Properties of Legendre Functions PhL 3.26.05

The last word here for me is Bateman, be careful to distinguish between on-the-cut and regular off cut versions of the functions, since properties are different. Other sources include Schaum, A&S, GR, Smythe, Jackson, see comments below.

Legendre Functions on the Cut ; Using the Bateman Tables. 1
Wronskians 2
General Properties 3
Comparison of Bateman with different data sources. 6
Summary of Results on the Associated Legendre Polynomials $P_n^m(z)$ -- integer m,n 7
Appendix A: Alternate forms for values at 0 9
The P calculation 9
The Q calculation 11
Appendix B. Other formula derivations: 14
Appendix C. More derivations 17
Appendix D. A study of the condition $Q_n^m(j0+) = 0$ 18
Appendix E. A study of the condition $P_n^m(z) = 0$ for all z. 21
Appendix F. More general results for derivatives at z = 0. 24

Lower index is called the **degree**, upper index is called the **order**.

Legendre Functions on the Cut ; Using the Bateman Tables.

Prescriptions are found p I.143 Bateman, where, in our notation here, no subscript means regular, and "cut" means we are talking about the "on the cut" version of the function (Bateman uses italic for regular, non italic for cut versions)

$$P_\nu^\mu(x)_{cut} \equiv (1/2)\left[e^{+i\pi\mu/2} P_\nu^\mu(x+i\varepsilon) + e^{-i\pi\mu/2} P_\nu^\mu(x-i\varepsilon) \right]$$

$$e^{-i\mu\pi} Q_\nu^\mu(x)_{cut} \equiv (1/2)\left[e^{-i\pi\mu/2} Q_\nu^\mu(x+i\varepsilon) + e^{+i\pi\mu/2} Q_\nu^\mu(x-i\varepsilon) \right]$$

Since both RHS's are of the form B + B*, we see that the cut functions are real on the cut, whereas the original functions are real on the real axis to the right of x = 1. Whereas the regular functions are defined for general complex z, the on-the-cut P and Q functions are only defined for real x in (-1,1).

When μ=m and ν=n, the $P_n^m(z)$ functions have no cut on (-1,1), and in this case we have $P_n^m(z)_{cut} = P_n^m(z)$. But the $Q_n^m(z)$ still have a cut so the two forms remain different.

There are many different ways to write P and Q in terms of F functions, and they all involve factors of this form $(z-1)^\alpha$. These are shown in the huge table of Bateman starting on page 124. It is convenient to use a compact symbolic form for these various expressions, as follows:

$$P_\nu^\mu(z) = e^{+i\pi a} A(z-1)^\alpha F + e^{+i\pi b} B(z-1)^\beta F'$$
$$e^{-i\pi\mu} Q_\nu^\mu(z) = e^{-i\pi a} A(z-1)^\alpha F + e^{-i\pi b} B(z-1)^\beta F'$$

We show the HG functions just as F or F', and we gather up constant factors and those involving z^α or $(1+z)^\alpha$ into the "constants" A and B. Obviously the A and B are not meant to be the same for the different entries in the table, nor are they meant to be the same for P and Q, and the same comment applies to F, F', α, β. We simply want to focus on the $(z-1)^\alpha$ type factors, and this compact notation helps us do that.

The issue is dealing with the phase of factors of the form $(z-1)^\alpha$ as we swing around onto the branch point at z = 1 from a point on the real axis to the right z = 1. Obviously the phase is different above versus below. If we carefully apply our "prescriptions" shown above to the functions above (see doc on this subject), we get these results:

P:
$$P(z) = e^{+i\pi a} A(z-1)^\alpha F + e^{+i\pi b} B(z-1)^\beta F'$$
$$P(x) = \cos\pi(\alpha+a+\mu/2)\, A(1-x)^\alpha F + \cos\pi(\beta+b+\mu/2)\, B\,(1-x)^\beta F'$$

Q:
$$e^{-i\pi\mu} Q(z) = e^{+i\pi a} A(z-1)^\alpha F + e^{+i\pi b} B(z-1)^\beta F'$$
$$Q(x) = \cos\pi(\alpha+a-\mu/2)\, A(1-x)^\alpha F + \cos\pi(\beta+b-\mu/2)\, B(1-x)^\beta F'$$

This says that a rule of simply replacing $(z-1)^\alpha$ with $(1-z)^\alpha e^{\pm i\pi\alpha}$ is *not correct* to convert a table entry into a form that applies to on-the-cut functions. You have to examine specifically the α,β,a,b exponents and then supply the cosine factors as shown. In some cases you will get cos(0) = 1. As examples, p 143 (6) shows an expression for P_{cut} where there is only one term, cos=1, so the only change is replacing z-1 by 1-x. But p 144 (10) shows an expression for Q_{cut} that involves a non-zero cosine factor in the second term only.

HOWEVER: The above rule is not valid if the Bateman table entry has a factor like $e^{\pm ic}$! In this case you have to figure things out by brute force.

Example 1: Consider Bateman p 126 (22). It has α = β = -μ/2 and a = b= 0, so for $P_\nu^\mu(x)$ we pick up a factor cos(0) = 1. So this particular relation is as stated with $(z-1)^\alpha$ replaced by $(1-x)^\alpha$. We use it below.

Example 2: Consider Bateman p 134 (40). It has α = β = -μ/2, a = (μ-ν-1)/2 and b = (μ-ν)/2. When we restate this for on-the-cut we get

$$\begin{aligned} Q(x) &= \cos\pi(-\mu/2+(\mu-\nu-1)/2-\mu/2)\, A(1-x)^\alpha F + \cos\pi(-\mu/2+(\mu-\nu)/2-\mu/2)\, B(1-x)^\beta F' \\ &= \cos\pi(-\mu+(\mu-\nu-1)-\mu)/2\, A(1-x)^\alpha F + \cos\pi(-\mu+(\mu-\nu)-\mu)/2\, B(1-x)^\beta F' \\ &= \cos\pi(-\mu-\nu-1)/2\, A(1-x)^\alpha F + \cos\pi(-\mu-\nu)/2\, B(1-x)^\beta F' \\ &= \cos\pi[(\mu+\nu+1)/2]\, A(1-x)^\alpha F + \cos\pi[(\mu+\nu)/2]\, B(1-x)^\beta F' \\ &= -\sin\pi[(\mu+\nu)/2]\, A(1-x)^\alpha F + \cos\pi[(\mu+\nu)/2]\, B(1-x)^\beta F' \end{aligned}$$

Bateman shows various properties of "on the cut" P and Q functions on pages 144-146 including the Wronskian on page 146.

Wronskians

(a) Regular: For the regular Legendre functions, Bateman p 123 gives the following Wronskian

$$W[\,P_\nu^\mu(z),\, Q_\nu^\mu(z)] \equiv P_\nu^\mu(z)\, Q'^\mu_\nu(z) - P'^\mu_\nu(z)\, Q_\nu^\mu(z)$$

$$= e^{i\mu\pi}\; 2^{2\mu}\; (1-z^2)^{-1}\; \Gamma(1 + \mu/2 + \nu/2)\, \Gamma(1/2 + \mu/2 + \nu/2) \,/\, [\Gamma(1 - \mu/2 + \nu/2)\, \Gamma(1/2 - \mu/2 + \nu/2)]$$

In my doc "the Smythe method for Green's.." I show this can be written in this simpler form

$$W[\,P_\nu^\mu(z),\, Q_\nu^\mu(z)] = e^{i\mu\pi}\,(1-z^2)^{-1}[\Gamma(1+\mu+\nu)/\,\Gamma(1-\mu+\nu)\,]$$

though I cannot find direct verification of this result from any source, web or otherwise. In the case that μ is an integer I get this confirmation from a Kent State PDF I found.

$$W(P_\nu^m(z), Q_\nu^m(z)) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\frac{(-1)^m}{1-z^2}.$$

If m and n are integers, both my and the Kent State result become

$$W[\,P_n^m(z),\, Q_n^m(z)] = (-1)^m\; (1-z^2)^{-1}\; [(n+m)!\,/\,(n-m)!] \qquad //\ \text{m,n integers}$$

(b) On the Cut: I have not done this for myself, but Bateman gives the result on page 146. Everything is exactly the same except there is no $e^{i\mu\pi}$ factor , and thus no $(-1)^m$ factor for integer m. So:

$$W[\,P_\nu^\mu(x)_{cut},\, Q_\nu^\mu(x)_{cut}]$$

$$= 2^{2\mu}\; (1-x^2)^{-1}\; \Gamma(1 + \mu/2 + \nu/2)\, \Gamma(1/2 + \mu/2 + \nu/2) \,/\, [\Gamma(1 - \mu/2 + \nu/2)\, \Gamma(1/2 - \mu/2 + \nu/2)]$$

$$= (1-x^2)^{-1}[\Gamma(1+\mu+\nu)/\,\Gamma(1-\mu+\nu)\,]$$

and

$$W[\,P_n^m(x)_{cut},\, Q_n^m(x)_{cut}] = (1-x^2)^{-1}\; [(n+m)!\,/\,(n-m)!]$$

General Properties

Reflection of the degree:

$$P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$$
$$P_{-\nu-1}^\mu(x) = P_\nu^\mu(x)$$

$$Q_{-\nu-1}^\mu(z)\,\sin\pi(\nu-\mu) = Q_\nu^\mu(z)\,\sin\pi(\nu+\mu) - \pi\, e^{i\pi\mu}\cos\pi\nu\, P_\nu^\mu(z)$$
$$Q_{-\nu-1}^\mu(x)\,\sin\pi(\nu-\mu) = Q_\nu^\mu(x)\,\sin\pi(\nu+\mu) - \pi\, e^{i\pi\mu}\cos\pi\nu\, P_\nu^\mu(x)\cos\pi\mu$$

Negation of the order:

