Legendre Functions on the Cut PhL 1.5.10

Motivation. Bateman has a huge chapter on Legendre functions which are defined with both cuts pulled off to the left. Then on page 143 they talk about separate versions of these functions being defined "on the cut", meaning on the (-1,1) interval. I know how they define these functions (their "prescription"), but I don't know how to modify the other 3,000 properties of Legendre functions they talk about. This needs sharpening. These on-the-cut functions are sometimes called "modified" Legendre functions.

We know that our ODE has regular singular points at +1 and -1 and ∞ and our P and Q solutions have branch points at these locations. From "MF ODE" doc we know that

$$X = P\begin{Bmatrix} 1 & -1 & \infty & \\ m/2 & m/2 & -n & \xi \\ -m/2 & -m/2 & n+1 & \end{Bmatrix}$$

so the exponents at z = 1 and -1 are ± μ/2 as we see so often in our Bateman reading.

Since cuts are always hard for me to think about, I have to build up with simple examples. I have this all in some other document, but I will just start over here. ["contours and cuts.doc" in ODEs]

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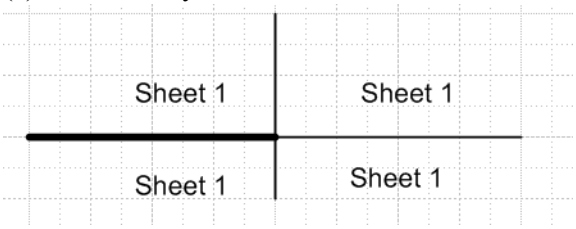
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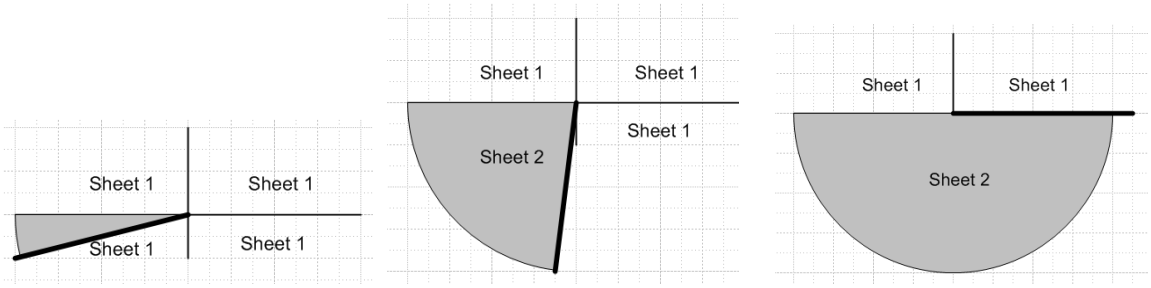
Warmup Example 1. $f = \sqrt{z}$

(a) We normally draw the cut like this:



This function has two sheets at are accessed by $f = \sqrt{R}\ e^{i\theta/2}$ as θ runs over a range of 4π. We could take this range to be (-2π, 2π). On the positive real access we set θ = 0 so $f = \sqrt{R}$ = real. Then (-π,π) gives us access to all of Sheet 1 (white). To get to Sheet 2 we need the two ranges (π,2π) and (-2π,π) sort of unioned together, but none of Sheet 2 is shown in the above picture. Over on the cut upper side of Sheet 1 we have $f = \sqrt{R}\ e^{i\pi/2} = i\sqrt{R}$, and on the lower side we have $-i\sqrt{R}$. One way to define Sheet 1 is to say that $\mathrm{Re}(f) = \sqrt{R}\cos(\theta/2)$ is positive everywhere on this sheet, and it is negative everywhere on Sheet 2.

(b) We can now talk about rotating the branch cut counter clockwise.



As we do this, we expose more and more of Sheet 2, shown gray. Eventually we have the cut going to the right. We are not changing the function, we are just exposing different parts of its sheets. So, for example, at a point z = 4+iε, we have f = 2 in all four pictures as ε→0. And at z = -4+iε, we have f = i 2. However, if we compare our first picture far above to our last picture on the right, the function has different values in the lower half plane, because in the first picture this lower half plane is Sheet 1, but in the last picture it is Sheet 2. For example, in the first picture at z = -4-iε we have f = -i 2. In the last picture, however, we have instead f = +i2 because we get to this point starting just above the cut and going below it.

Now someone might come along and attempt to define a function "f(z) on the cut",

$$f(z)_{\text{on the cut}} = (1/2)\,[\, e^{-i\pi/2} f(z=-R+i\varepsilon) + e^{+i\pi/2} f(z=-R-i\varepsilon)\,]$$

$$= (1/2)\,[\, e^{-i\pi/2}\, i\sqrt{R} + e^{+i\pi/2}\,(-i\sqrt{R})\,]$$

$$= (1/2)\,[\sqrt{R} + \sqrt{R}\,)] = \sqrt{R}$$

We would get the same result if we thought about this function:

g(z) = f(-z) with branch cut drawn to the right in the z plane

This function is then real for z on the negative real axis. This is NOT the same function we started with, which was f(z). So the "f(z) on the cut" function is not simply f(-|z|+iε). After all, the latter is imaginary, whereas "f(z) on the cut" is real. In fact, we have

$$f(z)_{\text{on the cut}} = f(-z) = \sqrt{-z}$$

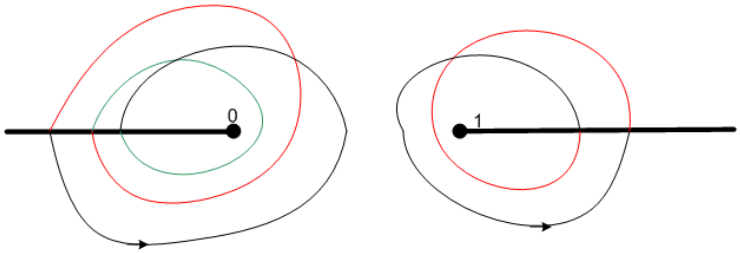
and his is real when z is on the negative real axis.

Warmup Example 2. $f(z) = z^{1/3}\,(1-z)^{1/2}$

Now things are much more complicated, and this is the example I discuss in contours and cuts. I will quote, with some editing:

Here is a useful picture:

m = 0,1,2 n = 0,1



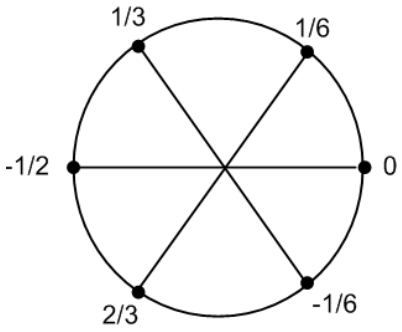
For the branch point on the left which arises from $z^{1/3}$, there are three sheets encountered as you rotate around the point z=0 many times. If we had f(t) = $z^{1/3}$, we would have just the left branch cut, and we would say that f(z) had 3 sheets. Similarly, if we just had f(z) = $(1-z)^{1/2}$, we would only have the branch cut on the right, and the function f(z) would have 2 sheets.

We can think of 3 winding numbers possible on the left, and 2 on the right. Call these m and n. Then the phase accrued by moving away from the black-black sheet is this

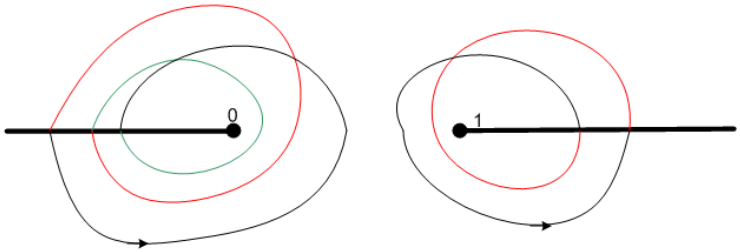
phase = 2π(m/3 - n/2)

m	n	(m/3 - n/2)	Sheet name:
0	0	0	black-black
1	0	1/3	red-black
2	0	2/3	green-black
0	1	-1/2	black-red
1	1	1/3 - 1/2 = -1/6	red-red
2	1	2/3 - 1/2 = +1/6	green-red

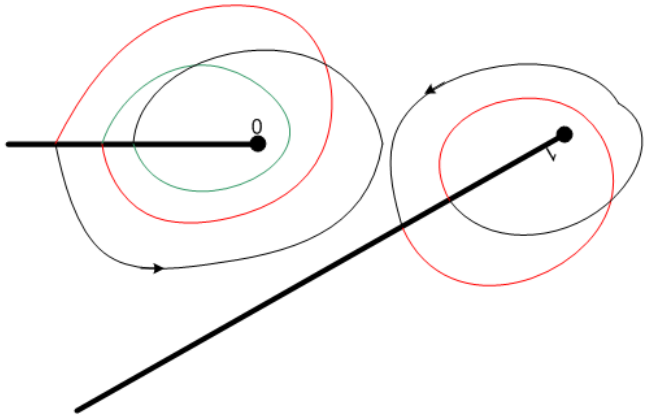
There are 6 sheets for $f(z) = z^{1/3}\,(1-z)^{1/2}$ because there are six different phases, and they are equally spaced around the circle:



End quote. Now let's re-examine the picture above



The colors are carefully drawn to match the branch cuts as drawn. But we really want the cut at z=1 to go off to the left. Imagine swinging that cut clockwise down, and as we do so, we adjust the colors on the two windings so they change at the cut. Well, it was not that hard to just rotate the right side of the picture, so here is something closer to what we want



So when you have multiple branch points in a function, you really need to label sheets with a set of winding numbers like the m.n considered here. It is very hard to draw a good picture (a parking garage somehow) showing the 6 sheets.

