This scrap was at the end of the Appendix:

Comments: So we start with our Legendre ODE and we end up with these solutions

$$w_{\pm}(z) = P_n^{\pm m}(z) = (z+1)^{\pm m/2}\ (z-1)^{\mp m/2}\ F(-n,n+1;\ 1\mp m;\ [1-z]/2)$$

as the two independent solutions for m ≠ integer. The expansion here is around z = 1, not around z = 0. If you want an expansion around z = 0, you can use p 126 P (22) where the first term is even in z and the second is odd in z, so in general $P_n^{\pm m}(z)$ are neither even nor odd. So I am trying to set a foundation for the functions P(z) before thinking about p and q and why they are even only.

So compare the ODEs

$$L_z = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2) \qquad \text{// modified from Leg prop doc}$$
$$L_z u(z) = n(n+1)u(z)$$

$$L_z = L_\zeta = +(1+\zeta^2)\,\partial_\zeta^2 - 2\zeta\,\partial_\zeta + m^2/(1+\zeta^2)$$
$$L_\zeta v(\zeta) = n(n+1)v(\zeta)$$

I think something must change in the ODE theory when your poles are off the real axis. Well, we think that for m ≠ integer, these should still be the two independent solutions:

$$v_{\pm}(\zeta) = P_n^{\pm m}(i\zeta)$$

but we know if we use P (22) we get something like this:

$$v_+(\zeta) = P_n^m(i\zeta) = e^{i\delta}\ \{\ F(\zeta^2) + i\zeta\, G(\zeta^2)\ \}$$

so for sure this thing is complex. But we could do this:

$$v_+(\zeta)^* = P_n^m(i\zeta)^* = e^{-i\delta}\ \{\ F(\zeta^2) - i\zeta\, G(\zeta^2)\ \}$$

Now, since L_ζ is real, both v and v* should be Legendre solutions! [Wrong!] So we are cutting new ground here. Then we can say:

$$f(\zeta) = e^{-i\delta}\, v_+(\zeta) + e^{i\delta}\, v_+(\zeta)^* = 2\, F(\zeta^2) = \text{real even solution}$$

$$g(\zeta) = (1/i)\ [\ e^{-i\delta}\, v_+(\zeta) - e^{i\delta}\, v_+(\zeta)^*] = 2\,\zeta\, G(\zeta^2) = \text{real odd solution}$$

I think f(ζ) is my p function, and here I have come up with an odd real function! So finally we are getting somewhere.

Are $v_+(\zeta)$ and $v_+(\zeta)^*$ independent? I cannot write one as a multiple of the other, so I guess so.

And this is the main scrap stuff from my earlier attempts at the "why are p and q" document. I think it can all be thrown out, but I will keep it for a while.

General Fact: In Bateman's tables for P and Q forms, there are usually two terms. In general, you can define the P and Q functions as a certain one-term expression (which solves the Legendre ODE). All the other forms are then obtained by the Kummer u_i definitions and triple-relations on pp 105-108. But all the six basic functions are solutions of the HGE, so each triple relation always replaces a solution with a linear combination of two solutions. This is reflected in the HGE world, so that each term of a two term form is always itself a solution of the Legendre ODE. I have not conclusively proven this claim, but I am 99% sure it is true. We have seen above an explicit example of the claim. And I showed if for Q (40) in a separate doc "A Maple study of Bateman Q (40).doc". [I tried again to prove this in recent doc " Deriving the Bateman P and Q forms.doc" but I did not succeed. I still think it is true.]