$[\ \Gamma(\nu+\mu+1)/\ \Gamma(\nu-\mu+1)\ ]\ Q_\nu^{-\mu}(z) = e^{-2i\pi\mu}\ Q_\nu^{\mu}(z)$
$[\ \Gamma(\nu+\mu+1)/\ \Gamma(\nu-\mu+1)\ ]\ Q_\nu^{-\mu}(x) = \cos\pi\mu\ Q_\nu^{\mu}(x) + (\pi/2)\ \sin\pi\mu\ P_\nu^{\mu}(x)$

$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ Q_\nu^{-m}(z) = Q_\nu^{m}(z)$
$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ Q_\nu^{-m}(x) = (-1)^m\ Q_\nu^{m}(x)$

$[\ \Gamma(\nu+\mu+1)/\ \Gamma(\nu-\mu+1)\ ]\ P_\nu^{-\mu}(z) = P_\nu^{\mu}(z) - (2/\pi)\ \sin\pi\mu\ Q_\nu^{\mu}(z)\ e^{-i\pi\mu}$
$[\ \Gamma(\nu+\mu+1)/\ \Gamma(\nu-\mu+1)\ ]\ P_\nu^{-\mu}(x) = \cos\pi\mu\ P_\nu^{\mu}(x) - (2/\pi)\ \sin\pi\mu\ Q_\nu^{\mu}(x)$

$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ P_\nu^{-m}(z) = P_\nu^{m}(z)$
$[\ \Gamma(\nu+m+1)/\ \Gamma(\nu-m+1)\ ]\ P_\nu^{-m}(x) = (-1)^m\ P_\nu^{m}(x)$

Negation of the argument:

$P_\nu^{\mu}(-z) = P_\nu^{\mu}(z)\ e^{\mp i\pi\nu} - (2/\pi)\ \sin\pi(\nu+\mu)\ Q_\nu^{\mu}(z)\ e^{-i\pi\mu}$
$P_\nu^{\mu}(-x) = P_\nu^{\mu}(x)\ \cos\pi(\nu+\mu) - (2/\pi)\ \sin\pi(\nu+\mu)\ Q_\nu^{m}(x)$

$P_\nu^{\mu}(-z) = P_\nu^{\mu}(z)\ e^{\mp i\pi\nu}$ // $\nu+\mu$ = integer
$P_\nu^{\mu}(-x) = P_\nu^{\mu}(x)(-1)^{\nu+\mu}$ // $\nu+\mu$ = integer

$Q_\nu^{\mu}(-z) = -\ Q_\nu^{\mu}(z)\ e^{\pm i\pi\nu}$
$Q_\nu^{\mu}(-x) = -\ Q_\nu^{\mu}(x)\ \cos\pi(\nu+\mu) - (\pi/2)\ P_\nu^{\mu}(x)\ \sin\pi(\nu+\mu)$

Value at z or x = ±1: *

In general, at z or x = ±1 we find that P = 0 or ∞ with one exception, $P_0^0(1) = 1$. For this reason, the results apply to both $P_\nu^{\mu}(z = \pm1)$ and $P_\nu^{\mu}(x = \pm1)$. Here is a list of the P = 0 cases.

<u>the P(+1) finite cases:</u> $P_\nu^{\mu}(1) = 0$ in these cases : [exception $P_\nu^0(1) = 1$] , else $P_\nu^{\mu}(1) = \infty$
ν = anything:
Re(μ) < 0 and $\mu \neq$ integer
μ = integer $P_\nu^{\mu}(1) = \delta_0^{\mu}$

<u>the P(-1) finite cases:</u> $P_\nu^{\mu}(-1) = 0$ in these cases : [exception $P_n^0(-1) = (-1)^n$] , else $P_\nu^{\mu}(-1) = \infty$
ν = integer:
Re(μ) > 0 and $\mu \neq$ integer
μ = 1,2,3.... // in this case, $P_\nu^{\mu}(z) \equiv 0$ for $-\mu \leq \nu < \mu$
μ = -1, -2, -3 and ν = integer and ![$-|\mu| \leq \nu < |\mu|$] // ie, ! [$\mu \leq \nu < -\mu$]
spectrum: μ = integer and $\nu = |\mu|, |\mu|+1, \ldots$ can use either $P_\nu^{\mu}(z)$ or $P_\nu^{|\mu|}(z)$
$P_\nu^{\mu}(z) \equiv 0$: $\mu = 1,2,3....$ ν = integer $(-\mu,\mu-1)$ as noted 5 lines above

ν = special integrally-spaced values shown here:
Re(μ) < 0 and $\mu \neq$ integer and [$\nu = \mu - J$ or $\nu = -\mu + I$]

spectrum: $\nu = -\mu, -\mu+1, \ldots$ and use $P_\nu^\mu(z)$

the Q(±1) finite cases: else $Q_\nu^\mu(\pm1) = \infty$

$\mathrm{Re}(\mu) < 0$ and $\mu \neq$ integer or half integer $Q_\nu^\mu(\pm1) = 0$ for $\nu = \mu - J$ only

* See "Legendre Functions Evaluated at z = +1 and -1.doc" summary section.
* See "Legendre SL Problem and Spherical Harmonics.doc"

Value at z or x = 0: [The notation 0± means 0 ±iζ . Smythe writes j0± for the same thing.]

$P_\nu^\mu(z=0\pm) = 2^\mu\sqrt{\pi}\ e^{\mp i\pi\mu/2} / [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$ // p 126 (22) + App E

$P_\nu^\mu(x=0) = 2^\mu\sqrt{\pi}\ / [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$ // Example 1 above

$e^{-i\pi\mu}Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\ 2^{\mu-1}\ e^{\mp i\pi(\nu+1)/2}\ [\ \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)]$ // p 134 (40) + App E.

$Q_\nu^\mu(x=0\ ) = -\sqrt{\pi}\ 2^{\mu-1}\ \sin\pi[(\mu+\nu)/2]\ [\ \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)]$ // Example 2 above

For $\mu = m$ and $\nu = n$ both integers, $n \geq |m|$, we get the following results: (see Appendix A)

$P_n^m(z=0\pm) = (-1)^{(n+m)/2}\ (n+m-1)!!/\ (n-m)!!\ e^{\mp i\pi m/2}$ // n+m = even

$P_n^m(z=0\pm) = 0$ // n+m = odd

$P_n^m(x=0) = (-1)^{(n+m)/2}\ (n+m-1)!!/\ (n-m)!!$ // n+m = even

$P_n^m(x=0) = 0$ // n+m = odd

$e^{-i\pi m}\ Q_n^m(z=0\pm) = (\pi/2)\ (\mp i)^{n+1}\ (n+m-1)!!/\ (n-m)!!$ // n+m = even

$e^{-i\pi m}\ Q_\nu^m(z=0\pm) = (\mp i)^{n+1}\ (n+m-1)!!/\ (n-m)!!$ // n+m = odd

$Q_n^m(x=0) = 0$ // n+m = even

$Q_n^m(x=0) = (-1)^{(n+m+1)/2}\ (n+m-1)!!/\ (n-m)!!$ // n+m = odd

Expressed as derivatives: (see "legendre on the cut" doc.)

$P_\nu^m(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ m = 1,2,3....

$P_\nu^m(x) = (-1)^m\ (1-x^2)^{m/2}\partial_m P_\nu(x)$ m = 1,2,3.... // "CS phase"

$Q_\nu^m(z) = (z^2-1)^{m/2}\partial_m Q_\nu(z)$ m = 1,2,3....

$Q_\nu^m(x) = (-1)^m\ (1-x^2)^{m/2}\ \partial_m Q_\nu(x)$ m = 1,2,3... // "CS phase"

$P_n(z) = (2^n n!)^{-1}\partial_n\ (z^2-1)^n$ n = 1,2,3... // Rodriguez

$P_n(x) = (2^n n!)^{-1}\partial_n\ (x^2-1)^n$ n = 1,2,3.. // Rodriguez (same)

$Q_n(z) = (1/2)\ P_n(z)\ \log[\ (z+1)/(z-1)]\ - W_{n-1}(z)$ // p 153

$Q_n(x) = (1/2)\, P_n(x)\, \log[(1+x)/(1-x)] - W_{n-1}(x)$ // p 153

First derivatives at x = z = 0: (see Appendix B)

$P_\nu^{m\,\prime}(x=0) = -P_\nu^{m+1}(x=0)$ $m \geq 0$ and integer
$Q_\nu^{m\,\prime}(x=0) = -Q_\nu^{m+1}(x=0)$ $m \geq 0$ and integer

$P_\nu^{m\,\prime}(0\pm) = \mp i\, P_\nu^{m+1}(0\pm)$ $m \geq 0$ and integer
$Q_\nu^{m\,\prime}(0\pm) = \mp i\, Q_\nu^{m+1}(0\pm)$ $m \geq 0$ and integer

$P_\nu^{-m\,\prime}(0\pm) = \mp i\, P_\nu^{m+1}(0\pm)\, \Gamma(\nu-m+1)/\Gamma(\nu+m+1)$ $m \geq 0$ and integer
$Q_\nu^{-m\,\prime}(0\pm) = \mp i\, Q_\nu^{m+1}(0\pm)\, \Gamma(\nu-m+1)/\Gamma(\nu+m+1)$ $m \geq 0$ and integer

First derivatives at z = 0: (see Appendix F, these are valid for general m and n)

$P_n^{m\,\prime}(0\pm i\varepsilon) = -e^{\mp i n m/2}\, \sqrt{\pi}\, 2^{m+1}/[\Gamma(1/2+n/2-m/2)\, \Gamma(-n/2-m/2)]$ // two derivations agree
$Q_n^{m\,\prime}(0\pm i\varepsilon) = e^{+i n m}\, e^{\mp i n n/2} \sqrt{\pi}\, 2^m\, [\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$ // two derivations agree

Large z limits: (the two Q limits are the same due to the gamma duplication formula)

$e^{-i\pi\mu} Q_\nu^\mu(z) = 2^\nu\, [\Gamma(1+\nu+\mu)\, \Gamma(1+\nu)/\Gamma(2+2\nu)]\, z^{-\nu-1}$ $\nu \neq -1,-2.....$ // Bateman p 132 (36)

$e^{-i\pi\mu} Q_\nu^\mu(z) = \sqrt{\pi}\, [\Gamma(1+\nu+\mu)/\Gamma(\nu+3/2)]\, (2z)^{-\nu-1}$ $\nu \neq -3/2,-5/2.....$ // Bateman p 134 (41)

$P_\nu^\mu(z) = (2^\nu/\sqrt{\pi})\, [\Gamma(\nu+1/2)/\Gamma(1+\nu-\mu)]\, z^\nu$ $\mathrm{Re}(\nu) > -1/2$ // Bateman p 126 (23)
// see p 164 for $\nu<-1/2$
$P_{-1/2}(z) = (\sqrt{2}/\pi)\, \ln[8z]/\sqrt{z}$ // see "the charged donut.doc", via K function

Parameter poles:

$Q_\nu^\mu(z)$ has a pole when $\nu = -\mu-1,-\mu-2$ .. for all z.