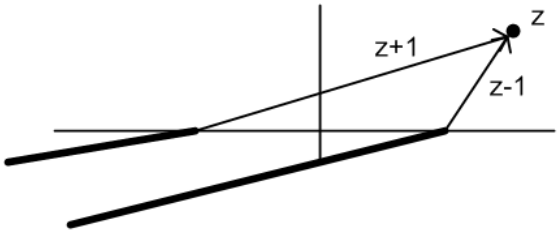
Hopefully these warmup examples are enough. In our general case below of course we have an infinite number of winding numbers possible at each branch point.

The P function on the cut (Bateman)

Let's start with $P_\nu^\mu(z)$ as given in Bateman p 122 [P (14)] top where we have, from a cut point of view,

$$P_\nu^\mu(z)_{\mathtt{start}} = [(z+1)/(z-1)]^{\mu/2}$$

where the F function converges in a disk of radius 2 about the point z = 1. So we can't really "get to" the cut below -1 with this form, so we shall restrict our interest to the portion of the cut that runs (-1,1). But still we can draw the cuts everywhere:



The reason we draw the cuts to the left is to have things be real on the positive real axis far off to the right, the usual "real analytic" thing I think.

Recall from Warmup Example 1 how we arranged to have things be real on the negative axis. We defined $f(z)_{\mathtt{on_the_cut}}$ by replacing z with -z. Here we can do that same thing just with the cut on the right by replacing z-1 with 1-z. Then we get

$$P_\nu^\mu(z)_{\mathtt{on_the_cut}} = [(z+1)/(1-z)]^{\mu/2} \qquad \text{// can also write as } [(1+z)/(1-z)]^{\mu/2}$$

Now notice that (the first phase is negative since (1-z) is in the denominator)

$$P_\nu^\mu(x+i\varepsilon)_{\mathtt{start}} = [(z+1)/(z-1)]^{\mu/2}\, e^{-i\pi\mu/2}$$
$$P_\nu^\mu(x-i\varepsilon)_{\mathtt{start}} = [(z+1)/(z-1)]^{\mu/2}\, e^{+i\pi\mu/2}$$

We can do this same thing "the long way" by using the rule Bateman gives on page 143:

$$P_\nu^\mu(z)_{\mathtt{on_the_cut}} = (1/2)\,[\, e^{+i\pi\mu/2}\, P_\nu^\mu(x+i\varepsilon)_{\mathtt{start}} + e^{-i\pi\mu/2}\, P_\nu^\mu(x-i\varepsilon)_{\mathtt{start}}\,]$$

which is *not* simply the average of $P_\nu^\mu(z)_{\mathtt{start}}$ above and below the cut!

So basically, we get to our "on the cut" version simply by changing z-1 → 1-z in our factor. We do NOT make any changes in the F function of course, since it has no cut. We are just dealing with the cuts here.

When we do as described above, we have

$$P_\nu^\mu(z)_{\mathtt{start}} = \text{p 122 (3)} \qquad = 1/\Gamma(1-\mu)\,[\,(z+1)/(z-1)]^{\mu/2}\, F[\; -\nu, \nu+1;\, 1-\mu;\, (1-z)/2\,]$$

$$P_\nu^\mu(z)_{\mathtt{on\ the\ cut}} = \text{p 143 (6)} \quad = 1/\Gamma(1-\mu)\,[\,(1+z)/(1-z)]^{\mu/2}\, F[\; -\nu, \nu+1;\, 1-\mu;\, (1-z)/2\,]$$

Note added 2.22.10. You can think of P (22) as "working everywhere" if you think of F as a function, not a series. In this sense F(a,b,c,z) has a cut (1,∞), so F (a,b,c,(1-z)/2) has a cut (-∞,-1). So one should keep this cut in mind, it is not JUST the outside powers that make cuts. More elsewhere.

The Q function on the cut (Bateman)

1. Let's apply the prescription of p 144 to the form p 130 (32) for the Q function.
We do this because this form has the same (1-z)/2 argument of F as the P form we used, which seems nice somehow. The symbolic form of (32) is this:

$$e^{-i\mu\pi}Q_\nu^\mu(z) = (z-1)^{\mu/2} + (z-1)^{-\mu/2}$$

Note added 1.26.10. In this *symbolic form*, we ignore constants of μ and ν, we ignore the F function, and we ignore factors of the form z^α or $(1+z)^\alpha$. The *reason* is that none of these factors is expressed differently if you are to the right of the branch point at z = 1, or if you are to the left of it in the range (-1,1) (above or below the cut). Thus, all these factors will be the same for on cut and off cut functions. We might choose to write z+1 and 1+z for on cut functions, but obviously there is no issue here. We never "wind" around the point z = -1 or z = 0.

Therefore we can say

$$e^{-i\mu\pi}Q_\nu^\mu(x+i\varepsilon) = e^{+i\pi\mu/2}|z-1|^{\mu/2} + e^{-i\pi\mu/2}|z-1|^{\mu/2}$$
$$e^{-i\mu\pi}Q_\nu^\mu(x-i\varepsilon) = e^{-i\pi\mu/2}|z-1|^{\mu/2} + e^{+i\pi\mu/2}|z-1|^{\mu/2}$$

The prescription then says

$$e^{-i\mu\pi}Q_\nu^\mu(x) \equiv (1/2)\left[e^{-i\pi\mu/2}\, Q_\nu^\mu(x+i\varepsilon) + e^{+i\pi\mu/2}\, Q_\nu^\mu(x-i\varepsilon)\right]$$

$$= (1/2)\, e^{-i\pi\mu/2}\, [Q_\nu^\mu(x+i\varepsilon)]$$
$$+ (1/2)\, e^{+i\pi\mu/2}\, [Q_\nu^\mu(x-i\varepsilon)]$$

$$= (1/2)\, e^{-i\pi\mu/2}\, [e^{+i\pi\mu/2}|z-1|^{\mu/2} + e^{-i\pi\mu/2}|z-1|^{\mu/2}]$$
$$+ (1/2)\, e^{+i\pi\mu/2}\, [e^{-i\pi\mu/2}|z-1|^{\mu/2} + e^{+i\pi\mu/2}|z-1|^{\mu/2}]$$

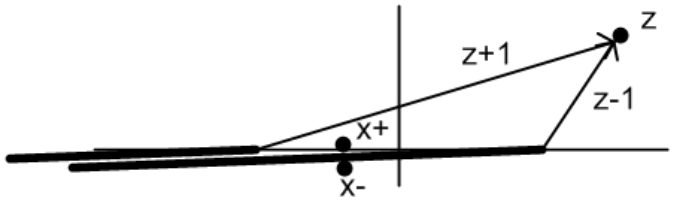
$$= (1/2)\, [|z-1|^{\mu/2} + e^{-i\pi\mu}|z-1|^{\mu/2}]$$
$$+ (1/2)\, [|z-1|^{\mu/2} + e^{+i\pi\mu}|z-1|^{\mu/2}]$$

$$= |z-1|^{\mu/2} + \cos(\pi\mu)\, |z-1|^{\mu/2}$$

and this agrees with the symbolic form of p 144 (10). Notice that this resulting form is *real* on the cut (-1,1) for real parameters and real argument! Surely this must be a major motivation for choosing this definition for the on-the-cut Q function. The on-the-cut P function has this same property.