Given the above fact, we may *generalize* our conclusions above to general z, not just real x, and say:

$$q_n^m(z)_{even} = (1-z^2)^{-m/2}\ 2A'\ F'(z^2) \qquad \text{m and n real}$$
$$q_n^m(z)_{odd} = (1-z^2)^{-m/2}\ 2\ B'\ z\ G'(z^2)$$

or, recalling that $(z^2-1)^{-m/2} = (\mp i)^{2m}\,(1-z^2)^{-m/2}$,

$$q_n^m(z)_{even} = (\pm i)^{2m}\,(z^2-1)^{-m/2}\ 2A'\ F'(z^2) \qquad \text{m and n real}$$
$$q_n^m(z)_{odd} = (\pm i)^{2m}\,(z^2-1)^{-m/2}\ 2\ B'\ x\ G'(z^2)$$

just to stay consistent. So these functions are now even and odd in general complex z, not just x in (-1,1). We can use the first form and consider these results when $z = i\zeta$ = in the imaginary z axis. then we have

$$q_n^m(i\zeta)_{even} = (1+\zeta^2)^{-m/2}\ 2A'\ F'(-\zeta^2) \qquad \text{m and n real}$$
$$(-i)q_n^m(i\zeta)_{odd} = (1+\zeta^2)^{-m/2}\ 2\ B'\ \zeta\ G'(-\zeta^2)$$

where we added the (-i) factor to kill of the i from $z = i\zeta$ on the right side linear factor. So let's then make up some new names here (by adding primes).

$$q'^m_n(\zeta)_{even} = (1+\zeta^2)^{-m/2}\ 2A'\ F'(-\zeta^2) \qquad \text{m and n real}$$
$$q'^m_n(\zeta)_{odd} = (1+\zeta^2)^{-m/2}\ 2\ B'\ \zeta\ G'(-\zeta^2)$$

Example: Let's try to compute these even and odd functions for a simple case n = 1 and m = 0 :

$$q_1^0(x)_{even} = 2\sqrt{\pi}\,(1/2)\,[\Gamma(1)/\Gamma(3/2)]\,F(-1/2,1,1/2;\,z^2) = 2\,F(-1/2,1,1/2;\,z^2)$$

= 2 times function shown below:

```
f := hypergeom([-1/2,1],[1/2],z^2);
```

$$f := \text{hypergeom}\left(\left[1, \frac{-1}{2}\right], \left[\frac{1}{2}\right], z^2\right)$$

```
f1 := convert(f,StandardFunctions);
```

$$f1 := 1 - z\,\text{arctanh}(z)$$

```
f2 :=convert(f1,ln);
```

$$f2 := 1 - z\left(\frac{1}{2}\ln(z+1) - \frac{1}{2}\ln(1-z)\right)$$

Notice that this f2 function is manifestly even because the (...) quantity is odd. The odd function is

$q_1^0(x)_{odd} = \ 2\,x\sqrt{\pi}\ \ F(0,3/2,3/2;\,x^2) \ = 2\sqrt{\pi}\ x$ // F = 1

We know that $P_1(x) = x$, so x is a good solution. And it is easy to show that the log term f2 is a solution:

```
n :=1; m := 0;
```

$$n := 1$$
$$m := 0$$

```
g := -(1-z^2)*diff(f2,z,z) + 2*z*diff(f2,z) + ( m^2/(1-z^2) - n*(n+1))*f2:simplify(%);
```

$$0$$

Compare our q_{even} result to the even q(ζ) function obtained in the P and Q doc.

I will start here by quoting some text from far above:

" Start with Q (40) in z^2, appropriate for pondering symmetry near z = 0, and we find:

$e^{-i\pi m}\ Q_n^m(z) \ = \ = (z^2\text{-}1)^{-m/2}\ (\pm i)^{m-n}\ \{\mp i\ \ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$

where the primed factors may be read from (40). Note that we have already extracted the phase factors.

One thing is (now) obvious just looking at the form: You won't be able to *isolate* either term in {...} by making a linear combination of Q(z) and Q(-z). The reason is this

$Q(+z) \ = f_1(z)\ \{\mp i\ \ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$

$Q(-\ z) \ = f_2(z)\{\pm i\ \ A'\ F'(z^2) - B'\ z\ G'(z^2)\} = -\ f_2(z)\ \{\mp i\ \ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$

$= \ f_3(z)\ \{\mp i\ \ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$

So we have to give up that idea at once."