Comparison of Bateman with different data sources.

1. Morse and Feshbach. See "legendre on cut" for derivation of this claim:

$P_\nu^\mu(x)_{MF} = e^{-i\mu\pi}\, P_\nu^\mu(x)_{cut}$

$Q_\nu^\mu(z)_{MF} = e^{-i\mu\pi}\, Q_\nu^\mu(z)$

For integer m we would say

$P_n^m(x)_{MF} = (-1)^m P_n^m(x)$
$Q_n^m(z)_{MF} = (-1)^m Q_n^m(z)$

where no subscript means Bateman, and z or x says which Bateman. So we see that MF have extra phase factors (the same one) for their P and Q functions, so one says that they "do not use the Condon Shortley phase", whereas Bateman "does use the Condon Shortley phase". Here is an example:

$P_1^1 = (1-x^2)^{1/2}$	Smythe and MF (non C-S phase)
$P_1^1 = -(1-x^2)^{1/2}$	Bateman (Condon-Shortley phase)

MF tend to use a certain T function in place of the P functions:

$$T_n^m(x) \equiv (1-x^2)^{-m/2} P_{n+m}^m(x)_{MF} = (1-x^2)^{-m/2} (-1)^m P_{n+m}^m(x)$$

where recall the Bateman result for integer μ,ν is the same for on cut or off cut versions.

2. Smythe . See same source. My conclusion is that for regular P and Q functions, Smythe is the same as Bateman. But for on-the-cut functions, Smythe is the same as MF and uses the "non CS phase". So I extend the on-the-cut MF results above:

$P_n^m(x)_{Smythe} = P_n^m(x)_{MF} = (-1)^m P_n^m(x)$
$Q_n^m(z)_{Smythe} = Q_n^m(z)_{MF} = (-1)^m Q_n^m(z)$

3. A&S. A glance at their basic definitions for P and Q shows they are same as Bateman. And their on the cut prescriptions also agree. AS also use x and z arguments to distinguish which function type they are talking about. So the answer here is " AS is the same as Bateman on everything. "

4. GR p 998. This source also agrees 100% with Bateman, for on and off cut functions.

5. Schaum p 149. This source works only with on-the-cut functions, and agrees with MF and Smythe, which is to say, they use the non-CS phase convention and differ in this way from Bateman.

6. Jackson. Uses only integer m and n, but always writes n as ℓ . Page 64 (3.49) shows agreement with Bateman, so Jackson uses the CS phase and even comments on that in his footnote.

Summary of Results on the Associated Legendre Polynomials $P_n^m(z)$ -- integer m,n

1. The ODE for associated Legendre functions is this: (take your choice)

$L = (1-z^2) D^2 - 2z D + [n(n+1) - m^2/(1-z^2)]$ $\quad Lu = 0$ Bateman p 121

$L' = D^2 - 2z/(1-z^2) D + [n(n+1)/ (1-z^2) - m^2/(1-z^2)^2] = D^2 + pD + q$ $\quad L'u = 0$

Notice that p has a pole at z = +1 and -1, and that q has a pole and a double pole at these two locations. There is also a regular singularity at z=∞. Think of m as a fixed non-negative integer. This ODE as we well know has three regular singular points at 1,-1 and ∞ and no irregular singular points, so it is very closely tied in with the hypergeometric ODE and its F function solutions. Recall that the HG ODE has three standard singular points 0,1,∞ but these are moved to -1,1,∞ for Legendre and the associated Legendre functions have this Papperitz (Riemann P function) designation

The Legendre equation therefore corresponds to the P-function

$$P\left\{\begin{array}{cccc} 1 & \infty & -1 & \\ m/2 & -n & -m/2 & z \\ -m/2 & n+1 & m/2 & \end{array}\right\}.$$

which shows the Frobenius exponents (indices) for solutions as power series about the singular points. Our most common expansion point is z = 1 where we are used to seeing $(z-1)^{\pm\mu/2}$ factors.

2. For m a fixed integer, the eigenfunctions of this S-L system are the $P_n^m(z)$ with n = m,m+1....∞.

3. The orthogonality of these eigenfunctions is given by

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = \delta_{n,k}\ (n+1/2)^{-1}\ [(n+m)!\ /\ (n-m)!\ ] \qquad n,k = m, m+1, m+2\ .....\ \infty$$

$$= \delta_{n,k}\, K_n^m \qquad K_n^m = (n+1/2)^{-1}\ (n+m)!\ /\ (n-m)!$$

You can think of $P_n^m(z) = 0$ for n < m if you want. $\qquad K_n^0 = (n+1/2)^{-1}$

4. The completeness of these eigenfunctions is given by:

$$\Sigma_{n=m}^{\infty}(1/K_n^m)\ P_n^m(z')\ P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\ (n+m)!\ /\ (n-m)!$$

4A. The above two items can be written in standard form for normalized $\varphi_n^m(z)$:

$$\Sigma_{n=m}^{\infty}\ \varphi_n^m(z')\ \varphi_n^m(z) = \delta(z'-z) \qquad \varphi_n^m(z) = P_n^m(z)\ /\ \sqrt{K_n^m} \qquad \text{// complete}$$

$$\int_{-1}^{1} dz\, \varphi_n^m(z)\varphi_k^m(z) = \delta_{n,k} \qquad n,k = m, m+1, m+2\ .....\ \infty \qquad \text{// orthonormal}$$

Think about QM bra-ket notation here and everything is perfect.

5. In the special case that m = 0 we get the following orthogonality from (3) and completeness from (4)

$$\int_{-1}^{1} dz\, P_n(z) P_k(z) = \delta_{n,k}\ (n+1/2)^{-1} \qquad n,k = 0,1...\ \infty$$

$$\Sigma_{n=0}^{\infty}(n+1/2)\ P_n(z')\ P_n(z) = \delta(z'-z)$$

6. The above results are compatible with Jackson's following statements about spherical harmonics,

$Y_{nm}(\theta,\varphi) \equiv [\ \{(2n+1)/4\pi\}\ (n-m)!/(n+m)!\ ]^{1/2}\ P_n^m(z)\ e^{im\varphi}$	definition	3.53 Jackson
$\Sigma_{n=0}^{\infty}\Sigma_{m=-n}^{n}\ Y_{nm}^*(\theta',\varphi')\ Y_{nm}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(z-z')$	completeness	3.56 Jackson
$\int d\Omega\ Y_{nm}^*(\theta,\varphi)\ Y_{n'm'}(\theta,\varphi) \ = \delta_{n,n'}\delta_{m,m'}$	orthogonality	3.55 Jackson
$P_n^{-m}(z) = (-1)^m(n-m)!/(n+m)!\ P_n^m(z)$	m reflection	3.51 Jackson

but it takes a bit of effort to verify this compatibility, as shown out in detail above.

Appendix A: Alternate forms for values at 0.

Our results from above were

$P_\nu^\mu(z=0\pm) \ = 2^\mu\sqrt{\pi}\ e^{\mp i\pi\mu/2} / [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$ // p 126 (22)

$P_\nu^\mu(x=0) \ = 2^\mu\sqrt{\pi}\ / [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$ // Example 1 above

$e^{-i\pi\mu}Q_\nu^\mu(z=0\pm) = \ \sqrt{\pi}\ 2^{\mu-1}\ e^{\pm i\pi(-\nu-1)/2}\ [\ \Gamma(1/2 + \nu/2+ \mu/2) / \Gamma(1/2 + \nu/2- \mu/2)]$ // p 134 (40)

$Q_\nu^\mu(x=0\ ) = -\sqrt{\pi}\ 2^{\mu-1}\ \sin\pi[(\mu+\nu)/2]\ [\ \Gamma(1/2 + \nu/2+ \mu/2) / \Gamma(1/2 + \nu/2- \mu/2)]$ // Example 2 above

We shall assume integers from now on that $\nu = n \geq 0$ and $\mu = m$ with the usual $|m| \leq n$. Let

$n + m = k \geq 0 \qquad n - m \ = j \geq 0 \qquad k-j = 2m$

The P calculation

$\Gamma(1/2 - n/2 - m/2) = \ \Gamma(1/2 - [n+m]/2) \ = \Gamma(1/2 - k/2)$

We use the reflection formula with $1-z = 1/2 - k/2$ so that $z = 1/2+k/2$:

$\Gamma(z)\ \Gamma(1-z) = \pi/\sin(\pi z)$

$\Gamma(1/2+k/2)\ \Gamma(1/2 - k/2) = \pi/\sin(\pi(1/2+k/2)$

$\Gamma(1/2 - k/2) \ = \ \pi/ [\ \sin(\pi(1/2+k/2)\ \Gamma(1/2+k/2)\ ] \ = \pi/ [\ \cos(\pi k/2)\ \Gamma(1/2+k/2)\ ]$

Meanwhile

$\Gamma(1 + n/2 - m/2) \ = \Gamma(1 + [n-m]/2) \ = \Gamma(1 +j/2)$

So we then have

$$[\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]\ =\ [\ \Gamma(1 + j/2)\ /\ \Gamma(1/2+k/2)]\ \ \pi/\ \cos(\pi k/2)$$

This appears in reciprocal in our formula above for P, so we write

$$[\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]^{-1}\ =\ (1/\pi)\ [\ \Gamma(1/2 + k/2)\ /\ \Gamma(1+j/2)]\ \ \cos(\pi k/2)$$

If k is an odd integer, $\cos(\pi k/2) = 0$. This tells us that $P_\nu^\mu(z=0\pm) = P_\nu^\mu(x=0) = 0$ for odd k, so we don't have to think about that case any more.