2. Does Q(x) solve the Legendre equation?

Consider $\lim_{\varepsilon\to 0}$ (x+iε). Let's say this takes you to a point on Sheet 1 of the ODE sheet structure, call it x+. At this point, certainly $Q_\nu^\mu(x+)$ is a solution of the ODE. If we, as an ant, crawl around the branch point at z = 1, and go to point x-, and from there up to the real axis, then we are at a point on Sheet 2. More precisely, if the starting sheet was m,n = 0.0, then x+ is on this sheet 0.0 and x- is on the sheet 0.1 because we wind CW one around z = 1 (making my own CW convention just locally here). Since $Q_\nu^\mu(z)$ is a solution of the ODE for z any point on any sheet m.n (OK, stay away from the singular points), then certainly $Q_\nu^\mu(x-)$ is also a solution of the ODE.



Therefore, the function which is defined to be " Q on the cut" is the sum of two ODE solutions, and must therefore itself be a solution at the point z = x. Question answered!

3. Are the many "relations" which were proven for Q(z) and P(z) still valid for Q(x) and P(x) ?

This question needs a systematic answer. Start on page 122 where we have various numbered "facts" about P(z) and Q(z) functions. Items (3) and (5) are two entries of the table of "joining relations" to follow. Items (7) and (8) come from parameter shifting forms of the F function. Then on page 123 we have an introduction to the tables, and I skip the Wronskian for the moment.

(3a) Table Relations (allowing for possible signed phases):

So basically, our question is this: are the many entries in "the tables" valid for the P(x) and Q(x) functions? Or do they need to be modified? A certain table entry has this symbolic form, where I show only (z-1) factors [see note on symbolic form above] and possible ± type phase factors

$$P = e^{\pm i\pi a}A(z-1)^\alpha F + e^{\pm i\pi b}B(z-1)^\beta F'$$

There might only be one term. The α,β exponents are usually 0, μ/2 or -μ/2, but other values do appear. Let's now consider the above sample relation at our special points. (the relation is valid at *all* points z)

$$P(x+) = e^{+i\pi a}A(z-1)_+^\alpha F + e^{+i\pi b}B(z-1)_+^\beta F'$$
$$P(x-) = e^{-i\pi a}A(z-1)_-^\alpha F + e^{-i\pi b}B(z-1)_-^\beta F'$$

where we have

$$(z-1)_+ = |z-1|\, e^{+i\pi} \quad \Rightarrow \quad (z-1)_+^\alpha = |z-1|^\alpha e^{+i\pi\alpha}$$
$$(z-1)_- = |z-1|\, e^{-i\pi} \quad \Rightarrow \quad (z-1)_-^\alpha = |z-1|^\alpha e^{-i\pi\alpha}$$

Rewrite the above then as (later we can set |z-1| = (1-x))

$$P(x+) = e^{+i\pi a} A\, e^{+i\pi\alpha} |z-1|^{\alpha} F + e^{+i\pi b} B e^{+i\pi\beta} |z-1|^{\beta} F'$$
$$P(x-) = e^{-i\pi a} A\, e^{-i\pi\alpha} |z-1|^{\alpha} F + e^{-i\pi b} B e^{-i\pi\beta} |z-1|^{\beta} F'$$

Now apply the phases to each equation according to our prescription

$$e^{+i\pi\mu/2}\, P(x+) = e^{+i\pi a} A\, e^{+i\pi\mu/2}\, [e^{+i\pi\alpha}|z-1|^{\alpha}]\, F + e^{+i\pi b}\, B\, e^{+i\pi\mu/2} [e^{+i\pi\beta}\, |z-1|^{\beta}]F'$$
$$e^{-i\pi\mu/2}\, P(x-) = e^{-i\pi a} A\, e^{-i\pi\mu/2}\, [e^{-i\pi\alpha}\, |z-1|^{\alpha}]\, F + e^{-i\pi b}\, B\, e^{-i\pi\mu/2} [e^{-i\pi\beta}\, |z-1|^{\beta}]F'$$

where I have put [..] around places where we use the "rules" that Bateman states. Now let's add and divide by 2.

$$P(x) = \{ e^{+i\pi a}\, e^{+i\pi\mu/2}\, e^{+i\pi\alpha} + e^{-i\pi a}\, e^{-i\pi\mu/2}\, e^{-i\pi\alpha} \}/2 * A\, |z-1|^{\alpha} F$$
$$+ \{ e^{+i\pi b}\, e^{+i\pi\mu/2}\, e^{+i\pi\beta} + e^{-i\pi b}\, e^{-i\pi\mu/2}\, e^{-i\pi\beta} \}/2 * B\, |z-1|^{\beta} F'$$

$$= \{ e^{+i\pi(\alpha+\mu/2+a)} + e^{-i\pi(\alpha+\mu/2+a)} \}/2 * A\, |z-1|^{\alpha} F$$
$$+ \{ e^{+i\pi(\beta+\mu/2+b)} + e^{-i\pi(\beta+\mu/2+b)} \}/2 * B\, |z-1|^{\beta} F'$$

$$= \cos\pi(\alpha+\mu/2+a) * A\, |z-1|^{\alpha} F + \cos\pi(\beta+\mu/2+b) * B\, |z-1|^{\beta} F'$$

So this is the new relation, and it is *not* the same, we have the extra cosine factors!!! So the exact relation for the P(x) functions depends on the values of α and β and a and b. I will do some examples in a moment. Bateman could have been clearer about this!

What about the Q functions? The table entries for those have this form:

$$e^{-i\pi\mu}\, Q(z) = e^{\pm i\pi a}\, A(z-1)^{\alpha} F + e^{\pm i\pi b}\, B(z-1)^{\beta} F$$

and following similar steps we quickly get

$$e^{-i\pi\mu}\, Q(x+) = A\, e^{+i\pi(\alpha+a)} |z-1|^{\alpha} F + B e^{+i\pi(\beta+b)}\, |z-1|^{\beta} F'$$
$$e^{-i\pi\mu}\, Q(x-) = A\, e^{-i\pi(\alpha+a)}\, |z-1|^{\alpha} F + B e^{-i\pi(\beta+b)}\, |z-1|^{\beta} F'$$

Now apply the *reverse* phases to those applied for the P function:

$$e^{-i\pi\mu/2} e^{-i\pi\mu}\, Q(x+) = A\, e^{-i\pi\mu/2}\, e^{+i\pi(\alpha+a)} |z-1|^{\alpha} F + B\, e^{-i\pi\mu/2} e^{+i\pi(\beta+b)}\, |z-1|^{\beta} F'$$
$$e^{+i\pi\mu/2} e^{-i\pi\mu}\, Q(x-) = A\, e^{+i\pi\mu/2}\, e^{-i\pi(\alpha+a)}\, |z-1|^{\alpha} F + B\, e^{+i\pi\mu/2}\, e^{-i\pi(\beta+b)}\, |z-1|^{\beta} F'$$

Then add and divide by 2 to get

$$Q(x) = \{ e^{-i\pi\mu/2}\, e^{+i\pi(\alpha+a)} + e^{+i\pi\mu/2}\, e^{-i\pi(\alpha+a)} \}/2 * A\, |z-1|^{\alpha} F$$
$$+ \{ e^{-i\pi\mu/2}\, e^{+i\pi(\beta+b)} + e^{+i\pi\mu/2}\, e^{-i\pi(\beta+b)} \}/2 * B\, |z-1|^{\beta} F'$$

which is the same as our previous result for P but with $\mu \rightarrow -\mu$. So here is our final result:

$$Q(x) = \cos\pi(\alpha + a - \mu/2) * A\,|z-1|^{\alpha} F \;+\; \cos(\beta + b - \mu/2) * B\,|z-1|^{\beta} F'$$

Summary of these results:

P: $P(z) = e^{\pm i\pi a} A(z-1)^{\alpha} F + e^{\pm i\pi b} B(z-1)^{\beta} F'$

$P(x) = \cos\pi(\alpha+a+\mu/2)\, A\,|z-1|^{\alpha} F + \cos\pi(\beta+b+\mu/2)\, B\,|z-1|^{\beta} F'$

Q: $e^{-i\pi\mu} Q(z) = e^{\pm i\pi a} A(z-1)^{\alpha} F + e^{\pm i\pi b} B(z-1)^{\beta} F'$

$Q(x) = \cos\pi(\alpha+a-\mu/2)\, A\,|z-1|^{\alpha} F + \cos\pi(\beta+b-\mu/2)\, B\,|z-1|^{\beta} F'$

Let's now do a P example. Select entry (14) from the table. We see that $\alpha = -\mu/2$ and $B = 0$ and $a=b=0$:

$P(x) = \cos\pi(0)\, A\,|z-1|^{\alpha} F = A\,|z-1|^{\alpha} F$ // agrees with 143 (6)

Let's now do a Q example. Select entry (32) from the table. We see that $\alpha = \mu/2$ and $\beta = -\mu/2$:

$Q(x) = \cos\pi(0)\, A\,|z-1|^{\alpha} F + \cos\pi(\mu)\, B\,|z-1|^{\beta} F'$ // agrees with 144 (10)

There are entries in the table that require special attention in two ways. First, look at (17). The second term has a ± factor that we would have to throw into our machinery, causing a change in what happened above. We would use the + sign for our x+iε value, and the - sign for our x-iε. Second, look at (23). This one has a factor z^{σ} so we have to work that into our analysis if we care about the (-1,0) part of the interval.

