Questions about Legendre Functions META PhL 3.26.05

Some of the work here is presented in better form in these later docs:
See "Legendre Functions Evaluated at z = +1 and -1.doc" summary section.
See "Legendre SL Problem and Spherical Harmonics.doc"

This doc is mostly about how the Legendre function lower index gets quantized when the Laplace equation is treated in various coordinate systems.

Quote from Raw: "When I started this document, I was very confused about these questions. I had just read about the S-L problem (ODE) and wanted to apply the S-L completeness idea to the associated Legendre equation, but I did not know how to do this for that ODE, nor did I know how to extend the idea to completeness in a multi-variable sense for solutions of a PDE. This document then stretched out to 36 pages of pretty raw notes, as new topics kept coming up. I eventually got all the answers, and I will now write a META version of this document to highlight what was learned and to filter out the massive amount of noise. I will also try to write a cleaner answer to the first question. "

Filing Problem: The topics discussed here are a bit hard to classify. Usually we are talking ODE's, so ODE is a good tag or keyword. On the other hand, these ODE's arise from the separation of variables in solutions to PDE's in various curvilinear coordinate systems, so PDE and curvilinear are other possible tags. Then of course we are dealing with special functions. For the moment, this doc is stored in the ODE world under a separate Legendre folder. This is the usual issue of how to file a multi-subject document.

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Question #1: In Spherical Coordinates, what causes ℓ to be quantized?

This was my starting point, a question I had often wondered about and no source really nailed it very well. When we have a full sphere to deal with, we know that m gets quantized to integer values by the azimuthal separated solution. The question then is this: we know that the general solution for the z variable has the form P_n^m and Q_n^m. I want to argue that for integer m, since we want a function that is finite everywhere on (-1,1), we have to completely throw out the Q function, and in the P function we have to set n = |m|, |m|+1, |m|+1. In hopes of finding someone just handing me the right discussion on a platter, I looked at Smythe and then MF. MF got me all tangled up with their T_α^β functions which are related to the P and also to Gegenbauer, so I did a huge digression on that subject, mostly in my MF ODE notes, but that did not really help me on my Question #1 here.

So instead of cataloging the confusing raw notes for Question #1, let's just deal with it right here, making use of all our Bateman ammunition.

The P Situation. Let's use form Bateman p 124 (16) which says

$$P_\nu^\mu(z) = 2^{-\nu}\,(z+1)^{\nu+\mu/2}\,(z-1)^{-\mu/2}/\Gamma(1-\mu)\;\; F(-\nu, -\nu-\mu; 1-\mu; [\,(z-1)/(z+1)])$$

(a) Negative integer m. Let's assume now that $\mu = m$ = *negative* integer like m = 0,-1,-2... so the F function does not have a c-pole. Now first look near z = +1 where we have

$$P_\nu^m(z) \approx 2^{-\nu}\,(2)^{\nu+m/2}\,(z-1)^{-m/2}/\Gamma(1-m)$$

Our first observation: since $m \le 0$, the $(z-1)^{-m/2}$ does not blow up, so our solution is finite at z = 1, being in fact 0 for m = -1, -2... The convergence range of our F function is $\mathrm{Re}(z) \ge 0$ if it is an infinite series. However, if we select $b = -\nu-\mu = -N = 0,-1,-2...$ we get truncation in the b coefficient, and maybe this will give us something finite at z = -1, we need to study it. This means $\nu = -\mu+N = |m| + N$, N = 0,1,2.. So let's see what happens for these values of ν. If we pick N=0, then F = 1. If we pick N = -1, then $F = 1 + ab\xi/c$ so in general, if we pick N, the highest term in F will be ξ^N where ξ means (z-1)/(z+1). At z = -1 we then have, as our worst behaved term in $P_\nu^\mu(z)$,

$$(z+1)^{\nu+\mu/2}\;(z+1)^{-N} \;=\; (z+1)^{\nu+m/2-N} \;=\; (z+1)^{[-m+N]+m/2-N} \;=\; (z+1)^{-m+m/2} \;=(z+1)^{-m/2}$$

and we see that, since $m \le 0$, this does NOT diverge. Therefore, $P_\nu^m(z)$ will be finite at z = -1 when ν takes these special values -μ+N and is in fact 0 there for $m \ne 0$. We already know it is finite at z = +1. And of course it is finite everywhere in the interval (-1,1).

So at this point we have shown that, if m = 0, -1, -2....., then the function $P_\nu^m(z)$ will be finite on the entire interval (-1,1) provided we select $\nu = |m| + N$, N = 0,1,2.. . And it is the -1 endpoint that is "the main problem". Also, we know that these $P_\nu^m(z)$ vanish at each endpoint, except when m = 0.

(b) Positive integer m. Now consider Bateman p 140 result (7) with m = 1,2,3... which says

$$P_\nu^m(z) = \Gamma(\nu+m+1)/\,\Gamma(\nu-m+1)\;\; * P_\nu^{-m}(z)$$

We have shown that the rightmost factor is finite on (-1,1) for m = 0,1,2.. when ν = m+N with N = 0,1,2.. [we have changed the sign of m relative to the above discussion]. That is,

$P_\nu^m(z) = \Gamma(N + 2m + 1)/\Gamma(N+1) \; * P_\nu^{-m}(z)$ $\qquad$ $m = 0,1,2....$

Neither gamma function has any poles for m in this range. The conclusion is that changing the polarity of m merely causes multiplication by some constant, so the function $P_\nu^m(z)$ for m = 0,1,2.. is finite on the entire interval (-1,1) as well. The two polarity functions are not independent since one is a multiple of the other. So one usually takes positive integer values for m and uses $P_\nu^m(z)$ with $\nu = m+N$ with N = 0,1,2. as the Sturm-Liouville eigenfunctions for this problem.

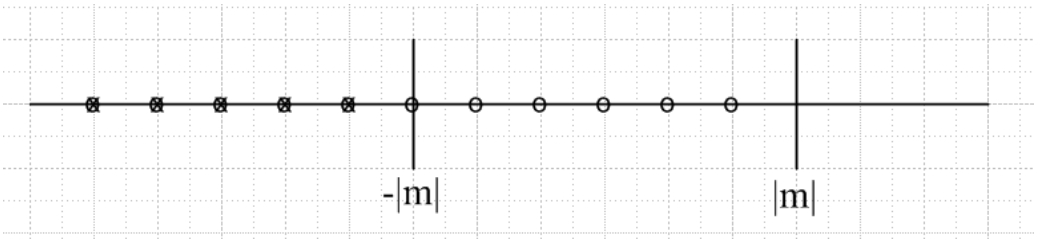
Now, we have found quantized values of ν that produce a P function P_ν^m which is finite on the entire interval (-1,1). We know that Q_ν^m is the second independent solution, and we wonder if it could perhaps be made finite on (-1,1) with some suitable selection of ν values, perhaps different from the choices we made for the P function. We suspect the answer is no, or if it is yes, then the Q functions are the same as the P functions somehow. So we now investigate this question.