If k is an even integer, $\cos(\pi k/2) = (-1)^{k/2}$. In this case, j is also even, since j+k = 2n. From our gamma page, we have, for even j,

$$j!! = 2^{j/2}\Gamma(j/2+1) \qquad \Rightarrow \qquad \Gamma(j/2+1) = j!!\ 2^{-j/2}$$

and we have for even k,

$$(k-1)!!\ =\ 2^{k/2}\ \Gamma(k/2+1/2)/\sqrt{\pi} \qquad \Rightarrow \qquad \Gamma(k/2+1/2) = \sqrt{\pi}\ \ (k-1)!!\ 2^{-k/2}$$

We then have

$$[...]^{-1}\ = (1/\pi)\ (-1)^{k/2}\ [\sqrt{\pi}\ \ (k-1)!!\ 2^{-k/2}]\ /\ [j!!\ 2^{-j/2}]$$

$$= (1/\sqrt{\pi}\ )\ (-1)^{k/2} 2^{-(k-j)/2}\ \ (k-1)!!/\ j!!$$

$$= (1/\sqrt{\pi}\ )\ (-1)^{(n+m)/2} 2^{-m}\ \ (n+m-1)!!/\ (n-m)!!$$

Recalling our P results from above

$$P_\nu^\mu(z=0\pm)\ \ = 2^\mu\sqrt{\pi}\ \ e^{\mp i m\mu/2}\ /\ [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$$
$$P_\nu^\mu(x=0)\ \ = 2^\mu\sqrt{\pi}\ \ /\ [\ \Gamma(1/2 - \nu/2 - \mu/2)\ \Gamma(1 + \nu/2 - \mu/2)\ ]$$

We install our results to find that

$$P_n^m(z=0\pm)\ \ = 2^m\sqrt{\pi}\ \ e^{\mp i m\mu/2}\ (1/\sqrt{\pi}\ )\ (-1)^{(n+m)/2} 2^{-m}\ \ (n+m-1)!!/\ (n-m)!!$$
$$P_n^m(x=0)\ \ = 2^m\sqrt{\pi}\ \ (1/\sqrt{\pi}\ )\ (-1)^{(n+m)/2} 2^{-m}\ \ (n+m-1)!!/\ (n-m)!!$$

$$P_n^m(z=0\pm)\ \ = (-1)^{(n+m)/2}\ (n+m-1)!!/\ (n-m)!!\ e^{\mp i n m/2} \qquad // \ n+m = \text{even}$$
$$P_n^m(x=0)\ \ = \ (-1)^{(n+m)/2}\ (n+m-1)!!/\ (n-m)!!$$

$$P_n^m(z=0\pm) = P_n^m(x=0)\ = 0 \qquad // \ n+m = \text{odd}$$

Let's now go dredging for any results Smythe might provide on this subject. On page 153 he has results for on-the-cut functions only:

$$(n+m \text{ even}) \qquad P_n^m(0) = (-1)^{\frac{1}{2}(n-m)} \frac{1 \cdot 3 \cdot 5 \cdots (n+m-1)}{2 \cdot 4 \cdot 6 \cdots (n-m)}$$

$$(n+m \text{ odd}) \qquad P_n^m(0) = 0$$

We disagree by $(-1)^m$. This is because precisely because Bateman uses CS and Smythe does not. We can surely assume that Smythe's results are for "on the cut" since he does not write j0 or 0+, etc.

So I now retstate my results

The Q calculation

The gammas of interest here are (again, n and m are integers)

$n + m = k \geq 0$ $\qquad$ $n - m = j \geq 0$ $\qquad$ $k-j = 2m$

$$[\Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)] = [\Gamma(1/2 + k/2) / \Gamma(1 + j/2)]$$

These are the same gamma factors we had last time, but this time we shall need then for k,j odd and for k,j even. For the even case, we know from above that

$$[\Gamma(1/2 + k/2) / \Gamma(1 + j/2)] = [\sqrt{\pi}\ (k-1)!!\ 2^{-k/2}] / [j!!\ 2^{-j/2}]$$

$$= \sqrt{\pi}\ 2^{-(k-j)/2}\ (k-1)!!/\ j!!$$

$$= \sqrt{\pi}\ 2^{-m}\ (n+m-1)!!/\ (n-m)!! \qquad n+m = \text{even}$$

For the odd case, we pull in these results from the gamma page,

$$(k-1)!! = 2^{(k-1)/2}\Gamma(k/2+1/2) \quad \Rightarrow \quad \Gamma(k/2+1/2) = 2^{-(k-1)/2}(k-1)!!$$

$$j!! = 2^{j/2}\Gamma(j/2+1)\ (2/\pi)^{1/2} \quad \Rightarrow \quad \Gamma(j/2+1) = (\pi/2)^{1/2}\ 2^{-j/2}\ j!!$$

Then we get

$$[\Gamma(1/2 + k/2) / \Gamma(1 + j/2)] = [2^{-(k-1)/2}(k-1)!!] / [(\pi/2)^{1/2}\ 2^{-j/2}\ j!!]$$

$$= 2^{-(k-j-1)/2}\ (2/\pi)^{1/2}\ (k-1)!!\ /\ j!!$$

$$= (\ 2/\sqrt{\pi}\ )\ 2^{-(k-j)/2}\ (k-1)!!\ /\ j!!$$

$$= (\ 2/\sqrt{\pi}\ )\ 2^{-m}\ (n+m-1)!!/\ (n-m)!! \qquad n+m = \text{odd}$$

Now recall our starting expressions

$e^{-i\pi\mu}Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\, 2^{\mu-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\,\Gamma(1/2+\nu/2+\mu/2)\,/\,\Gamma(1/2+\nu/2-\mu/2)]$

$Q_\nu^\mu(x=0\ ) = -\sqrt{\pi}\, 2^{\mu-1}\, \sin\pi[(\mu+\nu)/2]\ [\,\Gamma(1/2+\nu/2+\mu/2)\,/\,\Gamma(1/2+\nu/2-\mu/2)]$

We install our [..] first for the n+m = even case

$e^{-i\pi m}Q_n^m(z=0\pm) = \sqrt{\pi}\, 2^{m-1}\, e^{\pm i\pi(-n-1)/2}\sqrt{\pi}\ 2^{-m}\ (n+m-1)!!/\,(n-m)!!$

$Q_n^m(x=0\ ) = -\sqrt{\pi}\, 2^{m-1}\, \sin\pi[(m+n)/2]\,\sqrt{\pi}\ 2^{-m}\ (n+m-1)!!/\,(n-m)!!$

$e^{-i\pi m}Q_n^m(z=0\pm) = (\pi/2)\, e^{\pm i\pi(-n-1)/2}\,(n+m-1)!!/\,(n-m)!!$

$Q_n^m(x=0\ ) = -(\pi/2)\,\sin\pi[(m+n)/2]\,(n+m-1)!!/\,(n-m)!! \ = 0$

Now we install our [..] first for the n+m = odd case

$e^{-i\pi m}Q_\nu^m(z=0\pm) = \sqrt{\pi}\, 2^{m-1}\, e^{\pm i\pi(-n-1)/2}(\,2/\sqrt{\pi}\,)\,2^{-m}\ (n+m-1)!!/\,(n-m)!!]$

$Q_n^m(x=0\ ) = -\sqrt{\pi}\, 2^{m-1}\, \sin\pi[(m+n)/2]\ (\,2/\sqrt{\pi}\,)\,2^{-m}\ (n+m-1)!!/\,(n-m)!!$

$e^{-i\pi m}Q_\nu^m(z=0\pm) = e^{\pm i\pi(-n-1)/2}\ (n+m-1)!!/\,(n-m)!!]$

$Q_n^m(x=0\ ) = -\ \sin\pi[(m+n)/2]\,(n+m-1)!!/\,(n-m)!!$ $\qquad\qquad \sin\pi[(n+m)/2] = (-1)^{(n+m-1)/2}$

$e^{-i\pi m}Q_\nu^m(z=0\pm) = e^{\pm i\pi(-n-1)/2}\ (n+m-1)!!/\,(n-m)!!]$

$Q_n^m(x=0\ ) = -\ (-1)^{(n+m-1)/2}\,(n+m-1)!!/\,(n-m)!!$

So here is a summary of these results:

$e^{-i\pi m}Q_n^m(z=0\pm) = (\pi/2)\, e^{\pm i\pi(-n-1)/2}\,(n+m-1)!!/\,(n-m)!!$ // n+m = even

$Q_n^m(x=0\ ) \ = 0$ // n+m = even

$e^{-i\pi m}Q_\nu^m(z=0\pm) = e^{\pm i\pi(-n-1)/2}\ (n+m-1)!!/\,(n-m)!!$ // n+m = odd