(3b) Page 140 relations. These are based on the tables, so I think we have to consider each one as a separate problem! Luckily Bateman has done most these for us on page 144 (all involve arg = x). The one exception on page 144 is the equation showing that P is the discontinuity of Q, and the 2008 errata found a sign error on the RHS. Probably (9) is OK since not mentioned in the errata. But there might be errata which I am missing. [I looked into this and see that there is a source of errata at jstor that I cannot access, though I could walk to the library. I asked Jim if he could get me a password, doubt anything will come of that.]

(3c) The Wronskian. He gives this result on page 146, very close to the z result on page 123. The only difference is that the phase is missing and $(1-z^2)$ is now $(1-x^2)$.

(3d) Integer Cases.

[P] The various constants which appear in the "definition" of $P_{\nu}^{\mu}(z)$ come, no doubt, from the historical integral μ situation where, given our P(z) definition, it is true that

$$P_{\nu}^{m}(z) = (z^2-1)^{m/2}\partial_m P_{\nu}(z) \qquad m = 1,2,3....$$

Let's think of the above line as a case of our more general form [we can think of P_{ν} like F because P_{ν} has $\mu=0$, the prescription phases are both 1, and there is no cut for P_{ν}, so $P_{\nu}(z) = P_{\nu}(x)$],

$$P = A(z-1)^{\alpha} F + B(z-1)^{\beta} F'$$

where AF is $\partial_m P_\nu(z)$ and $\alpha = m/2$ and $B = 0$. We then know our on-the-cut form of this relation will be:

$$P(x) = \cos\pi(\alpha+m/2) * A\,|z-1|^{\alpha} F = \cos(\pi m)\, A\,|z-1|^{\alpha} F = (-1)^m A\,|z-1|^{\alpha} F$$

Therefore, we have that

$$P_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \partial_m P_\nu(x) \qquad m = 1,2,3....$$

Once we define $P_\nu^m(z)$, the $(-1)^m$ in the above equation is forced, it is not an optional phase. The presence of this $(-1)^m$ factor is called the **Condon-Shortley Phase**. You could define P(z) differently to get rid of this phase, but I don't think anyone would do that.

Side Question: Does $Q_\nu(z)$ have a cut? These functions all have $\ln[(z+1)/(z-1)]$ which *does* in fact have a branch point at z = 1. The generic "form" here is $\ln(z-1)$. When we set $(z-1)_+ = |z-1|\, e^{+i\pi}$, for example, we get form = $\ln|z-1| + i\pi$. then disc $\ln(z-1) = 2\pi i$. I have been ignoring this kind of cut in the above discussion! You cannot use (32) to express Q_ν since $\Gamma(\mu)$ blows up in both terms. In fact, none of the Q_ν "relations" exists showing a disk around z = +1. So at least we don't have any contradictions. That is to say, we don't have a relation saying Q_ν has a power branch point when we know it has a log one.

[Q] The various constants which appear in the "definition" of $Q_\nu^\mu(z)$ come, no doubt, from the historical integral μ situation where, given our Q(z) definition, it is true that

$$Q_\nu^m(z) = (z^2-1)^{m/2} \partial_m Q_\nu(z) \qquad m = 1,2,3....$$

which is identical to the P equation. We have just shown that $Q_\nu(z)$ has a cut from z=1 so we have to be more careful now. Let's construct our on-the-cut version of things here:

$$Q_\nu^m(z+i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, e^{im\pi/2} \partial_m\, Q_\nu(z+i\varepsilon)$$
$$Q_\nu^m(z-i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, e^{-im\pi/2} \partial_m\, Q_\nu(z-i\varepsilon)$$

Apply the proper prescription phases:

$$e^{-im\pi/2} Q_\nu^m(z+i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, e^{-im\pi/2} e^{im\pi/2}\, \partial_m Q_\nu(z+i\varepsilon)$$
$$e^{+im\pi/2} Q_\nu^m(z-i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, e^{+im\pi/2} e^{-im\pi/2} \partial_m Q_\nu(z-i\varepsilon)$$

$$e^{-im\pi/2} Q_\nu^m(z+i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, \partial_m Q_\nu(z+i\varepsilon)$$
$$e^{+im\pi/2} Q_\nu^m(z-i\varepsilon) = (z+1)^{m/2}\, |z-1|^{m/2}\, \partial_m Q_\nu(z-i\varepsilon)$$

The prescription p 143 (applied twice) then says

$$e^{im\pi}\, Q_\nu^m(x) = (z+1)^{m/2}\, |z-1|^{m/2}\, \partial_m Q_\nu(x)$$

$(-1)^m\ Q_\nu^m(x) = (1+x)^{m/2}\,(1-x)^{m/2}\,\partial_m Q_\nu(x)$

$(-1)^m\ Q_\nu^m(x) = (1-x^2)^{m/2}\,\partial_m Q_\nu(x)$

$Q_\nu^m(x) = (-1)^m\ (1-x^2)^{m/2}\,\partial_m Q_\nu(x)$

so in the end, we get an equation identical to the P equation.

4. Why does the grouping [$e^{-i\mu\pi}\,Q_\nu^\mu(z)$] always appear together?

You see this factor sort of "thrown in" in the fundamental definition p 122 (5) of Q. We have also seen how we sort of "throw it out" when we deal with Q "on the cut", according to the prescription. Here is a possible explanation. Consider our results above:

$P_\nu^m(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ m = 1,2,3....
$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....

Had we not included the $e^{i\pi\mu}$ factor in definition p 122 (5) of Q, then the second equation above would have an extra factor of $(-1)^m$ sitting in it, and then the second equation would not look like the first one. I will bet that is the historical reason!

Then once you put $e^{i\pi\mu}$ into the definition of Q to make the above Rodriguez things work right, in order for the on-the-cut Q function to work out right, you have to remove this phase in the prescription.

5. What causes the (1-z) to turn around inside the log of the Q functions?

Consider first some Smythe data,

$$Q_1^1(\mu) = (1-\mu^2)^{\frac{1}{2}}\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)$$
$$Q_2^1(\mu) = (1-\mu^2)^{\frac{1}{2}}\left(\frac{3}{2}\mu\ln\frac{1+\mu}{1-\mu} + \frac{3\mu^2-2}{1-\mu^2}\right)$$
$$Q_2^2(\mu) = (1-\mu^2)\left(\frac{3}{2}\ln\frac{1+\mu}{1-\mu} + \frac{5\mu-3\mu^3}{(1-\mu^2)^2}\right)$$

where things are already "turned around" so he is using on-the-cut functions. Let's just try this with Q_0 and see what happens. We start with

$Q_0(z) = (1/2)\log[\,(z+1)/(z-1)]$

Now we swing z around on the top so we have

$Q_0(z+i\varepsilon) = (1/2)\log[\,(z+1)/\{|z-1|e^{+i\pi}\})]$
$Q_0(z-i\varepsilon) = (1/2)\log[\,(z+1)/\{|z-1|e^{-i\pi}\})]$

and apply the prescription which in this case is just add and divide by 2. In doing this sum, the +iπ cancels the -iπ and you get

$$Q_0(x) = (1/2)\log[(z+1)/\{|z-1|\}] = (1/2)\log[(1+x)/(1-x)]$$

and THAT is how the (1-z) gets turned around inside a log. Once the turn around has happened, the log function is real on the (-1,1) cut region, so the statement made earlier that $Q_\nu^\mu(x)$ is real on the cut is born out in this example.