Now let's reconsider what happens if $z = i\zeta$ with ζ real. We then have:

$e^{-i\pi m}\ Q_n^m(i\zeta) \ = \ [(\mp i)^m\ \ (1+\zeta^2)^{-m/2}]\ \ (\pm i)^{m-n}\ \{\mp i\ \ A'\ F'(-\zeta^2) + B'\ i\zeta\ G'(-\zeta^2)\}$

$$= i\ [(\mp i)^m\ (1+\zeta^2)^{-m/2}]\ (\pm i)^{m-n}\ \{\mp A'\ F'(-\zeta^2) + B'\ \zeta\ G'(-\zeta^2)\}$$

where now the upper/lower sign refers to the sign of ζ, and where we use these results to deal with the power factor,

$$(z^2-1)^{-m/2} = (1-z^2)^{-m/2}\ e^{\pm i\pi(-m/2)} \qquad \text{// page 123 (12) rule}$$

so that

$$(-\zeta^2-1)^{-m/2} = (1+\zeta^2)^{-m/2}\ e^{\pm i\pi(-m/2)} = (1+\zeta^2)^{-m/2}\ e^{\mp i\pi m/2} = (\mp i)^m\ (1+\zeta^2)^{-m/2}$$

Our conclusion is now different! We *will* be able to isolate the terms in {}. Rewrite the above as the first line below, then take ζ → -ζ to get the second line

$$e^{-i\pi m}\ Q_n^m(i\zeta) = i\ (1+\zeta^2)^{-m/2}\ (\pm i)^{-n}\ \{\mp A'\ F'(-\zeta^2) + B'\ \zeta\ G'(-\zeta^2)\}$$
$$e^{-i\pi m}\ Q_n^m(-i\zeta) = i\ (1+\zeta^2)^{-m/2}\ (\mp i)^{-n}\ \{\pm A'\ F'(-\zeta^2) - B'\ \zeta\ G'(-\zeta^2)\}$$

Oops! things just collapsed! In fact, my entire P and Q real functions document has just crashed into nothingness (that I have been working on for 2 weeks) because I made a sign error! Look more closely.

$$e^{-i\pi m}\ Q_n^m(i\zeta) = i\ (1+\zeta^2)^{-m/2}\ (+i)^{-n}\ \{-A'\ F'(-\zeta^2) + B'\ \zeta\ G'(-\zeta^2)\} \qquad \text{// } \zeta > 0$$

$$e^{-i\pi m}\ Q_n^m(i\zeta) = i\ (1+\zeta^2)^{-m/2}\ (-i)^{-n}\ \{+A'\ F'(-\zeta^2) + B'\ \zeta\ G'(-\zeta^2)\} \qquad \text{// } \zeta < 0$$

Recall way back where we wrote

$$e^{-i\pi m}\ Q_n^m(z) = (z^2-1)^{-m/2}\ \{(\pm i)^{m-n-1}\ A'\ F'(z^2) + (\pm i)^{m-n}\ B'\ z\ G'(z^2)\}$$

$$= (z^2-1)^{-m/2}\ (\pm i)^{m-n}\ \{(\pm i)^{-1}\ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$$

$$= (z^2-1)^{-m/2}\ (\pm i)^{m-n}\ \{\mp i\ A'\ F'(z^2) + B'\ z\ G'(z^2)\}$$

****

Comment: Our problem above was to consider the Legendre ODE limiting to the interval z in (-1,1) where we call it x. In retrospect, finding purely real functions f(x) which solve the Legendre ODE and which have definite parity in the range x in (-1,1) was "no big deal" (though it was a very big deal when I started and didn't know how to do it!).

Now I want to repeat the Q work, but instead I want to be talking about the flipped ζ Legendre ODE and I want to find real definite-parity solutions of that ODE on ζ in (-∞,∞). This is the same as the regular Legendre ODE on the interval z = pure imaginary = iζ .

****

Finally I am seeing the light. Start with P (22) and Q (40) in z^2, appropriate for pondering symmetry near z = 0, and we find:

$$P_n^m(z) = (z^2-1)^{-m/2} \{ A\, F(z^2) + B\, z\, G(z^2)\}$$

$$e^{-i\pi m} Q_n^m(z) = (z^2-1)^{-m/2} \{ (\pm i)^{m-n-1} A' F'(z^2) + (\pm i)^{m-n} B' z G'(z^2)\}$$

$$= (z^2-1)^{-m/2} (\pm i)^{m-n} \{ (\pm i)^{-1} A' F'(z^2) + B' z G'(z^2)\}$$

$$= (z^2-1)^{-m/2} (\pm i)^{m-n} \{ \mp i A' F'(z^2) + B' z G'(z^2)\}$$

where A,B,F,G and primed versions are all real when n, m and z are real.