$Q_n^m(x=0\ ) = -\,(-1)^{(n+m-1)/2}\,(n+m-1)!!/\,(n-m)!!$ // n+m = odd

Once again we go dredging in Smythe for these results (p 153) which are for the on-cut case:

$$(n+m \text{ odd}) \qquad Q_n^m(0) = (-1)^{\frac{1}{2}(n-m+1)}\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(n+m-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot(n-m)}$$

$$(n+m \text{ even}) \qquad Q_n^m(0) = 0$$

We can fiddle with my phase:

$$-\,(-1)^{(n+m-1)/2} \ = \ (-1)^{(n+m+1)/2} \ = \ (-1)^{(n-m+1\ +2m)/2} \ = \ (-1)^{(n-m+1)/2}\,(-1)^m$$

which allows me to rewrite my odd answer as

$$Q_n^m(x=0) = (-1)^m \left[(-1)^{(n-m+1)/2} (n+m-1)!!/ (n-m)!! \right] = (-1)^m \; Q_n^m(x=0)_{\text{Smythe}}$$

so given Smythe's NCS phase convention for on the cut P and Q, we agree. Here then are my on-cut results with phases expressed simply:

$$Q_n^m(x=0) = 0 \qquad \text{// n+m = even}$$
$$Q_n^m(x=0) = (-1)^{(n+m+1)/2} (n+m-1)!!/ (n-m)!! \qquad \text{// n+m = odd}$$

Smythe does not give results for the off-cut P and Q associated functions, but when order = 0 he does as follows, from p 146:

$$(n \text{ even}) \; Q_n(j\cdot 0) = -\frac{1}{2} j\pi P_n(j\cdot 0) = (-1)^{\frac{1}{2}n} \frac{-j\pi}{2} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} \quad (3)$$

$$(n \text{ odd}) \; Q_n(j\cdot 0) = (-1)^{\frac{1}{2}(n+1)} \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n} \quad (4)$$

Here is what I get for this situation.

$$e^{-i\pi m} Q_n^m(z=0\pm) = (\pi/2)\, e^{\pm i\pi(-n-1)/2} (n+m-1)!!/ (n-m)!! \qquad \text{// n+m = even}$$
$$e^{-i\pi m} Q_\nu^m(z=0\pm) = e^{\pm i\pi(-n-1)/2} (n+m-1)!!/ (n-m)!! \qquad \text{// n+m = odd}$$

$$Q_n(z=0\pm) = (\pi/2)\, e^{\pm i\pi(-n-1)/2} (n-1)!!/ (n)!! \qquad \text{// n = even}$$
$$Q_\nu(z=0\pm) = e^{\pm i\pi(-n-1)/2} (n-1)!!/ (n)!! \qquad \text{// n = odd}$$

Now more phase fiddling. When n = odd we can say

$$e^{\pm i\pi(-n-1)/2} = (\pm i)^{-n-1} = \left[(\pm i)^2 \right]^{(-n-1)/2} = (-1)^{(-n-1)/2} = (-1)^{(n+1)/2}$$

which replicates Smythe's odd answer. For n even we write

$$e^{\pm i\pi(-n-1)/2} = e^{\pm i\pi(-n)/2} e^{\pm i\pi(-1/2)} = \left[(\pm i)^2 \right]^{-n/2} (\mp i)) = \mp i\, (-1)^{n/2}$$

My results then become

$$Q_n(z=0\pm) = (\pi/2)[\mp] \, i\, (-1)^{n/2} (n-1)!!/ (n)!! \qquad \text{// n = even}$$
$$Q_\nu(z=0\pm) = (-1)^{(n+1)/2} (n-1)!!/ (n)!! \qquad \text{// n = odd}$$

or

$$Q_n(z=0\pm) = \mp (i\pi/2) (-1)^{n/2} (n-1)!!/ (n)!! \qquad \text{// n = even}$$
$$Q_\nu(z=0\pm) = (-1)^{(n+1)/2} (n-1)!!/ (n)!! \qquad \text{// n = odd}$$

and these both agree exactly with Smythe.

I don't know how Smythe claims there is a connection between $Q_n(j0)$ and $P_n(j0)$ as he shows.
I cannot find web confirmation of my Q results for $m \neq 0$.

Here again are my results for general m

$e^{-i\pi m}Q_n^m(z=0\pm) = (\pi/2)\, e^{\pm i\pi(-n-1)/2}\, (n+m-1)!!/\,(n-m)!!$ // n+m = even

$e^{-i\pi m}Q_\nu^m(z=0\pm) = e^{\pm i\pi(-n-1)/2}\, (n+m-1)!!/\,(n-m)!!$ // n+m = odd

Maybe a more compact way to write the phase:

$$e^{\pm i\pi(-n-1)/2} = (\pm i)^{-n-1} = (\mp i)^{n+1}$$

$e^{-i\pi m}Q_n^m(z=0\pm) = (\pi/2)\, (\mp i)^{n+1}\, (n+m-1)!!/\,(n-m)!!$ // n+m = even

$e^{-i\pi m}Q_\nu^m(z=0\pm) = (\mp i)^{n+1}\, (n+m-1)!!/\,(n-m)!!$ // n+m = odd

Appendix B. Other formula derivations:

Part 1: Smythe's Claim? For on the cut P and Q functions, we know that, for *Smythe* NCS phase.

$P_\nu^m(x) = (1-x^2)^{m/2}\, \partial_m P_\nu(x)$ m = 1,2,3.... // "NCS phase"

$Q_\nu^m(x) = (1-x^2)^{m/2}\, \partial_m Q_\nu(x)$ m = 1,2,3... // "NCS phase"

Since the structure is the same, we work only with the first. The claim of Smythe is this:

$$[\partial_r P_\nu^m(x)]^{x=0} = P_\nu^{m+r}(0)$$

I think he is wrong, and this is only true for r = 1. I will show a counterexample below.

<u>First derivative:</u> Maple tells me that

$$[\partial_1 P_\nu^m(x)]^{x=0} = [\partial_{m+1} P_\nu(x)]^{x=0}$$

but

$$P_\nu^{m+1}(x) = (1-x^2)^{(m+1)/2}\, \partial_{m+1} P_\nu(x) \quad \Rightarrow \quad P_\nu^{m+1}(0) = [\partial_{m+1} P_\nu(x)]^{x=0}$$

so we conclude that

$$[\partial_1 P_\nu^m(x)]^{x=0} = P_\nu^{m+1}(0)$$

<u>Second derivative:</u> Maple tells me that

$$[\partial_2 P_\nu^m(x)]^{x=0} = -m\,[\partial_m P_\nu(x)]^{x=0} + [\partial_{m+2} P_\nu(x)]^{x=0}$$

but

$$P_\nu^{m+2}(x) = (1-x^2)^{(m+2)/2}\,\partial_{m+2}P_\nu(x) \qquad \Rightarrow \qquad P_\nu^{m+2}(0) = [\partial_{m+2}P_\nu(x)]^{x=0}$$

so we conclude that

$$[\partial_2 P_\nu^m(x)]^{x=0} = -m\,[\partial_m P_\nu(x)]^{x=0} + P_\nu^{m+2}(0)$$

This disagrees with Smythe I think. Suppose $\nu = 4$ and $m = 2$.

$$[\partial_2 P_4^2(x)]^{x=0} = -2\,[\partial_2 P_4(x)]^{x=0} + P_4^4(0)$$

$\partial_2 P_4(x) = (1/8)\partial_2[\,35x^4 - 30x^2 + 3\,] = (1/8)\partial_1[\,140x^3 - 60x\,] = (1/8)\,[\,420x^2 - 60\,]$

$[\partial_2 P_4(x)]^{x=0} = -60/8 = -15/2$

so we then have

$$[\partial_2 P_4^2(x)]^{x=0} = 15 + P_4^4(0)$$

so this gives a counterexample to Smythe's claim!

The same counter example from the horse's mouth. Smythe states that:

$$\begin{aligned}
P_1^1(\mu) &= (1-\mu^2)^{\frac{1}{2}} & P_3^3(\mu) &= 15(1-\mu^2)^{\frac{3}{2}} \\
P_2^1(\mu) &= 3(1-\mu^2)^{\frac{1}{2}}\mu & P_4^1(\mu) &= \tfrac{5}{2}(1-\mu^2)^{\frac{1}{2}}(7\mu^3 - 3\mu) \\
P_2^2(\mu) &= 3(1-\mu^2) & P_4^2(\mu) &= \tfrac{15}{2}(1-\mu^2)(7\mu^2 - 1) \\
P_3^1(\mu) &= \tfrac{3}{2}(1-\mu^2)^{\frac{1}{2}}(5\mu^2 - 1) & P_4^3(\mu) &= 105(1-\mu^2)^{\frac{3}{2}}\mu \\
P_3^2(\mu) &= 15(1-\mu^2)\mu & P_4^4(\mu) &= 105(1-\mu^2)^2
\end{aligned}$$

and he claims that

$$[\partial_r P_\nu^m(x)]^{x=0} = P_\nu^{m+r}(0)$$

so I will test this for the following case

$$[\partial_2 P_4^2(x)]^{x=0} = P_4^4(0)$$

Maple computes the LHS as follows:

```
restart;
P42 := (15/2)*(1-x^2)*(7*x^2-1);
```

$$P42 := \frac{15}{2}(1-x^2)(7x^2-1)$$

```
diff(P42,x,x);
```

$$-630\,x^2 + 120$$

```
eval(%,x=0);
```

$$120$$

But we can see that $P_4^4(0) = 105$, so these differ by the 15 I got earlier.

Part 2: For on the cut P and Q functions, we know that, for Bateman CS phase.