The Q Situation. Consider Q_ν^m in the region of z = +1. We need some convergence on the right end of (-1,1) so we really have to use Bateman p 130 (32). We have in this region (small disk around z = 1) no issue with the convergence of the F series. But we have a different and bad problem.

(a) Let's first consider the case μ = m = *negative* integer. The first term in (32) in general has a c-pole at all such negative integral values of m. The only way to fix this is to have the ratio of leading-factors gamma functions kill off the entire term, or to have the F function truncate before we hit the c-pole. In the latter case (select ν to get truncation), the term is still present and we have leading factor $(z-1)^{\mu/2}$ $= (z-1)^{-|m|/2}$ which diverges at z = +1. So our only hope in rescuing this term is the have the gamma functions kill off the entire term. But I am now going to show that this is impossible. Let's examine the gamma functions in question. We have

poles when $1+\nu+\|m\| = 0,-1,-2....$	$\nu = -\|m\|-1-N$	N=0,1,2,3
zeros when $1+\nu-\|m\| = 0,-1,-2...$	$\nu = \|m\| - 1 - M$	M = 0,1,2...

Here is a picture showing the poles and zeros of the first term due to these gamma functions



and we find that every single possible pole is cancelled by a zero. Thus, there is no value of ν that will kill off our first term in Bateman p 131 (32). The conclusion then is that, for negative integer m, there exists no value of ν such that the function $Q_\nu^m(z)$ will be finite at z = +1 because this term contains the diverging factor $(z-1)^{-|m|/2}$

(b) Next, lets consider μ = m = *positive* integers. In this case, the first term has no c-pole issues in F, but it has a Γ(-μ) divergence issue from a leading factor. Let's imagine that we can somehow rescue this situation maybe by selecting ν values such that the denominator poles in μ cancel the numerator ones. So let's put this first term on hold for a moment, and go over to the second term in (32). This term has c-poles in F, but let's imagine we could choose ν to truncate before we hit such a pole. The problem with this second term is that we have $(z\text{-}1)^{-\mu/2} = (z\text{-}1)^{-m/2}$ which diverges at z = +1, and there is no way to fix this with any gamma functions containing ν. So this second term kills us, and we can forget about the first term. The conclusion then is that, for positive integer m, there exists no value of ν such that the function $Q_\nu^m(z)$ will be finite at z = +1 because this term contains the diverging factor $(z\text{-}1)^{-|m|/2}$.

(c) Our total Q conclusion then is that for any integer m, there exists no value of ν such that the function $Q_\nu^m(z)$ will be finite at z = +1 because this term contains the diverging factor $(z\text{-}1)^{-|m|/2}$. Perhaps this is just obvious by simply staring at the leading factors in (32), but one has to consider the possibility of a term getting "gammad away". Barring this, for one sign of m, the one term diverges at z = +1, and for the other sign of m, the other term diverges at z = +1, so in either case, you are stuck with a divergence.

Comment: in atomic physics, say, we do have "the whole sphere" to work with. When we separate the SE in a central potential, we get the spherical harmonic separated equations exactly as discussed here (and more below) and exactly as in electrostatics problems. So it is precisely the ODE logic above that leads to $\boldsymbol{\ell}$ being quantized to the values $\boldsymbol{\ell}$ = 0,1,2,3.... We arrive at this same quantization of $\boldsymbol{\ell}$ by consideration of the abstract rotation group Lie Algebra. The two methods then merge when we realize that we have θ,φ representations of our Lie Algebra generators. We then associate the quantized $\boldsymbol{\ell}$ values with the phrase "angular momentum" since the L_i generators are the angular momentum generators **L** = **r** x **p** in QM.

Question #2: In what sense are the Spherical Harmonics "complete" ?

Consider the PDE $\hat{L}^2 Y = \boldsymbol{\ell}(\boldsymbol{\ell}+1)Y$ where the operator is the usual θ,φ operator. This is *not* the Laplace equation, but it is something that arises in solving the Laplace equation in spherical coordinates when you do the separation of variables. The Y functions are not eigenfunctions of the Laplace equation, but they are eigenfunctions of the $\hat{L}^2$ operator, and the functions are defined on the sphere, meaning θ and φ with the usual ranges.

If we were to consider this as a Stakgold PDE in two variables over region R = the sphere, we would wonder if the Y functions form a "complete" set of eigenfunctions on θ,φ. Stak only hints at completeness in the multiple-variable sense on page 138 of vol 2. The operator here $L = \hat{L}^2$ is self-adjoint, the boundary conditions are single valued in φ and finite S-L in θ. Stak would have us compute the Green's Function for this L, and then treat that as the kernel in an integral form of the PDE. This integral equation would then be Hilbert-Schmidt, and we would conclude that the EF's form a complete set for the sphere. I think I could write the Green's Function for this L as a sum over properly normalized Y functions in the manner of $g(\mathbf{x}|\boldsymbol{\xi}) = \Sigma_n \varphi_{\lambda n}(\mathbf{x})\bar{\varphi}_{\lambda n}(\boldsymbol{\xi}) / \lambda_n$. So this would be one approach to showing that the Y functions are a complete set on the sphere. The Y are the "angular momentum eigenfunctions".

But there is another way to show that the Y are a complete set. As hinted by MF, you show this by showing that each of the separated product functions is complete in its quantum number, and this is what I do below.

Completeness means that, when you include more and more terms in an infinite expansion series where you use unmoving "Fourier" coefficients as basis function multipliers, the "tail" of the series gets smaller and smaller in the L_2 norm sense, and in fact the tail $\rightarrow 0$. This causes "Bessel's Inequality" to become an equality. This notion applies to functions of one, or of more than one, variable. Once we know completeness is true, we can represent it as a sum which equals some product of delta functions. Both MF and CH and Stak think of completeness in this manner.

In our particular case we want to show that $Y_{nm} = P_n^m(z=\cos\theta)\,\Phi_m(\varphi)$ is a complete set. The Courant Hilbert method is to first show that $\Phi_m(\varphi)$ is complete for m = integers, and then to show that -- for each integer m -- the set of functions $P_n^m(z)$ are complete on (-1,1). CH show how you prove that if two (or more) function sets of a product are *separately* complete, then the product is complete on the multiple-variable domain. This was a key "theorem" that I needed to see and understand, and CH had it.