Before continuing, we have to understand the meaning of our first factor

$$f(z) = \text{"}(z^2-1)^{-m/2}\text{"} = (z+1)^{-m/2} (z-1)^{-m/2}$$

I put the function in quotes as a reminder that it is a "misnomer" because the quoted complex function differs from what is shown on the right. We then have

$$f(-z) = \text{"}([-z]^2-1)^{-m/2}\text{"} = (-z+1)^{-m/2} (-z-1)^{-m/2} = (1-z)^{-m/2} (-z-1)^{-m/2}$$

$$= [\, e^{\mp i\pi(-m/2)} (z-1)^{-m/2}]\, [\, e^{\mp i\pi(-m/2)} (z+1)^{-m/2}]$$

$$= e^{\mp i\pi(-m)} (z-1)^{-m/2} (z+1)^{-m/2} = e^{\pm i\pi m} \text{"}(z^2-1)^{-m/2}\text{"} = (\pm i)^{2m} \text{"}(z^2-1)^{-m/2}\text{"}$$

Therefore, when we take z→-z in our two equations, we take this into account and write

$$P_n^m(+z) = (z^2-1)^{-m/2} \{ A\, F(z^2) + B\, z\, G(z^2)\}$$
$$P_n^m(-z) = (\pm i)^{2m} (z^2-1)^{-m/2} \{ A\, F(z^2) - B\, z\, G(z^2)\}$$

$$e^{-i\pi m} Q_n^m(+z) = (z^2-1)^{-m/2} (\pm i)^{m-n} \{ \mp i A' F'(z^2) + B' z G'(z^2)\}$$
$$e^{-i\pi m} Q_n^m(-z) = (\pm i)^{2m} (z^2-1)^{-m/2} (\mp i)^{m-n} \{ \pm i A' F'(z^2) - B' z G'(z^2)\}$$

$$= (\pm i)^{2m} (z^2-1)^{-m/2} (\pm i)^{n-m} \{ \pm i A' F'(z^2) - B' z G'(z^2)\}$$

$$= \quad (z^2\text{-}1)^{-m/2}\,(\pm i)^{n+m}\,\{\pm i\;\; A'\,F'(z^2) - B'\,z\,G'(z^2)\}$$

$$= \quad (z^2\text{-}1)^{-m/2}\,(\pm i)^{m+n}\,\{\pm i\;\; A'\,F'(z^2) - B'\,z\,G'(z^2)\}$$

So our two Q's are these:

$$e^{-i\pi m}\;Q_n^m\,(+z) \;=\; (z^2\text{-}1)^{-m/2}\,(\pm i)^{m-n}\,\{\mp i\;\; A'\,F'(z^2) + B'\,z\,G'(z^2)\}$$
$$e^{-i\pi m}\;Q_n^m\,(-z) \;=\; (z^2\text{-}1)^{-m/2}\,(\pm i)^{m+n}\,\{\pm i\;\; A'\,F'(z^2) - B'\,z\,G'(z^2)\}$$

Let's do Q first since it is harder. In order to construct an even function , we need to cancel the second terms. So consider:

$$(\mp i)^{m-n}\,e^{-i\pi m}\;Q_n^m\,(+z) \;+\; (\mp i)^{m+n}\,e^{-i\pi m}\;Q_n^m\,(-z)$$

$$= (z^2\text{-}1)^{-m/2}\,\{\mp i\;\; A'\,F'(z^2) + B'\,z\,G'(z^2)\}$$
$$+ (z^2\text{-}1)^{-m/2}\;\; \{\pm i\;\; A'\,F'(z^2) - B'\,z\,G'(z^2)\}$$

$$= 0$$

For n, m, z real, these two functions have mixed symmetry, they are neither even nor odd functions of z. And the function P is real but Q is complex. We could construct some even and odd functions like this, AND have them all be real : [the sum of any two ODE solutions is also an ODE solution]

$$P_n^m(z) + P_n^m(-z) \;=\; (z^2\text{-}1)^{-m/2}\,2A\,F(z^2) \qquad \text{even}$$
$$P_n^m(z) - P_n^m(-z) \;=\; (z^2\text{-}1)^{-m/2}\,2B\,z\,G\,(z^2) \qquad \text{odd}$$