6. How could you make a table of the integral P and Q functions? Start with these formulas:

$$P_n^m(z) = (z^2-1)^{m/2}\partial_m P_n(z) \qquad m = 1,2,3....$$
$$Q_n^m(z) = (z^2-1)^{m/2}\partial_m Q_n(z) \qquad m = 1,2,3....$$

Then all you need is a formula for $P_n(z)$ and $Q_n(z)$. But Bateman page 152 shows that

$$P_n(z) = (2^n n!)^{-1}\partial_n (z^2-1)^n \qquad \text{// Rodriguez}$$

The Q_n functions are not so easy. The general form is this:

$$Q_n(z) = (1/2)\,P_n(z)\,\log[(z+1)/(z-1)] - W_{n-1}(z)$$

where W is a polynomial of degree n-1 in z. An expression on page 153 is given for this polynomial, but it is something you have to compute from a sum involving shifted P_n functions. Using Rodriquez I think I could cause Maple to generate all these functions.

<u>What about integer m < 0 values?</u>

From Bateman p 140 we know that (μ must be an integer or there is another term)

$$P_\nu^m(z)\,\Gamma(\nu-m+1) = P_\nu^{-m}(z)\,\Gamma(\nu+m+1) \qquad m = \text{any integer}$$

For the Q functions, the answer is given in page 140 (2)

$$e^{i\mu\pi}\,Q_\nu^{-\mu}(z)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi}\,Q_\nu^{\mu}(z)\,\Gamma(\nu-\mu+1)$$

This is true for any μ, not just integer μ.

<u>What about on-the-cut integer m < 0 values?</u>

Bateman doesn't give us these, so let's use the prescription and compute them:

$$P_\nu^m(z+i\varepsilon)\,\Gamma(\nu-m+1) = P_\nu^{-m}(z+i\varepsilon)\,\Gamma(\nu+m+1)$$
$$P_\nu^m(z-i\varepsilon)\,\Gamma(\nu-m+1) = P_\nu^{-m}(z-i\varepsilon)\,\Gamma(\nu+m+1)$$

Apply proper phases on the left

$$e^{+im\pi/2}P_\nu^m(z+i\varepsilon)\,\Gamma(\nu-m+1) = e^{+im\pi/2}P_\nu^{-m}(z+i\varepsilon)\,\Gamma(\nu+m+1)$$
$$e^{-im\pi/2}P_\nu^m(z-i\varepsilon)\,\Gamma(\nu-m+1) = e^{-im\pi/2}P_\nu^{-m}(z-i\varepsilon)\,\Gamma(\nu+m+1)$$

$$e^{+im\pi/2}P_\nu^m(z+i\varepsilon)\,\Gamma(\nu-m+1) = e^{+im\pi}e^{-im\pi/2}P_\nu^{-m}(z+i\varepsilon)\,\Gamma(\nu+m+1)$$
$$e^{-im\pi/2}P_\nu^m(z-i\varepsilon)\,\Gamma(\nu-m+1) = e^{-im\pi}e^{+im\pi/2}P_\nu^{-m}(z-i\varepsilon)\,\Gamma(\nu+m+1)$$

$$e^{+im\pi/2}P_\nu^m(z+i\varepsilon)\,\Gamma(\nu-m+1) = (-1)^m\, e^{-im\pi/2}P_\nu^{-m}(z+i\varepsilon)\,\Gamma(\nu+m+1)$$
$$e^{-im\pi/2}P_\nu^m(z-i\varepsilon)\,\Gamma(\nu-m+1) = (-1)^m\, e^{+im\pi/2}P_\nu^{-m}(z-i\varepsilon)\,\Gamma(\nu+m+1)$$

and now do the prescription to get

$$P_\nu^m(x)\,\Gamma(\nu-m+1) = (-1)^m\, P_\nu^{-m}(x)\,\Gamma(\nu+m+1)$$

Jackson on page 65 agrees. Let's try the Q functions now

$$e^{i\mu\pi}\,Q_\nu^{-\mu}(z+i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi}\,Q_\nu^{\mu}(z+i\varepsilon)\,\Gamma(\nu-\mu+1)$$
$$e^{i\mu\pi}\,Q_\nu^{-\mu}(z-i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi}\,Q_\nu^{\mu}(z-i\varepsilon)\,\Gamma(\nu-\mu+1)$$

Apply the phases: (carefully!)

$$e^{+i\mu\pi/2}\,e^{i\mu\pi}\,Q_\nu^{-\mu}(z+i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{+i\mu\pi/2}\,e^{-i\mu\pi}\,Q_\nu^{\mu}(z+i\varepsilon)\,\Gamma(\nu-\mu+1)$$
$$e^{-i\mu\pi/2}\,e^{i\mu\pi}\,Q_\nu^{-\mu}(z-i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi/2}\,e^{-i\mu\pi}\,Q_\nu^{\mu}(z-i\varepsilon)\,\Gamma(\nu-\mu+1)$$

$$e^{i\mu\pi}\,e^{+i\mu\pi/2}\,Q_\nu^{-\mu}(z+i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{+i\mu\pi/2}\,e^{+i\mu\pi/2}e^{-i\mu\pi}\,e^{-i\mu\pi/2}Q_\nu^{\mu}(z+i\varepsilon)\,\Gamma(\nu-\mu+1)$$
$$e^{i\mu\pi}\,e^{-i\mu\pi/2}\,Q_\nu^{-\mu}(z-i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi/2}\,e^{-i\mu\pi/2}e^{-i\mu\pi}\,e^{+i\mu\pi/2}Q_\nu^{\mu}(z-i\varepsilon)\,\Gamma(\nu-\mu+1)$$

$$e^{i\mu\pi}\,e^{+i\mu\pi/2}\,Q_\nu^{-\mu}(z+i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{+i\mu\pi}\,e^{-i\mu\pi}\,e^{-i\mu\pi/2}Q_\nu^{\mu}(z+i\varepsilon)\,\Gamma(\nu-\mu+1)$$
$$e^{i\mu\pi}\,e^{-i\mu\pi/2}\,Q_\nu^{-\mu}(z-i\varepsilon)\,\Gamma(\nu+\mu+1) = e^{-i\mu\pi}\,e^{-i\mu\pi}\,e^{+i\mu\pi/2}Q_\nu^{\mu}(z-i\varepsilon)\,\Gamma(\nu-\mu+1)$$

At this point, we have to assume μ = integer = m so both equations show the same factor at the start of the RHS:

$$e^{i\mu\pi}\,e^{+i\mu\pi/2}\,Q_\nu^{-\mu}(z+i\varepsilon)\,\Gamma(\nu+\mu+1) = (-1)^m\,e^{-i\mu\pi}\,e^{-i\mu\pi/2}Q_\nu^{\mu}(z+i\varepsilon)\,\Gamma(\nu-\mu+1)$$
$$e^{i\mu\pi}\,e^{-i\mu\pi/2}\,Q_\nu^{-\mu}(z-i\varepsilon)\,\Gamma(\nu+\mu+1) = (-1)^m\,e^{-i\mu\pi}\,e^{+i\mu\pi/2}Q_\nu^{\mu}(z-i\varepsilon)\,\Gamma(\nu-\mu+1)$$

Now we can apply the prescription on both sides to get

$$Q_\nu^{-\mu}(x)\,\Gamma(\nu+\mu+1) = (-1)^m\,Q_\nu^{\mu}(x)\,\Gamma(\nu-\mu+1)$$

by which we really mean

$$Q_\nu^{-m}(x)\,\Gamma(\nu+m+1) = (-1)^m\,Q_\nu^{m}(x)\,\Gamma(\nu-m+1)$$

So we can now summarize our results: (m = any integer)

$P_\nu^{-m}(z)\,\Gamma(\nu+m+1) = P_\nu^{m}(z)\,\Gamma(\nu-m+1)$ // general
$Q_\nu^{-m}(z)\,\Gamma(\nu+m+1) = Q_\nu^{m}(z)\,\Gamma(\nu-m+1)$

$P_\nu^{-m}(x)\,\Gamma(\nu+m+1) = (-1)^m P_\nu^{m}(x)\,\Gamma(\nu-m+1)$ // on the cut
$Q_\nu^{-m}(x)\,\Gamma(\nu+m+1) = (-1)^m Q_\nu^{m}(x)\,\Gamma(\nu-m+1)$

7. Pause to gather up all results so far:

Here is the "base form" for each function, z = general, x = on-the-cut version

$$P_\nu^{\mu}(z) = 1/\Gamma(1-\mu)\,[(z+1)/(z-1)]^{\mu/2}\;F[\;-\nu,\nu+1;\,1-\mu;\,(1-z)/2\;]$$
$$P_\nu^{\mu}(x) = 1/\Gamma(1-\mu)\,[\;(1+x)/(1-x)]^{\mu/2}\;F[\;-\nu,\nu+1;\,1-\mu;\,(1-x)/2\;]$$

$$Q_\nu^{\mu}(z) = (1/2)\,\Gamma(1+\nu+\mu)\,\Gamma(-\mu)/\Gamma(1+\nu-\mu) * (z-1)^{\mu/2}\,(z+1)^{-\mu/2}F[\;-\nu,1+\nu;\,1+\mu;\,(z-1)/2\;] + (1/2)\,\Gamma(\mu)\,(z-1)^{-\mu/2}(z+1)^{\mu/2}\;F[\;-\nu,1+\nu;\,1-\mu;\,(z-1)/2\;]$$

$$Q_\nu^{\mu}(x) = (1/2)\Gamma(1+\nu+\mu)\,\Gamma(-\mu)/\Gamma(1+\nu-\mu) * (1-x)^{\mu/2}\,(1+x)^{-\mu/2}\,F[\;-\nu,1+\nu;\,1+\mu;\,(1-x)/2\;] + \cos(\pi\mu)\,(1/2)\,\Gamma(\mu)\,(1-x)^{-\mu/2}\,(1+x)^{\mu/2}\,F[\;-\nu,1+\nu;\,1-\mu;\,(1-x)/2\;]$$

The on-the-cut versions of P and Q are real for z in (-1,1) if the parameters are real.