$P_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \partial_m P_\nu(x)$ m = 1,2,3.... // "CS phase"
$Q_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \partial_m Q_\nu(x)$ m = 1,2,3... // "CS phase"

We have shown above that

$$[\partial_1 P_\nu^m(x)_{Smythe}]^{x=0} = P_\nu^{m+1}(0)_{Smythe}$$

But we know that

$$P_\nu^m(x)_{Smythe} = (-1)^m P_\nu^m(x)$$
$$P_\nu^{m+1}(0)_{Smythe} = (-1)^{m+1} P_\nu^{m+1}(0)$$

Thus, we are going to pick up a minus sign, and we get

$$[\partial_1 P_\nu^m(x)]^{x=0} = -P_\nu^{m+1}(0)$$

Since Q has the same structure, we conclude that

$$[\partial_1 Q_\nu^m(x)]^{x=0} = -Q_\nu^{m+1}(0)$$

And the simple way to present these results is this:

$$P_\nu^{m\,\prime}(0) = -P_\nu^{m+1}(0)$$
$$Q_\nu^{m\,\prime}(0) = -Q_\nu^{m+1}(0)$$

Let's now repeat this for off the cut functions: (P and Q are again the same in form)

$P_\nu^m(z) = (z^2-1)^{m/2}\partial_m P_\nu(z)$ $m \geq 0$ and integer

$$\partial_z P_\nu^m(z) = (z^2-1)^{m/2}\partial_{m+1} P_\nu(z) + (m/2)(z^2-1)^{m/2-1}(2z)\,\partial_m P_\nu(z)$$

$$\partial_z P_\nu^m(0+) = e^{i\pi m/2}\,\partial_{m+1} P_\nu(0+)$$

$$P_\nu^{m+1}(z) = (z^2-1)^{(m+1)/2}\partial_{m+1}P_\nu(z)$$

$$P_\nu^{m+1}(0+) = e^{i\pi(m+1)/2}\partial_{m+1}P_\nu(0+)$$

so

$$\partial_z P_\nu^m(0+) = e^{i\pi m/2}\,\partial_{m+1}P_\nu(0+) = e^{i\pi m/2}\,e^{-i\pi(m+1)/2}\,P_\nu^{m+1}(0+) = e^{-i\pi/2}\,P_\nu^{m+1}(0+) = -i\,P_\nu^{m+1}(0+)$$

Conclusion is this:

$$\partial_z P_\nu^m(0\pm) = \mp i\,P_\nu^{m+1}(0\pm) \qquad m \geq 0 \text{ and integer}$$
$$\partial_z Q_\nu^m(0\pm) = \mp i\,Q_\nu^{m+1}(0\pm) \qquad m \geq 0 \text{ and integer}$$

What happens if $m < 0$? I don't know. For P we can use p 140 (7) to get

$$P_\nu^m(z) = f(\nu,m)\,P_\nu^{-m}(z) \qquad m = \text{integer} \qquad f(\nu,\mu) = \Gamma(\nu+m+1)/\Gamma(\nu-m+1)$$
$$\Rightarrow \quad P_\nu^{-m}(z) = f(\nu,-m)\,P_\nu^m(z)$$

Suppose in this last line $m \geq 0$. Then we have

$$\partial_z P_\nu^{-m}(z)\big|^{0\pm} = f(\nu,-m)\,\partial_z P_\nu^m(z)\big|^{0\pm} = f(\nu,-m)\,[\mp i\,P_\nu^{m+1}(0\pm)]$$

$$= \mp i\,f(\nu,-m)\,P_\nu^{m+1}(0\pm)$$

which I will just leave as is for the answer. As for Q, we start with p 140 (2) where we can ignore the phases because they cancel when m = integer. It then says

$$Q_\nu^m(z) = f(\nu,m)\,Q_\nu^{-m}(z) \qquad m = \text{integer}$$

which is the same as the Q formula. So the same derivation goes through. Our results:

$$\partial_z P_\nu^{-m}(0\pm) = \mp i\,P_\nu^{m+1}(0\pm)\,\Gamma(\nu-m+1)/\Gamma(\nu+m+1) \qquad m \geq 0 \text{ and integer}$$
$$\partial_z Q_\nu^{-m}(0\pm) = \mp i\,Q_\nu^{m+1}(0\pm)\,\Gamma(\nu-m+1)/\Gamma(\nu+m+1) \qquad m \geq 0 \text{ and integer}$$

Appendix C. More derivations

1. Facts:

(a) The function $Q_\nu^\mu(z)$ has a parameter pole when $\nu = -\mu-1,-\mu-2$ ... for all z

(b) The function $Q_\nu^\mu(z)$ behaves as $z^{-\nu-1}$ for all complex ν and μ, but for the special values $\nu = -\mu-1,-\mu-2$ ....$Q_\nu^\mu(z)$ has the pole shown above.

(c) The function $Q_\nu^\mu(z)/\Gamma(1+\nu+\mu)$ has no parameter poles at all.

(d) The function $Q_\nu^\mu(z)/\Gamma(1+\nu+\mu)$ behaves as $z^{-\nu-1}$ for all complex ν and μ

Proof: Bateman tells us this:

$$Q_\nu^\mu(z) = \sqrt{\pi}\ [\ \Gamma(1+\nu+\mu)/\ \Gamma(\nu+3/2)]\ (2z)^{-\nu-1} \qquad \nu \neq -3/2,-5/2..... \qquad \text{// Bateman p 134 (41)}$$

This formula says the large z behavior is $z^{-\nu-1}$ with the following two exceptions:

(1) at ν= -3/2,-5/2..... we must use the alternate formula:

$$Q_\nu^\mu(z) = 2^\nu\ \ [\Gamma(1+\nu+\mu)\ \Gamma(1+\nu)/\ \Gamma(2+2\nu)]\ z^{-\nu-1} \qquad \nu \neq -1,-2..... \qquad \text{// Bateman p 132 (36)}$$

in which case we still get the $z^{-\nu-1}$ behavior.

(2) at $\nu = -\mu-1,-\mu-2$ .... the limit blows up because numerator Γ has a pole.

Our conclusion then is this: the function $Q_\nu^\mu(z)$ behaves as $z^{-\nu-1}$ for all complex ν and μ, but for the special values $\nu = -\mu-1,-\mu-2$ ....$Q_\nu^\mu(z)$ blows up.

The fact that $Q_\nu^\mu(z)$ has a pole for $\nu = -\mu-1,-\mu-2$ ... is a completely general fact at least outside $|z| = 1$ according to Bateman p 134 (41). Inside $|z| = 1$ (40) the first term also has a pole here:

$\Gamma(1/2+\nu/2+\mu/2)$ has pole when $1/2+\nu/2+\mu/2 = 0,-1,-2$ or $1+\nu-\mu = 0,-2,-4$ or $\nu-\mu$= -1,-3, -5...

and the second term has a pole here

$\Gamma(1+\nu/2+\mu/2)$ has pole when $1+\nu/2+\mu/2 = 0,-1,-2$ or $2+\nu-\mu = 0,-2,-4$ or $\nu-\mu = -2,-4,-6$...

So between these two terms, $Q_\nu^\mu(z)$ has a pole for $\nu = -\mu-1,-\mu-2$ ... when $|z| < 1$.

Fact: The function $Q_\nu^\mu(z)$ has a pole when $\nu = -\mu-1,-\mu-2$ .. for all z.

I guess this is something I did not realize.

Appendix D. A study of the condition $Q_n^m(j0+) = 0$.

(a) We know that

$$Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\ 2^{\mu-1}\ e^{\pm i\pi(-\nu-1)/2}\ [\ \Gamma(1/2 + \nu/2+ \mu/2) / \Gamma(1 + \nu/2- \mu/2)] \quad \text{// p 134 (40)}$$

In the Bateman formula, the second term has a linear z factor, so that term gives nothing, and that is why we have only one term above. The F function has a fixed "c" argument, so it is good for all small z^2 in both terms. The only doubt we have for the above is when there is a pole in this factor

$$[\ \Gamma(1 + \nu/2 + \mu/2) / \Gamma(1/2 + \nu/2 - \mu/2)]$$

because such a pole blows up the second term we have thrown out. So our "region of doubt" is then

$1 + \nu/2 + \mu/2 = 0,-1,$ => $\nu/2 + \mu/2 = -1,-2,$ => $\nu+\mu = -2,-4,$ => $\nu = -\mu-2,-\mu-4..$

(b) As we see from above (m,n *general*), requiring $Q_n^m(j0+) = 0$ requires $\Gamma(1 + n/2 - m/2)$ to have a pole which in turn requires that n = m-N, N=2,4,6..... . However, we know from Legendre properties Appendix C that Q_n^m has a parameter pole when n = -m-1,-m-2. Notice that this string of values includes our "region of doubt" found above, concerning $Q_\nu^\mu(z=0\pm) = 0$. So since we are going to throw out values where Q blows up, we can cease worrying about the region of doubt.

So we now want to find those values of n for which Q = 0 and Q is also finite. We have to include the first string of values, but exclude the second string:

$Q_n^m(j0+) = 0$	when n = m-2, m-4, m-6....
$Q_n^m(j0+)$ has pole	when n = -m-1,-m-2 ....

Now, for the first time, we assume m = integer for the rest of this little discussion. We shall consider four cases: m >0, m < 0, m even, m odd. We start with m > 0 and consider m odd then m even:

In this case, there are some values of n where $Q_n^m(j0+) = 0$ and Q_n^m is finite. First here is m odd:

```
        -m                  0                  +m                  // m odd
*       *       *       *       *       *       *       *       *       *
p
        0               0               0
```

which tells us for m > 0 and odd, we have the following places where $Q_n^m(j0+) = 0$ and is finite:

n = m-2,m-4.... -m

Now for m even the picture is this

```
                -m              0               +m                          // m even
*       *       *       *       *       *       *       *       *       *
p       p
                0               0
```

which tells us for m > 0 and even, we have the following places where $Q_n^m(j0+) = 0$ and is finite:

n = m-2,m-4.... -m

which is the same as the odd-m rule.

Now let's consider m < 0.

```
        +m                    0                    -m                    // m odd
*       *       *       *       *       *       *       *       *       *
p       p       p       p       p       p       p
        0               0               0
```

The poles come right up to -m-1 but the zeros don't start until -m-2, so there are NO happy n values!
For m even we will have the same situation:

```
                +m              0               -m                    // m odd
*       *       *       *       *       *       *       *       *       *
p       p       p       p       p       p
0               0               0
```

So what about m = 0 ? The first Q = 0 point would be n = -2, but this is a pole point.

We are thus led to the following conclusion:

Theorem 1: If m = integer, then if we want $Q_n^m(j0+) = 0$, this can only happen when m > 0 and in this case $Q_n^m(j0+) = 0$ only when n = m-2,m-4.... -m .