For Q, things are so messy now that I have to write it all out longhand. Here we go:

$$e^{-i\pi m}\;[e^{\mp i\sigma}\,Q_n^m(z) + e^{\pm i\sigma}\,Q_n^m(-z)] \;=$$

$$= \; e^{\mp i\sigma}\;\{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\pm i\delta}\;\,A'\,F'(z^2) + e^{\pm i\sigma}\,B'\,z\,G'(z^2)\}$$
$$+\, e^{\pm i\sigma}\;\{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\mp i\delta}\;\,A'\,F'(z^2) - \,e^{\mp i\sigma}\,B'\,z\,G'(z^2)\}$$

$$= \quad \{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\pm i(\delta-\sigma)}\;\,A'\,F'(z^2) + B'\,z\,G'(z^2)\}$$
$$+ \;\{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\mp i(\delta-\sigma)}\;\,A'\,F'(z^2) - \,B'\,z\,G'(z^2)\}$$

$$= \quad \{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\pm i(\delta-\sigma)}\;\,A'\,F'(z^2)\}$$
$$+ \;\{\,(z^2\text{-}1)^{-m/2}\,\{\,e^{\mp i(\delta-\sigma)}\;\,A'\,F'(z^2)\}$$

$$= \quad \{\,(z^2\text{-}1)^{-m/2}\,A'\,F'(z^2)\{\,e^{\pm i(\delta-\sigma)} + \;\,e^{\mp i(\delta-\sigma)}\,\}$$

$= \{ (z^2-1)^{-m/2} A' F'(z^2) \{ 2\cos(\delta-\sigma)\}$

$= \{ (z^2-1)^{-m/2} A' F'(z^2) \{ 2\cos(-\pi/2)\}$

$= \{ (z^2-1)^{-m/2} A' F'(z^2) \{ 0)\}$

$= 0$

$= (z^2-1)^{-m/2} [e^{\pm i(\delta-\sigma)} + e^{\mp i(\delta-\sigma)}] A' F'(z^2)$

$= (z^2-1)^{-m/2} \cos(\delta-\sigma)\ 2A' F'(z^2)$

$= 0$ even

(i) $e^{-i\pi m} [e^{\mp i\delta} Q_n^m(z) - e^{\pm i\delta} Q_n^m(-z)] = (z^2-1)^{-m/2}$ B' z G'(z^2) $[e^{\mp i(\delta-\sigma)} - e^{\pm i(\delta-\sigma)}]$(i) odd

$= (z^2-1)^{-m/2} \sin(\delta-\sigma)\ 2A' F'(z^2)$

$= -(z^2-1)^{-m/2}\ 2A' F'(z^2$

To summarize:

$p_e(z) \equiv P_n^m(z) + P_n^m(-z)$ $= (z^2-1)^{-m/2}$ $2A\ F(z^2)$ even

$p_o(z) \equiv P_n^m(z) - P_n^m(-z)$ $= (z^2-1)^{-m/2}$ $2B\ z\ G\ (z^2)$ odd

$q_e(z) \equiv e^{-i\pi m} [Q_n^m(z) + Q_n^m(-z)]$ $= (z^2-1)^{-m/2} \cos(\delta-\sigma)\ 2A' F'(z^2)$ even

$q_o(z) \equiv$ (i) $e^{-i\pi m} [e^{\mp i\delta} Q_n^m(z) - e^{\pm i\delta} Q_n^m(-z)]$ $= (z^2-1)^{-m/2} \sin(\delta-\sigma)\ B' z G'(z^2)$ odd

In this way, we can generate four real solutions of the Legendre ODE which have definite parity. For general m, any two solutions (or our 6 shown above) can be taken as the two independent functions. For integer m, you have to be more careful about which pairs of the 6 solutions are really independent. I cannot remember ever seeing these four new solutions mentioned.