PL proof of the Courant and Hilbert "Linkage Theorem of Completeness"

It was so important to me, that I wrote out the proof of this little theorem in my own words. It has to do with showing "Bessel's Equality" as just mentioned above. The idea is that if we know this equality separately for two functions, then it is true for the product. The slightly strange aspect of things for the Y functions is that, if we can show that the P_n^m are complete for a fixed integer m (in which case n takes on the values n = m, m+1....), we might still be concerned that the product was not complete because the m quantum number belongs to the $\Phi_m(\varphi)$ function, and yet we find it sort of "tacked on" as a rider onto our P function. The theorem as I prove it in the raw notes shows that a "bystander" label like this m on P_n^m has no effect on the CH theorem that if the two sets are separately complete, then so is the product set. That was a major discovery for me.

Orthogonality and Completeness of the $P_n^m(z)$ functions.

I then turned my attention to showing that this set of functions (for each fixed m) is complete. First, I knew that the associated Legendre equation was self-adjoint and we have finiteness Sturm-Liouville boundary conditions at the two end points of (-1,1) and we expect this to result in a set of solution functions F_n^m that are a complete set. The CH authors claim that any ODE like this has solutions which form a complete set. The quantization of n comes from the S-L aspect (regular singular boundary points at z = -1 and 1) and not from explicit homo BC's which are the usual cause of quantization. In these meta notes above, I showed in great detail how the n of P_n^m gets quantized so the functions will be finite at z = -1, and that the Q_n^m are not finite, so P_n^m are the full solutions. I showed that n = m, m+1... are the right quantum numbers for n. I then wrote down a guess at the completeness relation and got it filled in correctly, using the known orthogonality of the P_n^m functions (for fixed integer m). I don't think I realized this orthogonality was true before doing this.

I then showed that my "new" completeness relation for the P_n^m is exactly consistent with the much more famous completeness statement for the Y functions. This was a very detailed verification and it worked out perfectly. I then decided to summarize what I learned, and I copy that entire summary right here:

Summary of Results on the Associated Legendre Polynomials $P_n^m(z)$ -- integer m,n

1. The ODE for associated Legendre functions is this: (take your choice)

$L = (1-z^2)\,D^2 - 2z\,D + [\,n(n+1) - m^2/(1-z^2)]$ $\qquad Lu = 0$ $\qquad$ Bateman p 121

$L' = D^2 - 2z/(1-z^2)\,D + [\,n(n+1)/(1-z^2) - m^2/(1-z^2)^2] = D^2 + pD + q$ $\qquad L'u = 0$

Notice that p has a pole at z = +1 and -1, and that q has a pole and a double pole at these two locations. There is also a regular singularity at z=∞. Think of m as a fixed non-negative integer. This ODE as we well know has three regular singular points at 1,-1 and ∞ and no irregular singular points, so it is very closely tied in with the hypergeometric ODE and its F function solutions.

2. For m a fixed integer, the eigenfunctions of this S-L system are the $P_n^m(z)$ with n = m,m+1....∞.

3. The orthogonality of these eigenfunctions is given by

$$\int_{-1}^{1} dz\, P_n^m(z) P_k^m(z) = \delta_{n,k}\,(n+1/2)^{-1}\,(n+m)!/(n-m)! \qquad n,k = m, m+1, m+2 \ldots\ldots \infty$$

$$= \delta_{n,k}\,K_n^m \qquad K_n^m = (n+1/2)^{-1}\,(n+m)!/(n-m)!$$

You can think of $P_n^m(z) = 0$ for n < m if you want. $\qquad K_n^0 = (n+1/2)^{-1}$

4. The completeness of these eigenfunctions is given by:

$$\Sigma_{n=m}^{\infty}(1/K_n^m)\,P_n^m(z')\,P_n^m(z) = \delta(z'-z) \qquad K_n^m = (n+1/2)^{-1}\,(n+m)!/(n-m)!$$

4A. The above two items can be written in standard form for normalized $\varphi_n^m(z)$:

$$\Sigma_{n=m}^{\infty}\varphi_n^m(z')\,\varphi_n^m(z) = \delta(z'-z) \qquad \varphi_n^m(z) = P_n^m(z)/\sqrt{K_n^m} \qquad \text{// complete}$$

$$\int_{-1}^{1} dz\,\varphi_n^m(z)\varphi_k^m(z) = \delta_{n,k} \qquad n,k = m, m+1, m+2 \ldots\ldots \infty \qquad \text{// orthonormal}$$

Think about QM bra-ket notation here and everything is perfect.

5. In the special case that m = 0 we get the following orthogonality from (3) and completeness from (4)

$$\int_{-1}^{1} dz\, P_n(z) P_k(z) = \delta_{n,k}\,(n+1/2)^{-1} \qquad n,k = 0,1... \infty$$

$$\Sigma_{n=0}^{\infty}(n+1/2)\,P_n(z')\,P_n(z) = \delta(z'-z)$$

6. The above results are compatible with Jackson's following statements about spherical harmonics,

$Y_{nm}(\theta,\varphi) \equiv [\,\{(2n+1)/4\pi\}\,(n-m)!/(n+m)!\,]^{1/2}\,P_n^m(z)\,e^{im\varphi}$ $\qquad$ definition $\qquad$ 3.53 Jackson

$\Sigma_{n=0}^{\infty}\Sigma_{m=-n}^{n} Y_{nm}^*(\theta',\varphi')\, Y_{nm}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(z-z')$ completeness 3.56 Jackson

$\int d\Omega\, Y_{nm}^*(\theta,\varphi)\, Y_{n'm'}(\theta,\varphi) = \delta_{n,n'}\delta_{m,m'}$ orthogonality 3.55 Jackson

$P_n^{-m}(z) = (-1)^m (n-m)!/(n+m)!\, P_n^m(z)$ m reflection 3.51 Jackson

but it takes a bit of effort to verify this compatibility, as shown out in detail above.

Some comments about completeness on the sphere. // quoted in full from raw notes.

We have seen that the set of functions $h_{nm}(\theta,\varphi) = P_n^m(z)\, e^{-im\varphi}$ (for usual range of n and m) forms a complete set for expanding an arbitrary reasonable function $f(\theta,\varphi)$ on the sphere. But we also know that the set of functions $k_{nm}(\theta,\varphi) = P_n(z)\, e^{-im\varphi}$ would also be a complete set for the same purpose, and this set is not "cross linked".

