(\xi)+1)^{(-\nu-1)}\ \mathrm{hypergeom}\left([\nu+1, \nu+1], [2+2\nu], 2\,\frac{1}{\cosh(\xi)+1}\right)}{\Gamma(2+2\nu)}\right)$$

Now show that these two functions agree at least somewhere!

```
> evalf(Q(1,1));
```

.1911607780

```
> evalf(LegendreQ(1,cosh(1)));
```

.191160779

Then I can construct the kinds of sums we get in donut potential!