Here is a general method to modify most of the joining relations, shown here in "symbolic form"

P: $P(z) = e^{\pm i\pi a}\,A(z-1)^{\alpha}\,F + e^{\pm i\pi b}\,B(z-1)^{\beta}\,F'$
$P(x) = \cos\pi(\alpha+a+\mu/2)\,A\,|z-1|^{\alpha}\,F + \cos\pi(\beta+b+\mu/2)\,B\,|z-1|^{\beta}\,F'$

Q: $e^{-i\pi\mu}\,Q(z) = e^{\pm i\pi a}\,A(z-1)^{\alpha}\,F + e^{\pm i\pi b}\,B(z-1)^{\beta}\,F'$
$Q(x) = \cos\pi(\alpha+a-\mu/2)\,A\,|z-1|^{\alpha}\,F + \cos\pi(\beta+b-\mu/2)\,B\,|z-1|^{\beta}\,F'$

The Wronskian for regular P and Q is page 123 (13).

The Wronskian for on-the-cut P and Q is page 146 (25).

When μ = m, an integer, we have these results:

$P_\nu^{m}(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ m = 1,2,3....
$P_\nu^{m}(x) = (-1)^m\,(1-x^2)^{m/2}\partial_m P_\nu(x)$ m = 1,2,3.... // "CS phase"

$Q_\nu^{m}(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....
$Q_\nu^{m}(x) = (-1)^m\;(1-x^2)^{m/2}\;\partial_m Q_\nu(x)$ m = 1,2,3... // "CS phase"

$P_n(z) = (2^n n!)^{-1} \partial_n (z^2-1)^n$ n = 1,2,3... // Rodriguez
$P_n(x) = (2^n n!)^{-1} \partial_n (x^2-1)^n$ n = 1,2,3.. // Rodriguez (same)

$Q_n(z) = (1/2)\, P_n(z)\, \log[(z+1)/(z-1)] - W_{n-1}(z)$ // p 153
$Q_n(x) = (1/2)\, P_n(x)\, \log[(1+x)/(1-x)] - W_{n-1}(x)$ // p 153

$P_\nu^{-m}(z)\, \Gamma(\nu+m+1) = P_\nu^m(z)\, \Gamma(\nu-m+1)$ // any integer m
$P_\nu^{-m}(x)\, \Gamma(\nu+m+1) = (-1)^m P_\nu^m(x)\, \Gamma(\nu-m+1)$ // any integer m

$Q_\nu^{-m}(z)\, \Gamma(\nu+m+1) = Q_\nu^m(z)\, \Gamma(\nu-m+1)$ // any integer m
$Q_\nu^{-m}(x)\, \Gamma(\nu+m+1) = (-1)^m Q_\nu^m(x)\, \Gamma(\nu-m+1)$ // any integer m

8. Comparison of Bateman/me to Morse Feshbach

First I throw a bunch of MF clips into Appendix A below, then fetch up things in some reasonable order:

1. The following Rodriguez agrees with mine above, could be either regular or on the cut.

$$P_n(z) = T_n^0(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

2. Below agrees with NEITHER my regular or cut versions! The exponent is really ½ m, just hard to see. This looks like on-the-cut but without the Condon-Shortley phase!

$$P_n^m(z) = (1 - z^2)^{\frac{1}{2}m} \frac{d^m}{dz^m} P_n(z)$$

$P_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \partial_m P_\nu(x)$ // my Bateman result on cut

3. I think this may be the key result below, which I rewrite

$$P_n^m(\cos\vartheta) = \sin^m\vartheta\, T_{n-m}^m(\cos\vartheta)$$

$P_n^m(x) = (1-x^2)^{m/2}\, T_{n-m}^m(x)$
$P_n(x) = T_n^0(x)$

Set n = n'+m in the above first line to get

$P_{n'+m}^m(x) = (1-x^2)^{m/2}\, T_{n'}^m(x)$

$P_{n+m}^m(x) = (1-x^2)^{m/2}\, T_n^m(x)$

$T_n^m(x) = (1-x^2)^{-m/2}\, P_{n+m}^m(x)$

and then my last line agrees with the MF statement below

$$T_n^m(z) = (1 - z^2)^{-\frac{1}{2}m} P_{n+m}^m(z)$$

Now, using his formula above, MF would say that

$$P_1^1(x) = (1-x^2)^{1/2}\, T_0^1(x) = (1-x^2)^{1/2} * 1 = (1-x^2)^{1/2}$$

Now according to Bateman, this should be

$$P_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \partial_m P_\nu(x)$$

$$P_1^1(x) = (-1) (1-x^2)^{1/2}$$

so yes, MF do use a different phase from Condon-Shortley!
And they use "on the cut" functions I would say.

4. This is a puzzling one for me:

$$(1-z^2)^{-\frac{1}{2}\beta} P^{\beta}_{\alpha+\beta}(z) = \frac{(z-1)^{-\beta}}{\Gamma(1-\beta)} F(-\alpha-\beta, \alpha+\beta+1 | 1-\beta | \tfrac{1}{2}-\tfrac{1}{2}z)$$

Let me first set $\beta = \mu$ and then $\alpha = \nu - \beta$ to get

$$(1-z^2)^{-\mu/2} P_\nu^\mu(z) = (z-1)^{-\mu} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-z)/2)$$

This is very confusing because they have both the 1-z and z-1 forms appearing together! Lets take a stab and write $(z-1)^{-\mu} = (1-z)^{-\mu} e^{-i\mu\pi}$ where I go above rather than below. Then we have

$$(1-z^2)^{-\mu/2} P_\nu^\mu(z) = (1-z)^{-\mu} e^{-i\mu\pi} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-z)/2)$$

$$(1-z)^{-\mu/2}(1+z)^{-\mu/2} P_\nu^\mu(z) = (1-z)^{-\mu} e^{-i\mu\pi} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-z)/2)$$

$$P_\nu^\mu(z) = (1-z)^{\mu/2} (1-z)^{-\mu} (1+z)^{\mu/2} e^{-i\mu\pi} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-z)/2)$$

$$P_\nu^\mu(z) = e^{-i\mu\pi} * (1-z)^{-\mu/2} (1+z)^{\mu/2} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-z)/2)$$

$$P_\nu^\mu(x) = e^{-i\mu\pi} * (1-x)^{-\mu/2} (1+x)^{\mu/2} / \Gamma(1-\mu) * F(-\nu, \nu+1; 1-\mu; (1-x)/2)$$

$$\Rightarrow \quad P_\nu^\mu(x)_{\mathbf{MF}} = e^{-i\mu\pi} P_\nu^\mu(x)_{\mathbf{Bateman}}$$

So there is that same sign factor I found in the example above. The sign in the phase is not clear, I just picked one. For integer μ of course get just $(-1)^m$ as in the example above.

5. How about this baby:

$$Q_n^m = (-1)^m(z^2 - 1)^{\frac{1}{2}m} \frac{d^m}{dz^m} Q_n(z)$$

which I can compare to my results

$Q_\nu^m(z) = (z^2\text{-}1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....
$Q_\nu^m(x) = (-1)^m\ (1\text{-}x^2)^{m/2}\ \partial_m Q_\nu(x)$ m = 1,2,3...