What is the difference between these two sets of 2D basis functions?

(a) Either one allows you to project and recover (expand) a reasonable $f(\theta,\varphi)$ on the sphere. Let's call the projections H_{nm} for the first basis, and K_{nm} for the second basis.

(b) However, if you are looking for *solutions to the Laplace equation*, then you must use the first set. For example, inside a Dirichlet sphere specified at some arbitrary $f(\theta,\varphi)$ you will find that

$$V(r,\theta,\varphi) = \Sigma_{nm}\, r^n\, H_{nm}\, h_{nm}(\theta,\varphi)$$

$$V(1,\theta,\varphi) = \Sigma_{nm}\, H_{nm}\, h_{nm}(\theta,\varphi) = f(\theta,\varphi)$$

The projections K_{nm} are useless for solving this problem. The reason is that when we do separation of the Laplace equation, we get the cross-linked general solution terms $r^n P_n^m(z)\, e^{-im\varphi} = r^n h_{nm}(\theta,\varphi)$ where the n is cross linked as shown. Each term of this form solves Laplace, so any lincomb also solves Laplace. The function $r^n k_{nm}(\theta,\varphi)$ does NOT solve Laplace, that's why I say it is "useless" for this purpose.

This brings us to page 21 in the raw notes (page 7 in our meta notes), so doing 3:1 compression so far.

Question #3: Eigenfunctions of Associated Legendre for m = 1/3. [p 21 raw notes]

Although I say $\mu = m = 1/3$, the real interest here is in non-integral values of m, such as m = 1/3. I know these will arise for example in "orange slice Laplace problems" where V = 0 on the two slice surfaces. This investigation led me far and wide, but I think some definite conclusions were reached which I will report out here.

The first conclusion is easily demonstrated. Start with the P function, written for general complex n and m, and we are interested in a disk near z = +1 where $F \to 1$,

$$P_n^m(z) = 1/\Gamma(1-m)\, [\,(z+1)/(z-1)]^{m/2}\, F[\; -n, n+1; 1-m; (1-z)/2\,]$$

$P_n^m(z) = 1/\Gamma(1-m)\,[\,(2)/(z-1)]^{m/2}$ near $z = 1$

If $\mathrm{Re}(m) > 0$, this blows up *for sure* at $z = 1$, regardless of the nature of n. For $\mathrm{Re}(m) < 1$, this blow up will be square integrable, for what it's worth (ie, finite s norm).

So this was somewhat of a surprise to me: for any non-integral $\mathrm{Re}(m)>0$, P_n^m blows up at $z = 1$ for all possible values of n. I do note that things can blow up and still be "finite s norm" ($s = 1$), and that would be the case here for $0 < \mathrm{Re}(m) < 1$.

At this point, I went off and found a Sturm-Liouville paper which talks about various ODEs including the associated Legendre. It claims that for $0 \le \mu < 1$ both endpoints are LC (meaning there are two finite s norm solutions) but for $\mu \ge 1$ both ends are LP (only one s norm solution). I think the endpoints are treated "on their own" and no claim is being made that there is a solution that is finite s norm at both endpoints at the same time.

Next, I took a look at the corresponding Q function. Near $z = 1$ that form becomes

$$Q_\nu^\mu(z) = \Gamma(1 + \nu + \mu)\,\Gamma(-\mu)\,(z-1)^{\mu/2}\,(2)^{-\mu/2}/[2\Gamma(1+\nu-\mu)] \quad + \quad (1/2)\,\Gamma(\mu)\,(z-1)^{-\mu/2}\,(2)^{\mu/2}$$

Regardless of what happens in the first term, the second term blows up at $z = +1$ for $\mathrm{Re}(m) > 0$. I was again amazed and started writing down some intermediate conclusions, to wit:

(1) both functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ blow up at $z = +1$ if $\mathrm{Re}(\mu) \ge 0$ non integer, for any value of ν.

(2) both P and Q are nevertheless finite-s-norm (at respective ends) provided $\mathrm{Re}(\mu) < 1$. ($s(x) = 1$)

(3) The Wronskian of P and Q is shown Bateman p 123 bottom. For general μ and ν, this is not zero. For certain values of ν, it will vanish, however, and then for those values P and Q are not independent.

(4) So, for $\mu = 1/3$ and ν = general value, <u>there is NO solution of the associated Legendre equation</u> (including P and Q) which is finite at $z = +1$.

(5) Therefore we don't even have to think about the point $z = -1$, since we already have "a problem".

(6) For the Sturm-Liouville problem where you look for functions that are finite on the interval, we find that when $\mu = 1/3$, there are NO solutions!!! But I think there might still be finite s-norm solutions at least at the respective endpoints and maybe over the entire range (-1,1).

(7) For our orange slice type problem, we seem to need $\mu = M(\alpha/\pi)$ which is a set of spaced values that go out forever, so we won't just have $\mathrm{Re}(\mu) < 1$. [I keep neglecting the negative μ values!]

OK, I think these conclusions are valid. For $\mathrm{Re}(\mu) > 1$ there remained a **mystery**, however. Since both the P and Q solutions are non-finite-s-norm for $\mu > 1$, how could an endpoint be LP, indicating at least one finite-s-norm solution? My conclusion, with some help from Maple to confirm it, is that you can find a specific linear combination of P and Q causing the blow up at $z = +1$ (say) to cancel between the terms, and then that aP + bQ solution could be the single finite-s-norm solution at $z =1$. I conjectured and Maple

confirmed that that same solution would NOT be finite-s-norm at z = -1, a different lincomb would be needed there. So I now think that for Re(μ) ≥ 0, there are no finite s norm solutions on the entire interval (-1,1).

So I was quite amazed by these conclusions. I naturally started with m > 0 thinking there is more or less symmetry between m > 0 and m < 0, but I now realize this is not the case when m ≠ integer. But in my notes at this point, I was still thinking only of m > 0. I started flailing around for a way out, because I knew the orange slice problem must have a solution that is finite everywhere. I scanned the web for hours, trying to find someone solving the associated Legendre equation for non-integer m. I wanted to see the problem attacked as an ODE alone, without being part of a larger PDE problem. I found nothing. I briefly wondered if somehow the T functions would help, but concluded no. I wondered why m > 0 integers worked, and found out why that was. I realized that in the full orange slice problem, you really must have some aP+bQ that vanishes at z = 1 and at z = -1 to match V=0 of the two slice faces. In order to match a general Dirichlet f(θ,φ) out on the orange slice skin surface, the P and Q would have to vanish by themselves at z = 1 and -1, since you cannot control the a and b coefficients to cancel a blow up, as I did above in the LP discussion. But I had shown this was impossible for any Re(m) > 0.