<u>Maple Verifications:</u>

		Prediction	Maple
Some predictions	m = 2	$Q_2^2(0) \neq 0$	-4.7i
		$Q_1^2(0) \neq 0$	-2
		$Q_0^2(0) = 0$	0
		$Q_{-1}^2(0) \neq 0$	-1
		$Q_{-2}^2(0) = 0$	0
		$Q_{-3}^2(0)$ = pole	singularity encountered
Some predictions	m = 3	$Q_3^3(0) \neq 0$	f
		$Q_2^3(0) \neq 0$	f
		$Q_1^3(0) = 0$	0
		$Q_0^4(0) \neq 0$	f
		$Q_{-1}^3(0)= 0$	0
		$Q_{-2}^3(0) \neq 0$	f
		$Q_{-3}^3(0) = 0$	0
		$Q_{-4}^3(0) \neq 0$	singularity encountered
Some predictions	m = 4	$Q_4^4(0) \neq 0$	finite

$Q_3^4(0) \neq 0$	f
$Q_2^4(0) = 0$	0
$Q_1^4(0) \neq 0$	f
$Q_0^4(0) = 0$	0
$Q_{-1}^4(0) \neq 0$	f
$Q_{-2}^4(0) = 0$	0
$Q_{-3}^4(0) \neq 0$	f
$Q_{-4}^4(0) = 0$	0
$Q_{-5}^4(0) = 0$	singularity encountered

Maple verifies the predictions of the theorem.

Appendix E. A study of the condition $P_n^m(z) = 0$ for all z.

Right now, m and n are general complex values. If we stay away from m = 1,2... then Bateman p 124 (14) is valid ["c" in F does not hit a negative integer] , and it says

$$P_n^m(\xi) = (\xi+1)^{m/2}\,(\xi-1)^{-m/2} / \Gamma(1-m) \;*\; F(-n,1+n;\,1-m;\,(1-\xi)/2) \qquad m \neq 1,2,3...$$

or

$$P_n^{-m}(\xi) = (\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\,1+m;\,(1-\xi)/2) \qquad m \neq -1,-2,-3...$$

As the first line shows, as long as we stay away from m = 1,2... the function $P_n^m(\xi)$ exists and is finite for all complex n in our disk of convergence for ξ. It is obvious that $P_n^m(\xi) = P_{-n-1}^m(\xi)$.

Now, for the first time, we restrict our interest to m = integer. In this case, Bateman p 140 (7) says

$$P_\nu^m(\xi) = f(\nu,m)\,P_\nu^{-m}(\xi) \qquad m = 0,1,2,3... \qquad f(\nu,\mu) = \Gamma(\nu+\mu+1)/\Gamma(\nu-\mu+1)$$

Since our second line above is valid for m = 0,1,2,3, we can write

$$P_\nu^m(\xi) = f(\nu,m)\,(\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\,1+m;\,(1-\xi)/2) \qquad m = 0,1,2.3$$

Therefore, we have these results:

$$P_n^{-m}(\xi) = \qquad (\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\,1+m;\,(1-\xi)/2) \qquad m = 0,1,2,3$$

$$P_\nu^m(\xi) = f(\nu,m)\,(\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\,1+m;\,(1-\xi)/2) \qquad m = 0,1,2.3$$

The point then is that we now have an expression for $P_\nu^m(\xi)$ for *all* integers m.

The first line tells us that, if we start off with some negative upper index -m, the function $P_n^{-m}(\xi)$ is finite for all complex n, and in particular for all integers n. It does not vanish for any value of n. For example, the function $P_2^{-3}(\xi)$ does not vanish. But Maple disagrees!

```
evalf(LegendreP(2,-3,0.7213));
                                    0
```

Both I and Maple and Bateman are doing "off the cut" P functions right now. Going back to our second line above, we have

$$P_n^{-m}(\xi) = (\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\, 1+m;\, (1-\xi)/2) \qquad m \neq -1,-2,-3...$$

Setting m = +3, and n = 2, we get

$$P_2^{-3}(\xi) = (\xi+1)^{-3/2}\,(\xi-1)^{3/2} / \Gamma(1+3) \;*\; F(-2,1+2;\, 1+3;\, (1-\xi)/2) \qquad m \neq -1,-2,-3...$$
$$= (\xi+1)^{-3/2}\,(\xi-1)^{3/2}/\Gamma(4) * F(-2,3;4;\, (1-\xi)/2)$$

Let's write out the F function in the above case:

$$F = 1 + (-2)(3)/4 * (1-\xi)/2 + (-2)(-1)(3)(4)/[2*(4)(5)] * (1-\xi)^2/4 + \text{ all other terms } 0$$

$$=1 + (-1)(3)/4 * (1-\xi) + (-1)(-1)(3)/[(5)] * (1-\xi)^2/4$$

$$=1 - (3/4)(1-\xi) + (3/20) * (1-\xi)^2 \qquad \neq 0 \quad !!!$$

It is true that $P_2^{-3}(\xi)$ will diverge at $\xi = -1$, but this doesn't mean we set the entire function to 0 ! However, if you are doing "spherical harmonics" and you are on the cut, then this function is not allowed in the set of complete functions, and maybe this is why Maple is "reaching in" and setting it to 0. Here is some evidence that this might be true. I consider here the value of the function $P_2^{-3}(0.7)$:

```
evalf(LegendreP(2.000000000000100,-3,0.7));
                          -.009742242761082911750 1 I
evalf(LegendreP(2.000000000000010,-3,0.7));
                          -.009742242761083089619 8 I
evalf(LegendreP(2.000000000000001,-3,0.7));
                          -.009742242761083107406 8 I

evalf(LegendreP(2.000000000000000,-3,0.7));
                                    0
```

.

It seems pretty clear that it is not zero and is really equal to the number shown - 0.00974 . But if you make ν be exactly 2, it overrides and sets the thing to 0.

So after this little scare from Maple, let's continue where we left off. We found that for m = 0,1,2,3...

$$P_n^{-m}(\xi) = (\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\, 1+m;\, (1-\xi)/2)$$
$$P_\nu^{m}(\xi) = \Gamma(\nu+m+1)/\Gamma(\nu-m+1)\,(\xi+1)^{-m/2}\,(\xi-1)^{m/2} / \Gamma(1+m) \;*\; F(-n,1+n;\, 1+m;\, (1-\xi)/2)$$

The two gamma functions do this:

$\Gamma(\nu+m+1)$ makes a pole when $\nu = -m-1, -m-2, ...$

$1/\Gamma(\nu-m+1)$ makes a zero when $\nu = m-1, m-2...$

As usual, the two gamma functions can cause poles, zeros, or be finite:

			-m		0		+m				// m even
ν	*	*	*	*	*	*	*	*	*	*	
	p	p									
	z	z	z	z	z	z					

This says that for $m \geq 0$, we get $P_\nu^m(\xi) = 0$ when $\nu = m-1, m-2....-m$.

Example: $P_{-2}^2(x) = 0$ // Maple agrees

Of course this is the same as $P_1^2(x)$ which we know is 0.

So we want to somehow "gather up" all these results an make a solid theorem of them.

Theorem 2: If m is an integer,

(a) $P_n^m(\xi)$ = finite for all complex n, including all integer n.

(b) $P_n^m(\xi) = 0$ only for $m \geq 1$ *and* $n = m-1, m-2, ... -m$. [n integer and $-m \leq n < m$]

Examples: $P_0^1(\xi) = P_{-1}^1(\xi) = 0$

$P_2^3(\xi) = P_1^3(\xi) = P_0^3(\xi) = P_{-1}^3(\xi) = P_{-2}^3(\xi) = P_{-3}^3(\xi) = 0$

$P_2^3(\xi) = P_1^3(\xi) = P_0^3(\xi) = P_0^3(\xi) = P_1^3(\xi) = P_2^3(\xi) = 0$

Due to the reflection rule $P_\nu^\mu(\xi) = P_{-\nu-1}^\mu(\xi)$, the ones with negative n can always be reflected as shown on the last line above. So if you restrict to $n \geq 0$, then $P_n^m(\xi) = 0$ when $m \geq 1$ and $m > n$. This then is the classical idea that m is "out of range" for n. But when $m < 0$, there is no such rule!

Theorems about $f(\nu,m) = \Gamma(\nu+m+1)/\Gamma(\nu-m+1)$

I am sick of doing this again and again, so make a theorem out of it please.

$\Gamma(\nu+m+1)$ has poles when $\nu = -m-1, -m-2...$

$1/\Gamma(\nu-m+1)$ has zeros when $\nu = m-1, m-2...$

Assume now that m = integer and $m > 0$. We then have this picture:

			-m		0		+m			
ν	*	*	*	*	*	*	*	*	*	*
	p	p								
	z	z	z	z	z	z				

We conclude that $f(\nu,m) = 0$ when $\nu = m-1, m-2...-m$ for m integer and $m > 0$.

Assume now that m = integer and m < 0. We then have this picture:

			m		0		-m			
ν	*	*	*	*	*	*	*	*	*	*
	z	z								
	p	p	p	p	p	p				

We conclude that f(ν,m) = ∞ when ν = |m|-1,|m|-2...-|m|.

Theorem 3: Let f(ν,m) = Γ(ν+m+1)/Γ(ν-m+1) and assume m = integer. then
(a) if m > 0, f(ν,m) = 0 when ν = m-1,m-2...-m . // there are 2m zeros
(b) if m < 0, f(ν,m) = ∞ when ν = |m|-1, |m|-2...- |m| . // there are 2|m| poles
(c) if m = 0, f(ν,m) = 1.

Appendix F. More general results for derivatives at z = 0.

Values at 0.