So this agrees with neither of my/Bateman results. I suspect they are NOT using the on-the-cut version of the Q functions as evidenced by some of their examples below, where you always see z first in the log. So I would say again that we have

$Q_\nu^\mu(z)_{MF} = e^{-i\mu\pi}\, Q_\nu^\mu(z)_{Bateman}$
$Q_\nu^m(z)_{MF} = (-1)^m\, Q_\nu^m(z)_{Bateman}$

so we have that same "extra" phase factor here. There is a certain logic here, however, in that the MF version is simpler in that it does not carry around all that $e^{-i\mu\pi}$ baggage that you see in Bateman.

6. My conclusions comparing MF and Bateman:

$P_\nu^\mu(x)_{MF} = e^{-i\mu\pi}\, P_\nu^\mu(x)_{Bateman}$ where Bateman is the on-the-cut version

$Q_\nu^\mu(z)_{MF} = e^{-i\mu\pi}\, Q_\nu^\mu(z)_{Bateman}$ where Bateman is the regular version

I think this then resolves the massive confusion one has when reading MF on this subject.

9. Comparison of Bateman/me to Smythe

His page 148 shows up these formulas,

where if $-1 < \mu < +1$, $P_n^m(\mu)$ and $Q_n^m(\mu)$ are defined by

$$P_n^m(\mu) = (1 - \mu^2)^{\frac{1}{2}m}\frac{d^m P_n(\mu)}{d\mu^m} \quad (6)$$

$$Q_n^m(\mu) = (1 - \mu^2)^{\frac{1}{2}m}\frac{d^m Q_n(\mu)}{d\mu^m} \quad (7)$$

For $-1 < \mu < +1$ Hobson introduces the factor $(-1)^m$ on the right. When μ is imaginary or real, but greater than unity, we define these functions by the equations

$$P_n^m(\mu) = (\mu^2 - 1)^{\frac{1}{2}m}\frac{d^m P_n(\mu)}{d\mu^m} \quad (8)$$

$$Q_n^m(\mu) = (\mu^2 - 1)^{\frac{1}{2}m}\frac{d^m Q_n(\mu)}{d\mu^m} \quad (9)$$

I should compare the first two to my on-the-cut versions

$P_\nu^m(x) = (-1)^m\,(1\text{-}x^2)^{m/2}\partial_m P_\nu(x)$ m = 1,2,3.... // "CS phase"
$Q_\nu^m(x) = (-1)^m\ (1\text{-}x^2)^{m/2}\ \partial_m Q_\nu(x)$ m = 1,2,3...

So Smythe *also* uses the non-CS phase, and even comments on it being in "Hobson". Then in the last two equations above, Smythe is talking about "general" z = μ, and I compare to Bateman

$P_\nu^m(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ m = 1,2,3....
$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....

So, Smythe *agrees* with Bateman for the general functions, and disagrees by $(-1)^m$ otherwise on the cut. (If I assume from the top, I could decide what this phase is for general μ, but let's not do that.)

So when Smythe writes "examples" for P and Q, how do you know which he is using? Well, just looking at these things, I would say they are ALL on the cut versions since 1 comes first:

$$
\begin{aligned}
P_1^1(\mu) &= (1-\mu^2)^{\frac{1}{2}} & P_3^3(\mu) &= 15(1-\mu^2)^{\frac{3}{2}} \\
P_2^1(\mu) &= 3(1-\mu^2)^{\frac{1}{2}}\mu & P_4^1(\mu) &= \tfrac{5}{2}(1-\mu^2)^{\frac{1}{2}}(7\mu^3-3\mu) \\
P_2^2(\mu) &= 3(1-\mu^2) & P_4^2(\mu) &= \tfrac{15}{2}(1-\mu^2)(7\mu^2-1) \\
P_3^1(\mu) &= \tfrac{3}{2}(1-\mu^2)^{\frac{1}{2}}(5\mu^2-1) & P_4^3(\mu) &= 105(1-\mu^2)^{\frac{3}{2}}\mu \\
P_3^2(\mu) &= 15(1-\mu^2)\mu & P_4^4(\mu) &= 105(1-\mu^2)^2
\end{aligned}
$$

$$
\begin{aligned}
Q_1^1(\mu) &= (1-\mu^2)^{\frac{1}{2}}\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right) \\
Q_2^1(\mu) &= (1-\mu^2)^{\frac{1}{2}}\left(\frac{3}{2}\mu\ln\frac{1+\mu}{1-\mu} + \frac{3\mu^2-2}{1-\mu^2}\right) \qquad (10) \\
Q_2^2(\mu) &= (1-\mu^2)\left(\frac{3}{2}\ln\frac{1+\mu}{1-\mu} + \frac{5\mu-3\mu^3}{(1-\mu^2)^2}\right)
\end{aligned}
$$

We can now compare the three authors for on-the-cut P_1^1 :

$P_1^1 = (1-x^2)^{1/2}$ Smythe
$P_1^1 = (1-x^2)^{1/2}$ MF
$P_1^1 = -(1-x^2)^{1/2}$ Bateman (Condon-Shortley phase)

For Q functions, Smythe would agree with MF, except MF uses regular and Smythe on-the-cut. As I think I mentioned elsewhere, Smythe never uses F and has the HG series written out only one time for something, as you find by searching his index.

Let's do one Bateman calculation to compare with the above.

$Q_1(z) = (1/2)\, z\, \ln([(z+1)/(z-1)] - 1$ regular

$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....

$Q_1^1(z) = (z^2-1)^{1/2}\partial_1 Q_1(z)$ m = 1,2,3....

$\partial_1 Q_1(z) = z\,(1/2)\,[\,1/(z+1) - 1/(z-1)] + (1/2)\ln([(z+1)/(z-1)]$

$= (1/2)\,[\,-z/(z^2-1)] + (1/2)\ln([(z+1)/(z-1)]$

$= - z/(z^2-1) + (1/2)\ln[(z+1)/(z-1)]$

$\Rightarrow Q_1^1(z) = (z^2-1)^{1/2}[- z/(z^2-1) + (1/2)\ln[(z+1)/(z-1)]$

$= (z^2-1)^{1/2}[(1/2)\ln[(z+1)/(z-1)] - z/(z^2-1)$ // Bateman general

Let's compare this first to MF who does general but with the other phase:

$$Q_1^1 = \sqrt{z^2-1}\left[\frac{z}{z^2-1} - \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\right]$$

So Bateman and MF agree except for a sign, as I expected.

Now how would I convert my Bateman result above to on-the-cut? Write it out:

$Q_1^1(z+i\varepsilon) = |z^2-1|^{1/2} e^{i\pi/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{i\pi}\}]$
$Q_1^1(z-i\varepsilon) = |z^2-1|^{1/2} e^{-i\pi/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{-i\pi}\}]$

As usual, add the prescription phases

$e^{-i\pi/2}Q_1^1(z+i\varepsilon) = |z^2-1|^{1/2} e^{-i\pi/2} e^{i\pi/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{i\pi}\}]$
$e^{+i\pi/2}Q_1^1(z-i\varepsilon) = |z^2-1|^{1/2} e^{+i\pi/2} e^{-i\pi/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{-i\pi}\}]$

$e^{-i\pi/2}Q_1^1(z+i\varepsilon) = |z^2-1|^{1/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{i\pi}\}]$
$e^{+i\pi/2}Q_1^1(z-i\varepsilon) = |z^2-1|^{1/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|e^{-i\pi}\}]$

In the difference the $i\pi$ will cancel so prescription then says

$Q_1^1(x) = e^{-i\pi 1} |z^2-1|^{1/2} [- z/(z^2-1) + (1/2)\ln[(z+1)/\{|z-1|\}]$

$= (-1) (1-x^2)|^{1/2} [x/(1-x^2) + (1/2)\ln[(1+x)/(1-x)|]$ // Bateman on the cut

Smythe above has

$$Q_1^1(\mu) = (1-\mu^2)^{\frac{1}{2}}\left(\frac{1}{2}\ln\frac{1+\mu}{1-\mu} + \frac{\mu}{1-\mu^2}\right)$$

so *as expected*, we differ only by an overall sign.

Here are some other Smythe facts just to get them included here.