Think about m < 0. And so in mid page 26, after thrashing for 5 pages, it finally dawned on me to start thinking about m < 0. This time I looked at the "on the cut" versions of P and Q, since that is what you really want in the context of a (-1,1) solution -- this only changes some phases and has nothing to do with things blowing up. The general forms of interest are

$$P_\nu^\mu(z) = 1/\Gamma(1-\mu)\,(1+z)^{\mu/2}(1-z)^{-\mu/2}\,F[\ -\nu, \nu+1;\ 1-\mu;\ (1-z)/2\ ]$$

$$Q_\nu^\mu(z) = \Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(1-z)^{\mu/2}\,(1+z)^{-\mu/2}/[2\Gamma(1+\nu-\mu)]\,F[\ -\nu, 1+\nu;\ 1+\mu;\ (1-z)/2\ ]$$

$$+ (1/2)\,\Gamma(\mu)\cos(\pi\mu)\,(1-z)^{-\mu/2}\,(1+z)^{\mu/2}\,F[\ -\nu, 1+\nu;\ 1-\mu;\ (1-z)/2\ ]$$

If we go near z = +1 we get

$$P_\nu^\mu(z) = 1/\Gamma(1-\mu)\,(2)^{\mu/2}(1-z)^{-\mu/2}$$
$$Q_\nu^\mu(z) = \Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(1-z)^{\mu/2}\,(2)^{-\mu/2}/[2\Gamma(1+\nu-\mu)] + (1/2)\,\Gamma(\mu)\cos(\pi\mu)\,(1-z)^{-\mu/2}\,(2)^{\mu/2}$$

The P function looks good for Re(μ) < 0, at least at z = +1, and this for general ν. The Q function blows up at z = +1 for Re(μ)<0 unless you can kill the first term with ν selection, so we put Q on hold.

I then looked at the z = -1 situation first for P.

$$P_\nu^\mu(z) = \Gamma(-\mu)\,(1+z)^{\mu/2}\,(1-z)^{-\mu/2}/\,[\,\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)]\ F(-\nu, 1+\nu;\ 1+\mu,\ [1+z]/2)$$

$$- (1/\pi)\sin(\nu\pi)\,\Gamma(\mu)\,(1-z)^{\mu/2}(1+z)^{-\mu/2}\,e^{(-1)^{(\pm1)}i\mu\pi}\ F(-\nu, 1+\nu;\ 1-\mu,\ [1+z]/2)$$

and near z = -1 this becomes:

$$P_\nu^\mu(z) = \Gamma(-\mu)\,(z+1)^{\mu/2}\,(-2)^{-\mu/2}/\,[\,\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)]\ - (1/\pi)\sin(\nu\pi)\,\Gamma(\mu)(-2)^{\mu/2}(z+1)^{-\mu/2}\,e^{(-1)^{(\pm1)}i\mu\pi}$$

The second term is OK, but the first term is a problem at z = -1, and the only rescue is to select ν to kill off the first term! There are two "strings" of ν values that do this, but since $P_\nu^\mu = P_{-\nu-1}^\mu$, we can think of just one of these strings, one stepping off to the right from $-\mu = +|\mu|$.

$\nu = -\mu + N$ N = 1,2,3... μ = some negative real value

But this string of ν values does not kill off the first Q term above, so Q diverges at z = +1 and we have to completely rule it out. [I think a more careful argument would confirm this.] So as in the integer m situation, for Re(m) < 0 non-integer, we find that only the P function is finite at both endpoints. It is in fact 0 at both endpoints which is just what we need for the orange slice problem! The quantized values for ν are ν = |m|, |m| + 1, |m| + 2 , but we need m < 0. Apart from the negativity requirement of m for the non-integral case, these are the same ν values that work for integer m of either sign! In fact, we could write the ν values just as shown here for either case! Of course when m is non-integer, so is ν. It is the difference between ν and |m| that must be a positive integer.

So we pause for another round of summary conclusions, verbatim from raw notes"

Comments on the orange slice Dirichlet problem.

(1) I have never seen this problem solved anywhere.

(2) If we take just the *negative* quantized m values allowed by the azimuthal problem, we find that there are some values of ν such that $P_\nu^m(z) = 0$ at both ends of the (-1,1) interval. This is what we need for the orange slice problem. These are ν = |m|, |m| + 1, |m| + 2, same as for the full sphere integer m problem.

(3) The general orange slice Laplace cross-linked solution will then have this form

$\mu_i = -i\pi/\alpha$ i = 0,1,2,3.... (0,α) is the orange slice azimuthal range
let $\nu_{ij} = |\mu_i| + j$ j = 0,1,2,3..
$\psi(r,z,\varphi) = \Sigma_{i,j=0}^{\infty} A_{\mu i,\nu ij}\ r^{\nu ij}\ P_{\nu ij}^{\mu i}(z) \sin(\mu_i \varphi)$

and you would then try to compute the A coefficients from the surface f(θ,φ) Dirichlet condition.

Now, something is missing here from the raw notes, we need a statement of orthogonality and completeness of the P functions for this non-integral m case. I will requote my summary above and make the appropriate edits. We know this works because SL ODE theory says it does and we are "complete":

Summary of Results on the Associated Legendre Polynomials $P_n^m(z)$: non-integral negative m

2. For m a fixed non-integer negative real number, the eigenfunctions of this S-L system are the $P_n^m(z)$ with n = |m|, |m| +1....∞. where of course |m| = -m > 0.

3. The orthogonality of these eigenfunctions is given by

$$\int_{-1}^{1} dz\ P_n^m(z) P_k^m(z) = \delta_{n,k}\ (n+1/2)^{-1}\ (n+m)!\ /\ (n-m)! \qquad n,k = |m|, |m|+1, |m|+2\ .....\ \infty$$

$$= \delta_{n,k}\, K_n^m \qquad\qquad K_n^m = K_n^m = (n+1/2)^{-1}\,(n+m)!\,/\,(n-m)!$$

$$\int_{-1}^{1} dz\, P_n^{-|m|}(z) P_k^{-|m|}(z) = \delta_{n,k}\,(n+1/2)^{-1}\,(n-|m|)!\,/\,(n+|m|)! \qquad n,k = |m|, |m|+1, |m|+2 \ldots\ldots \infty$$

Notice in the second form that $(n-|m|)! = \Gamma(1+n-|m|)$ and this will be $\Gamma(1)$, $\Gamma(2)$ and so on, so we never get a pole, so the normalization never blows up (in case you were worrying about that). The denominator factor is $(n+|m|)! = \Gamma(1+n+|m|)$ and it also never has a pole, so we never get a vanishing of the orthogonality result when n = k, also a good thing.