First, let's derive both our claims that

$$P_\nu^\mu(z=0\pm) = 2^\mu\sqrt{\pi}\; e^{\mp i\pi\mu/2} / [\,\Gamma(1/2 - \nu/2 - \mu/2)\,\Gamma(1 + \nu/2 - \mu/2)\,] \qquad // \text{ p 126 (22) + App}$$

$$e^{-i\pi\mu}Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\,2^{\mu-1}\,e^{\mp i\pi(\nu+1)/2}\,[\,\Gamma(1/2 + \nu/2+ \mu/2) / \Gamma(1 + \nu/2- \mu/2)] \quad // \text{ p 134 (40)}$$

The only issue is the phase of (z^2-1). We know that

$$(z^2-1)^{-\mu/2} = e^{\pm i\pi(-\mu/2)}\;(1-z^2)^{-\mu/2} \qquad // \text{ Bateman p 123 (12)}$$

so when this is evaluated at z = 0 it gives $e^{\pm i\pi(-\mu/2)} = e^{\mp i\pi\mu/2}$, confirming our P result above. The same factor appears in p 134 (40), and we then have to do this phase combination

$$e^{\mp i\pi\mu/2}\,e^{\pm i\pi(\mu-\nu-1)/2} = e^{\mp i\pi(\nu+1)/2}$$

and this confirms the Q result

Derivatives

Case 1: Q

Let's try this using Q (40) which seems nice near z = 0:

$$e^{-i\pi m}Q_n^m(z) = K_1\,(z^2-1)^{-\mu/2}\,F(a,b,c,z^2) + K_2\;z\,(z^2-1)^{-\mu/2}\,F(a',b',c';\,z^2)$$

$$= (z^2-1)^{-\mu/2}\,\{\,K_1\,F(a,b,c,z^2) + K_2\;z\,F(a',b',c';\,z^2)\,\}$$

Now we know that

$F'(a,b,c\ z) = ab/c + O(z)$

$F'(a,b,c\ z^2) = (ab/c + O(z)) * 2z$

$[F'(a,b,c\ z^2)]^{z=0} = 0$

So we can ignore terms that involve F'. Treat F as a constant We will then need:

$$\partial_z (z^2-1)^{-\mu/2} = (-\mu/2)(z^2-1)^{-\mu/2-1}(2z)$$

$$[\partial_z (z^2-1)^{-\mu/2}]|^{z=0} = 0$$

so treat $(z^2-1)^{-\mu/2}$ as a constant as well. Then our answer is just this:

$$\partial_z [e^{-i\pi m} Q_n^m(z)]^{z=0} = (z^2-1)^{-\mu/2} K_2$$

In other words, it is the second term in (40) with F = 1 and the z removed.

Probably before I did this derivative, I should have done this:

$$(z^2-1)^{-\mu/2} = e^{\pm i\pi(-\mu/2)} (1-z^2)^{-\mu/2}$$ // Bateman p 123 (12)

This does not change the fact that we treat this term as a constant. But now we get

$$\partial_z [e^{-i\pi m} Q_n^m(z)]^{z=0} = e^{\pm i\pi(-m/2)} K_2$$

$$= e^{\mp i\pi m/2} K_2$$

Looking at (40) we have

$$K_2 = \sqrt{\pi}\, 2^m\, g_2(n,m)\, e^{\pm i\pi(m-n)/2}$$

where $g_2(n,m) = \Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)$

So our result is

$$\partial_z [e^{-i\pi m} Q_n^m(z)]^{z=0} = e^{\mp i\pi m/2} \sqrt{\pi}\, 2^m\, g_2(n,m)\, e^{\pm i\pi(m-n)/2}$$

$$= e^{\mp i\pi n/2} \sqrt{\pi}\, 2^m\, g_2(n,m)$$

One more time with vigor:

$$e^{-i\pi m}\, Q_n^m{}'(0\pm i\varepsilon) = e^{\mp i\pi n/2}\sqrt{\pi}\, 2^m\, [\,\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

Case 2: P

Now let's try this same idea using (22) for P

$$P_n^m(z) = C_1 (z^2-1)^{-\mu/2} F(a,b,c,z^2) + C_2\ z\,(z^2-1)^{-\mu/2} F(a',b',c';\, z^2)$$

$$= (z^2-1)^{-\mu/2}\ \{\, C_1 F(a,b,c,z^2) + C_1\ z\, F(a',b',c';\, z^2)\,\}$$

This has the same form as above, and we end up with

$$P_n^m{}'(0\pm i\varepsilon) = e^{\mp i\pi m/2}\, C_2$$

Now we have

$$C_2 = -\sqrt{\pi}\, 2^{m+1} / [\,\Gamma(1/2+n/2-m/2)\Gamma(-n/2-m/2)]$$

so we get

$$P_n^m{}'(0\pm i\varepsilon) = -\, e^{\mp i\pi m/2}\sqrt{\pi}\, 2^{m+1} / [\,\Gamma(1/2+n/2-m/2)\Gamma(-n/2-m/2)]$$

Summary of this appendix:

$$P_n^m{}'(0\pm i\varepsilon) = -\, e^{\mp i\pi m/2} \qquad \sqrt{\pi}\, 2^{m+1} / [\,\Gamma(1/2+n/2-m/2)\ \Gamma(-n/2-m/2)]$$

$$Q_n^m{}'(0\pm i\varepsilon) = e^{+i\pi m}\ e^{\mp i\pi n/2}\sqrt{\pi}\, 2^m\ [\,\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

Second Method:

Case 1: P

GR give "functional relations" , and so does Bateman page 161 (10). We have

$$(z^2-1)\, P_n^m{}'(z) = nz\, P_n^m(z) - (n+m) P_{n-1}^m(z)$$

So we can just read off what we want at z = 0

$$P_n^m{}'(0) = (n+m) P_{n-1}^m(0)$$

From above we have

$$P_\nu^\mu(z=0\pm) = 2^\mu\sqrt{\pi}\ e^{\mp i\pi\mu/2} / [\,\Gamma(1/2 - \nu/2 - \mu/2)\,\Gamma(1 + \nu/2 - \mu/2)\,] \qquad \text{// p 126 (22)}$$

$P_{n-1}^{m}(z=0\pm) = 2^{m}\sqrt{\pi}\ e^{\mp i n m/2} / [\ \Gamma(1/2 - [n-1]/2 - m/2)\ \Gamma(1 + [n-1]/2 - m/2)\ ]$

$= 2^{m}\sqrt{\pi}\ e^{\mp i n m/2} / [\ \Gamma(1 - n/2 - m/2)\ \Gamma(1/2 + n/2 - m/2)\ ]$

Then we get:

$P_{n}^{m}\,'(0\pm i\varepsilon) = (n+m)\ 2^{m}\sqrt{\pi}\ e^{\mp i n m/2} / [\ \Gamma(1 - n/2 - m/2)\ \Gamma(1/2 + n/2 - m/2)\ ]$

which we can compare with our previous result:

$P_{n}^{m}\,'(0\pm i\varepsilon) = -\ e^{\mp i n m/2}\ \sqrt{\pi}\ 2^{m+1} / [\ \Gamma(1/2+n/2-m/2)\ \Gamma(-n/2-m/2)]$

These would be equal if we could show that

$(n+m) / \Gamma(1 - n/2 - m/2) = -2 / \Gamma(-n/2-m/2)$

$\Gamma(1 - n/2 - m/2) / (n+m) = -\Gamma(-n/2-m/2)/2$

$\Gamma(1 - n/2 - m/2) = -(n/2+m/2)\Gamma(-n/2-m/2)$

$\Gamma(1 - n/2 - m/2) = (-n/2-m/2)\Gamma(-n/2-m/2) = \Gamma(1 -n/2-m/2)$

so this result is well confirmed!

Case 2: Q

As for Q, Bateman says (10) applies to that as well, so we get

$Q_{n}^{m}\,'(0) = (n+m)Q_{n-1}^{m}(0)$

From above we have

$Q_{\nu}^{\mu}(z=0\pm) = e^{+i n m}\ \sqrt{\pi}\ 2^{\mu-1}\ e^{\pm i n(-\nu-1)/2}\ [\ \Gamma(1/2 + \nu/2+ \mu/2) / \Gamma(1 + \nu/2- \mu/2)]$ // p 134 (40)

$Q_{n-1}^{m}(z=0\pm) = e^{+i n m}\sqrt{\pi}\ 2^{m-1}\ e^{\pm i n(-[n-1]-1)/2}\ [\ \Gamma(1/2 + [n-1]/2+ m/2) / \Gamma(1 + [n-1]/2- m/2)]$

$= e^{+i n m}\sqrt{\pi}\ 2^{m-1}\ e^{\pm i n(-n/2)}[\ \Gamma(n/2+ m/2) / \Gamma(1/2 + n/2- m/2)]$

$= e^{+i n m}\sqrt{\pi}\ 2^{m-1}\ e^{\mp i n n/2}\ [\ \Gamma(n/2+ m/2) / \Gamma(1/2 + n/2- m/2)]$

Then we get

$Q_{n}^{m}\,'(0) = (n+m)\ e^{+i n m}\sqrt{\pi}\ 2^{m-1}\ e^{\mp i n n/2}\ [\ \Gamma(n/2+ m/2) / \Gamma(1/2 + n/2- m/2)]$

$$= e^{+i\pi m}\, 2\sqrt{\pi}\, 2^{m-1}\, e^{\mp i\pi n/2}\ \ [(n/2+ m/2)\Gamma(n/2+ m/2) / \Gamma(1/2 + n/2- m/2)]$$

$$= e^{+i\pi m}\, \sqrt{\pi}\, 2^{m}\, e^{\mp i\pi n/2}\ \ [\Gamma(1+n/2+ m/2) / \Gamma(1/2 + n/2- m/2)]$$

which we can compare with our previous result

$$Q_n^m{}'(0\pm i\varepsilon) = e^{+i\pi m}\ e^{\mp i\pi n/2}\, \sqrt{\pi}\, 2^{m}\ [\,\Gamma(1+n/2+m/2) / \Gamma(1/2+n/2-m/2)]$$

and again they agree.