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \tfrac{1}{2}(3\mu^2-1)$$
$$P_3(\mu) = \frac{1}{2}(5\mu^3-3\mu), \quad P_4(\mu) = \frac{(35\mu^4-30\mu^2+3)}{8}$$

$$P_5(\mu) = \frac{(63\mu^5 - 70\mu^3 + 15\mu)}{8}, \qquad P_6(\mu) = \frac{(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)}{16}$$

$$P_7(\mu) = \frac{(429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu)}{16}$$

$$P_8(\mu) = \frac{(6435\mu^8 - 12{,}012\mu^6 + 5930\mu^4 - 1260\mu^2 + 35)}{128}$$

$$Q_0(\mu) = \frac{1}{2} \ln \frac{1+\mu}{1-\mu}$$

$$Q_1(\mu) = \frac{1}{2}\mu \ln \frac{1+\mu}{1-\mu} - 1$$

$$Q_2(\mu) = \frac{3}{2}\mu Q_1(\mu) - \frac{1}{2}Q_0(\mu) = \frac{1}{4}(3\mu^2 - 1) \ln \frac{1+\mu}{1-\mu} - \frac{3\mu}{2}$$

(n even) $\qquad Q_n(0) = 0 \qquad (1)$

(n odd) $\qquad Q_n(0) = (-1)^{\frac{1}{2}(n+1)} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n} \qquad (2)$

(any n) $\qquad Q_n(-\mu) = (-1)^{n+1} Q_n(\mu) \qquad (3)$

$$Q_n(1) = \infty \qquad Q_n(\infty) = 0 \qquad (4)$$

My conclusions comparing Smythe and Bateman/me and MF

$P_\nu^\mu(x)_{\texttt{Smythe}} = e^{-i\mu\pi} P_\nu^\mu(x)_{\texttt{Bateman}} = P_\nu^\mu(x)_{\texttt{MF}}$ where Bateman is the on-the-cut version

$Q_\nu^\mu(x)_{\texttt{Smythe}} = e^{-i\mu\pi} Q_\nu^\mu(x)_{\texttt{Bateman}}$ where Bateman is on-the-cut version

$Q_\nu^\mu(z)_{\texttt{Smythe}} = Q_\nu^\mu(z)_{\texttt{Bateman}}$ where Bateman is regular

$Q_\nu^\mu(z)_{\texttt{MF}} = e^{-i\mu\pi} Q_\nu^\mu(z)_{\texttt{Bateman}}$ where Bateman is the regular version

and here I have just copied down a line from the MF to Bateman comparison.

Appendix A: Selections for MF regarding Legendre functions.

$$T_n^m(z) = \frac{1}{2^{n+m}(n+m)!} \frac{d^{n+2m}}{dz^{n+2m}} (z^2 - 1)^{n+m}$$

The following Rodriguez agrees with mine above, could be either regular or on the cut.

$$P_n(z) = T_n^0(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n$$

$$T_n^m(z) = \frac{d^m}{dz^m} P_{n+m}(z)$$

$$(1 - z^2)^{-\frac{1}{2}\beta} P^{\beta}_{\alpha+\beta}(z) = \frac{(z-1)^{-\beta}}{\Gamma(1-\beta)} F(-\alpha - \beta, \alpha + \beta + 1 | 1 - \beta | \tfrac{1}{2} - \tfrac{1}{2}z)$$

$$T^m_n(z) = (1 - z^2)^{-\frac{1}{2}m} P^m_{n+m}(z) = \frac{d^m}{dz^m} T^0_{m+n}(z)$$

$$= \frac{(n+2m)!}{2^{n+m} n!(n+m)!} \frac{1}{(z^2-1)^m} \frac{d^n}{dz^n} (z^2 - 1)^{n+m}$$

$$= \frac{1}{2^{n+m}(n+m)!} \frac{d^{n+2m}}{dz^{n+2m}} (z^2 - 1)^{n+m} \qquad (5.3.37)$$

$$T^{\beta}_{-\alpha-2\beta-1}(z) = \left[\frac{\sin \pi(\alpha + 2\beta)}{\sin(\pi\alpha)}\right] T^{\beta}_{\alpha}(z); \quad P^{\beta}_{\alpha+\beta}(z) = P^{\beta}_{-\alpha-\beta-1}(z)$$

Relation with hyygeomctric function:

$$T^{\beta}_n(z) = \frac{\Gamma(n + 2\beta + 1)}{2^{\beta} n! \Gamma(\beta + 1)} F\left(-n, n + 2\beta + 1 | 1 + \beta | \frac{1 - z}{2}\right)$$

$$= (-1)^n \frac{\Gamma(n + 2\beta + 1)}{2^{\beta} n! \Gamma(\beta + 1)} F\left(-n, n + 2\beta + 1 | 1 + \beta | \frac{1 + z}{2}\right)$$

Normalization integral:

$$\int_{-1}^{1} (1 - z^2)^{\beta} T^{\beta}_m(z) T^{\beta}_n(z)\, dz = \delta_{mn} \frac{2\Gamma(n + 2\beta + 1)}{(2n + 2\beta + 1)\Gamma(n + 1)}$$

$$P^m_n(\cos\vartheta) = \sin^m \vartheta T^m_{n-m}(\cos\vartheta)$$

$$Y^e_{mn} = \cos(m\phi) P^m_n(\cos\vartheta); \quad Y^0_{mn} = \sin(m\phi) P^m_n(\cos\vartheta) \quad (10.3.25)$$

$$P^m_n(z) = (1 - z^2)^{\frac{1}{2}m} \frac{d^m}{dz^m} P_n(z) = (1 - z^2)^{\frac{1}{2}m} T^m_{n-m}(z)$$

$$= \frac{(1 - z^2)^{\frac{1}{2}}}{2^n n!} \frac{d^{m+n}}{dz^{m+n}} (z^2 - 1)^n; \quad m, n = 0, 1, 2, 3, \ldots; \quad m \le n$$

$$= \frac{(n+m)!(1-z^2)^{\frac{1}{2}m}}{2^m m!(n-m)!} F\left(m - n, m + n + 1 | m + 1 | \frac{1 - z}{2}\right)$$

$$= \frac{i^m (2n)!}{2^n n!(n-m)!} (z^2 - 1)^{\frac{1}{2}m} z^{n-m} F\left(\frac{m-n}{2}, \frac{m-n+1}{2} | \tfrac{1}{2} - n | \frac{1}{z^2}\right)$$

$$\int_{-1}^{1} P^m_n(z) P^m_l(z)\, dz = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nl}$$

$$\int_{-1}^{1} \frac{P^m_n(z) P^k_n(z)}{1 - z^2}\, dz = \frac{1}{m} \frac{(n+m)!}{(n-m)!} \delta_{mk}$$

$$Q_n^m = (-1)^m(z^2-1)^{\frac{1}{2}m}\frac{d^m}{dz^m}Q_n(z) = (-1)^m(z^2-1)^{\frac{1}{2}m}V_{n-m}^m(z)$$

$$= (z^2-1)^{\frac{1}{2}m}\frac{2^m\Gamma(\frac{1}{2})(n+m)!}{\Gamma(n+\frac{3}{2})(2z)^{n+m+1}}F\left(\frac{n+m+1}{2},\frac{n+m+2}{2}\,|n+\tfrac{3}{2}|\,\frac{1}{z^2}\right)$$

$$Q_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[(z^2-1)^n\ln\left(\frac{z+1}{z-1}\right)\right] - \tfrac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right)$$

For $z = \cosh\mu$; $\coth(\frac{1}{2}\mu) = \sqrt{\dfrac{z+1}{z-1}}$;

$\frac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right) = \ln[\coth(\frac{1}{2}\mu)] = 2\tanh^{-1}(e^{-\mu})$:

$$Q_0^0 = \tfrac{1}{2}\ln\left(\frac{z+1}{z-1}\right) = \ln[\coth(\tfrac{1}{2}\mu)];$$

$$Q_1^0 = \tfrac{1}{2}z\ln\left(\frac{z+1}{z-1}\right) - 1 = \cosh\mu\ln[\coth(\tfrac{1}{2}\mu)] - 1$$

$$Q_2^0 = \tfrac{1}{4}(3z^2-1)\ln\left(\frac{z+1}{z-1}\right) - \tfrac{3}{2}z$$

$$= \tfrac{1}{8}[3\cosh(2\mu)+1]\ln[\coth(\tfrac{1}{2}\mu)] - \tfrac{3}{2}\cosh\mu$$

$$Q_1^1 = \sqrt{z^2-1}\left[\frac{z}{z^2-1} - \tfrac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\right] = \coth\mu - \sinh\mu\ln[\coth(\tfrac{1}{2}\mu)]$$

$$Q_2^1 = \sqrt{z^2-1}\left[\frac{3z^2-2}{z^2-1} - \tfrac{3}{2}z\ln\left(\frac{z+1}{z-1}\right)\right]$$

$$= \operatorname{csch}\mu + 3\sinh\mu - \tfrac{3}{2}\sinh(2\mu)\ln[\coth(\tfrac{1}{2}\mu)]$$