4. The completeness of these eigenfunctions is given by:

$$\Sigma_{n=|m|}^{\infty}(1/K_n^m)\, P_n^m(z')\, P_n^m(z) = \delta(z'-z)$$

$$\Sigma_{n=|m|}^{\infty}(1/K_n^{-|m|})\, P_n^{-|m|}(z')\, P_n^{-|m|}(z) = \delta(z'-z)$$

4A. The above two items can be written in standard form for normalized $\varphi_n^m(z)$:

$$\Sigma_{n=|m|}^{\infty}\, \varphi_n^m(z')\, \varphi_n^m(z) = \delta(z'-z) \qquad \varphi_n^m(z) = P_n^m(z) / \sqrt{K_n^m} \qquad \text{// complete}$$

$$\int_{-1}^{1} dz\, \varphi_n^m(z)\varphi_k^m(z) = \delta_{n,k} \qquad n,k = |m|, |m|+1, |m|+2 \ldots\ldots \infty \qquad \text{// orthonormal}$$

Comments: The requirement that $m < 0$ is key here, and not obvious if you were to try and simply "continue" the integer n results. The non-integer results are not the same as the integer results quoted earlier.

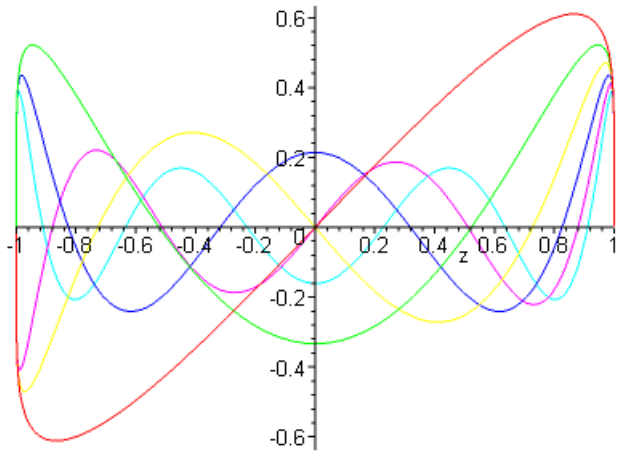
Maple Test: Here I check numerically on the orthogonality of functions $P_{4/3}^{-1/3}(z)$ and $P_{7/3}^{-1/3}(z)$ where we have m = -1/3 and so |m| = 1/3 and allowed ν values are 1/3, 4/3, 7/3 and so on.

```
evalf(Int(LegendreP(4/3,-1/3,z)*LegendreP(7/3,-1/3,z),z = -1..1));
```

$$-.3\ 10^{-13}$$

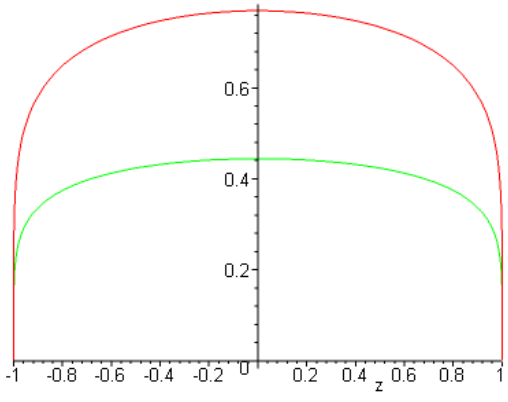
So yes, we get basically 0 as expected. Here are plots of 6 of the polynomials:

```
plot([seq(LegendreP(1/3+n,-1/3,z),n=1..6)], z = -1..1);
```



They all vanish at both ends. There is something going wrong with $P_{1/3}^{-1/3}(z)$ which has a single hump. Maple says it is complex, but I think Maple might have a bug.

```
plot([Re(LegendreP(1/3,-1/3,z)),Im(LegendreP(1/3,-1/3,z))], z = -1..1);
```



In any event, it is zero at the endpoints. If you look at the form above for P near z = 1,

$P_\nu^\mu(z) = 1/\Gamma(1-\mu)\,(1+z)^{\mu/2}(1-z)^{-\mu/2}\,F[\ -\nu, \nu+1;\ 1-\mu;\ (1-z)/2\ ]$ // Bateman on cut p 143 bot

you can see that $P_{1/3}^{-1/3}(z)$ on the cut is completely real, so Maple does have a bug and I recall reading about Legendre bugs in Maple. I *have* used _EnvLegendreCut := 1..infinity; prior to plotting which is suppose to clear out the (-1,1) region and does so for all the higher ν functions. Here is a sample point claiming the function is complex: (but let's not worry about Maple bugs right now!)

```
LegendreP(1/3,-1/3,.9);
                    .5836306242 + .3369592979 I
```

Question #4: How does all this apply to oblate spheroidal coordinates?

Mostly verbatim with details knocked out:

The cross linked separated solutions terms have this form (from Smythe + me):

$[\, A\, P_n^m(i\zeta) + B\, P_n^m(i\zeta)\,]\; P_n^m(\xi)\, \Phi_m(\varphi)$ r, θ, φ type ordering

Compared to spherical coordinates, here we see a double linkage instead of a single linkage between the left two functions, that is one big difference in the nature of the solution terms!

So we can do the cross link "trace" as before. For a full spheroid (perhaps Dirichlet problem on entire spheroid) we get m = integer quantization. Then at once for the θ type coordinate ξ we require n = integer in appropriate range on the middle P function, and also, it has to be a P! This linkage works out exactly as in sphericals. If you are running the full range of ξ (as in my full spheroid surface example), then the middle function has to be a P, and the lower index has to be an appropriate integer to m, just as in sphericals, so that the function is finite at both ξ endpoints. [If you don't run a full range of ξ, then if V = 0 on the two conical surfaces, you will get some quantized values of n that related to the zeros of P at the one or two bounding cone angles. In this case, perhaps the Q functions can also appear in the ξ world, though I have not shown them above.]

So assuming this full ξ range case, we know that the $\Phi_m(\varphi)$ are "complete" in φ, and we know that for each m, the functions $P_n^m(\xi)$ form a complete set in ξ, as n runs it range n = m, m+1..... (its "string" of values). So the combined functions $P_n^m(\xi)\, \Phi_m(\varphi)$ form a complete set for any function f(ξ,φ) on the spheroid (using the MF linkage completeness theorem) ! Hurray! Again, just as in sphericals.

At this point, there is no freedom left on the indices. Smythe points out these limits of the two functions in the square bracket. I think he really means "large ζ" limits (which I verify in raw notes)

$$A_1 P_n^m(j\zeta) \underset{c_1 \to 0}{\longrightarrow} A_2 r^n \qquad \text{and} \qquad B_1 Q_n^m(j\zeta) \underset{c_1 \to 0}{\longrightarrow} \frac{B_2}{r^{n+1}}$$

$P_n^m(i\zeta) = (2i)^n/\sqrt{\pi}\;\; \Gamma(1/2+n) / \Gamma(1+n-m)\;\; (r/c_1)^n$

$Q_n^m(i\zeta) = (-1)^m\, (2i)^{-n-1}\sqrt{\pi} / \Gamma(n+3/2)\, (r/c_1)^{-n-1}$

Main point: the role of the P and Q functions in the first factor is like the role of r^n and r^{-n-1} in sphericals. If you are doing an interior problem, you will pick the P only, and for exterior only the Q.

Notice that the large argument behavior does not depend so much on the fact that we are going off in the imaginary axis direction. We could go off in any direction and get a similar result for P and Q using our large z formula where F→ 1.

Comments: Let's look back at the confusion I was having a week ago on this subject. I kept wanting to understand the ODE system where the "range" was (0,+i∞). It is just the associated Legendre equation where z is in this range. I was wondering what the "eigenfunctions" were of this ODE in this range. I thought that this EF problem might somehow be setting values at least for n.

So here is how I now think this works out. Yes, we have an ODE system on (0,+i∞) as the interval. The 0 endpoint is regular (in the BV problem sense of Stakgold) because the p and q functions are smooth at that point, and the i∞ end is singular. As such, it is either LC or LP. I think it is LP because only the Q function will have finite s norm at i∞. The P function will not. As usual for this ODE, s(z) = 1 is the weight function, and λ = ν(ν+1) is the eigenvalue. As appropriate for a problem with a singular endpoint, we find that the spectrum of eigenvalues for ν is continuous. I think it is the entire real axis ν > -1/2 because this range makes finite s norm. This is true for any value of m which has the relationship to n noted above, and which we used in finding the large arg behavior of P. [The conclusion might hold for

any value of m as well.] So for any m, I think we have this continuous spectrum for ν from this ODE. There is no quantization of ν to get values to be finite. If you look at the small z expansions of P and Q, neither blows up at z = 0, so no quantization from that endpoint either.

So that ODE exists, but does not determine anything in this problem. The m and n values are determined completely from the "angular world".

We could consider an orange slice spheroidal problem and then we would I think have m = string of equally spaced negative values, then n = string just as we found above in the spherical case, and so the solution form is exactly the same idea as in the spherical case, and your only concern is exterior or interior. If you have an annular spheroid, then both P and Q can get involved and you specify a f(angles) on both surfaces!

So finally I think I am happy with this scenario! I imagine the prolate spheroidal situation is similar.

One extra note. We are talking the Laplace Dirichlet situation here. If we wanted to think about eigenfunctions inside a 3D metal spheroid, or orange slice, that is a different problem and we then have another eigenvalue constant. We can just think of that as the k_1^2 Helmholtz constant that MF carry around all the time. In the spherical case, this extra constant converted the simple powers like r^n into spherical $j_n(\beta^{(n+1/2)}_k r)$ Bessel functions. The m and n quantization here is the usual integer situation, and the new element is that, for each integer n, we have to find a set of zeros of j_n , and these are indexed by the letter k, and this makes all the $j_n = 0$ on the surface.

Now in the spheroidal eigenfunction inside all-metal V = 0 problem, if we add this extra element like k_1^2 or $\lambda_{n,k} = \beta^{(n+1/2)}_k$, the functions P and Q are going to be replaced by some more complicated functions and we are going to have k_1^2 get quantized by the requirement that these analogs of j_n have zeros on the spheroidal surface. I don't know what these functions are in the oblate case, but I would expect them to be more complicated than P and Q, just as the j_n are more complicated that r^n. Perhaps Smythe talks about this somewhere in his book, or perhaps MF do. These eigenfunctions could then be used to create a formal Green's Function for the spheroidal shell or orange slice etc etc using the bilinear summation series.

Question #5: How does all this apply to ellipsoidal coordinates? // verbatim

Looking back at my ellipsoidal doc #2, I am reminded of this piece of info

These equations are all of the same form:

$$\frac{d^2F}{dz^2} + \left[\frac{z}{z^2-a^2} + \frac{z}{z^2-b^2}\right]\frac{d\bar{F}}{dz} + \left[\frac{\kappa - m(m+1)z^2}{(z^2-a^2)(z^2-b^2)}\right]F = 0 \quad (10.3.83)$$

$$[z^4 - (a^2+b^2)z^2 + a^2b^2]F'' + z[2z^2 - (a^2+b^2)]F' + [\kappa - m(m+1)z^2]F = 0$$

where the solutions in the range $z > a$ are for the coordinate ξ_1, those in the range $a > z > b$ are for ξ_2, and those in the range $b > z > -b$ are for ξ_3. This equation is called the *Lamé equation*. It has the following regular singular points:

At $z = \pm b$; with indices 0 and $\frac{1}{2}$
At $z = \pm a$; with indices 0 and $\frac{1}{2}$
At $z = \infty$; with indices $m+1$ and $-m$

OK, this equation really does have all 5 regular singular points! We can compare this to WW page 197 where they had 5 points at a_1.....a_4 at the 5th at ∞, so our ellipsoidal equation is a special case of the "generalized Lame". The indices are given for each point, and I note the difference is 1/2 for four of the points, as WW commented. The z=∞ indices are as shown, functions of m. The two separation constants are called κ and m. The equation is the same equation for all three variables, just different ranges as we know, all shown in the clip above.

So what can I say about the problem in the three intervals: (a,∞), (b,a), (-b,b) or (0,b) as the case may be. Is this even a SL operator? Maybe I should look now at that last WW chapter in their last edition. I am now browsing this chapter, exactly on topic. Lame is dated 1839. But I see this is going to be quite complicated, so this is not the right place to take notes.

I took some notes and found WW to be completely irrelevant, not to mention opaque.

I then read some of my ellipsoidal META notes (doc #3) and that led me to write another doc called " Finding Solutions to the Lame Equation.doc" where I guess I understood for the first time how the separation constants get double-quantized in that world. It is all understood now, and here is my concluding comment from that doc:

Closing Comments:

I looked first at spherical, then at spheroidal coordinates, pondering the question of the cross-linkage of the separation constants and how they got quantized. In these two cases, quantization started with the azimuthal world, usually an index called m taking equally spaced values. For each such m, we found that the second separation constant ν on P_ν^m had *also* to be quantized. This happens a little differently in different cases. For a full sphere or spheroid, ν is the simple string ν = m, m+1, m+2... as this causes P to be finite at z = -1 and +1. For an orange slice of a sphere, ν = a similar string but is not integers. For a spherical section with one or two conical surfaces having V = 0, ν = a string that causes the P function to have zeroes on the conical surface(s).

When we turn to ellipsoidals, there is no azimuth to start quantization with, but as we have seen, it turns out that there is still an integral quantum number m = 0,1,2,3... just as in the azimuthal systems. In ellipsoidals, however, the second separation constant κ takes extremely strange values, not just an equally spaced string of values as in the other two cases and the ν coordinate. The mechanism which causes the double quantization in ellipsoidals is still that the solution functions be finite on their intervals, or shall we say, the solution functions exist on their intervals. In the ellipsoidal case, the three separated ODE's are all the same, so we need a solution valid on the entire interval (0,∞) and this requires truncation, and that requires double quantization.

Finally, I have to say there is really no mystery about these Lame solutions and the four species and the second kind partners. It is all completely straightforward, and there is no need to bring in the Weierstrass P function or Neville's method and all that stuff that WW get into.

Just a reminder about the exterior potential of a metal ellipsoid held at constant V. We find the solution with the lowest level functions E_0^0 = 1 for ξ_2 and ξ_3, and the F_0^0 second partner for ξ_1. This partner happens to be a Jacobi type elliptic integral function sn^{-1} of F(z,k) thing. Both MF and Kelvin and I am sure Byerly do this solution, and compress to get the elliptical plate as well. It is not rocket science, I have to keep saying that over and over.

Question #6: How does all this apply to Cartesian coordinates?

I sort of forget about these fellas. What happens here if you separate Laplace?

$$-(\partial_x^2 + \partial_y^2 + \partial_z^2)\, XYZ = k_h^2\, XYZ$$

Here I throw in a Helmholtz just for fun, so we can think eigenfunctions as well as Laplace solutions.

$$(YZ\partial_x^2 X + ZX\partial_y^2 Y + XY\partial_z^2 Z) = -k_h^2\, XYZ$$

$$\partial_x^2 X/X + \partial_y^2 Y/Y + \partial_z^2 Z/Z = -k_h^2$$

Let's set two of the separation constants to k_x^2 and k_y^2 so we have

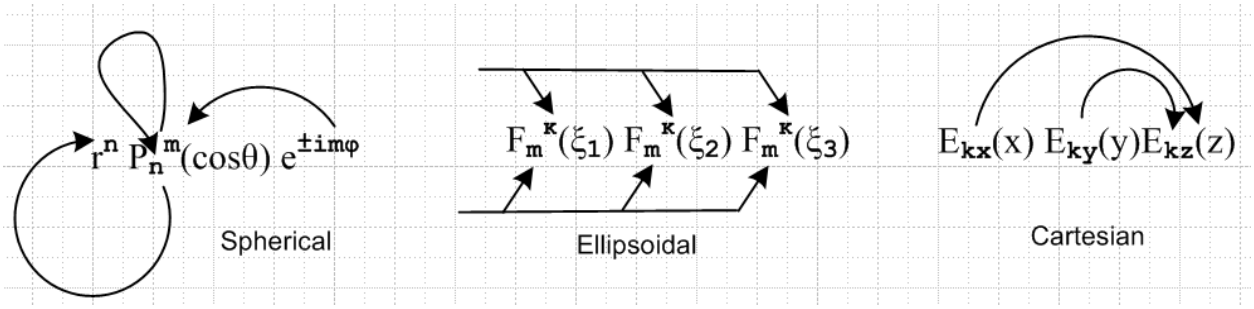
$$\partial_x^2 X/X = -k_x^2 \qquad \partial_y^2 Y/Y = -k_y^2 \qquad \partial_z^2 Z/Z = -k_h^2 + k_x^2 + k_y^2$$

The X solutions are $X(x) = \exp(\pm ik_x x)$ where we can stay hazy about the nature of k_x. The X equation has no finite singular points so series solutions (ie, these functions) converge in entire complex x plane. We then have

$$\text{solution term} = \exp(\pm ik_x x)\exp(\pm ik_y y)\exp(\pm ik_z z) = E_{kx}(x)\,E_{ky}(y)E_{kz}(z)$$

where $k_z^2 \equiv k_h^2 - k_x^2 - k_y^2$

This situation is a in a sense more like the ellipsoidals than the sphericals or spheroidals, because there is no azimuth. Without having a specific problem, all we can say is that for Laplace, there are two separation constants which I have called k_x and k_y (and $k_h = 0$ in this case), and k_z is a function of these two. Here is a comparison of the causality cross linkage structure for three coordinate systems based on the simplest basic problems (sphere, ellipsoid, box with one face free)



In our simple spherical problem, quantization begins with the azimuth m, causes integer n, and that is then forced onto the r^n third function. For Cartesian box, we get *two* of these independent starting integers n_x and n_y (see below), and these then force the quantum number for the z direction. In ellipsoidals, we don't "start" with any of the three coordinates, the m,κ quantized values sort of come from the outside -- the requirement that things truncate.

Cartesian Laplace and Eigenfunction Sample Problems

Here is just did "the box" for these two problems. The results are these:

$$u(x,y,c) = \Sigma_{nx,ny} \{ f_{nx,ny} / [sh([ik_z](n_x, n_y)c) K_{nx,ny}] \} \sin(n_x \pi x) \sin(n_y \pi y) \quad \text{// Dirichlet 1 face}$$

EF: $\varphi_{nx,ny,nz}(x,y,z) = \sin(n_x \pi x) \sin(n_y \pi y) \sin(n_z \pi z)$ $\quad \nabla^2 \varphi_{nx,ny,nz} = k_{nx,ny,nz}^2 \varphi_{nx,ny,nz}$

$$k_h^2 = (n_x \pi/a)^2 + (n_y \pi/b)^2 + (n_z \pi/c)^2 \equiv k_{nx,ny,nz}^2$$

End of raw notes page 36 there, page 16 here.