Even and odd P and Q functions of ζ.

Let's just rewrite the four equations above as

$p_e(i\zeta) \equiv P_n^m(i\zeta) + P_n^m(-i\zeta) = (-\zeta^2-1)^{-m/2}$ 2A F($-\zeta^2$) even

$p_o(i\zeta) \equiv P_n^m(i\zeta) - P_n^m(-i\zeta) = (-\zeta^2-1)^{-m/2}$ 2B $i\zeta$ G ($-\zeta^2$) odd

$q_e(i\zeta) \equiv e^{-i\pi m} [Q_n^m(i\zeta) + Q_n^m(-i\zeta)] = (-\zeta^2-1)^{-m/2}$ cos(δ) 2A' F'($-\zeta^2$) even

$q_o(i\zeta) \equiv (i)\, e^{-i\pi m} [\, e^{\mp i\delta} Q_n^m(i\zeta) - e^{\pm i\delta} Q_n^m(-i\zeta)] = (-\zeta^2-1)^{-m/2}$ sin(δ) B' $i\zeta$ G'($-\zeta^2$) odd

where now ± refers to the sign of ζ. Now we can back up and write

$(z^2-1)^{-m/2} = (1-z^2)^{-m/2}\, e^{\pm i\pi(-m/2)}$ // page 123 (12) rule

so that

$(-\zeta^2-1)^{-m/2} = (1+\zeta^2)^{-m/2}\, e^{\pm i\pi(-m/2)} = (1+\zeta^2)^{-m/2}\, e^{\mp i\pi m/2} = (\mp i)^m\, (1+\zeta^2)^{-m/2}$

We install this change, AND we left divide the 2nd and 4th equations by i , AND then we multiply all four equations by $(\pm i)^m$;

$p_e(i\zeta) \equiv P_n^m(i\zeta) + P_n^m(-i\zeta) = (\mp i)^m\, (1+\zeta^2)^{-m/2}$ 2A F($-\zeta^2$) even

$p_o(i\zeta)/i \equiv [P_n^m(i\zeta) - P_n^m(-i\zeta)]/i = (\mp i)^m\, (1+\zeta^2)^{-m/2}$ 2B ζ G ($-\zeta^2$) odd

$q_e(i\zeta) \equiv e^{-i\pi m} [Q_n^m(i\zeta) + Q_n^m(-i\zeta)] = (\mp i)^m\, (1+\zeta^2)^{-m/2}$ cos(δ) 2A' F'($-\zeta^2$) even

$q_o(i\zeta)/i \equiv e^{-i\pi m} [\, e^{\mp i\delta} Q_n^m(i\zeta) - e^{\pm i\delta} Q_n^m(-i\zeta)] = (\mp i)^m\, (1+\zeta^2)^{-m/2}$ sin(δ) B' ζ G'($-\zeta^2$) odd

Start with P (22) and Q (40) in z^2, appropriate for pondering symmetry near z = 0, and we find:

$P_n^m(z) = (z^2-1)^{-m/2} \{ A\, F(z^2) + B\, z\, G(z^2)\}$

$e^{-i\pi m} Q_n^m(z) = (z^2-1)^{-m/2} \{ e^{\pm i\delta}\, A' F'(z^2) + e^{\pm i\sigma} B'\, z\, G'(z^2)\}$ $\delta = (\pi/2)(m-n-1)$

Now install z = iζ [note that A' and B' are here defined to include the usual Q function $e^{+i\pi m}$ factor]

$P_n^m(i\zeta) = (-\zeta^2-1)^{-m/2} \{ A\, F(-\zeta^2) + B\, i\zeta\, G(-\zeta^2)\}$

$$e^{-i\pi m} Q_n^m(i\zeta) = (-\zeta^2-1)^{-m/2} \{ e^{\pm i\delta} A' F'(-\zeta^2) + e^{\pm i\sigma} B' i\zeta G'(-\zeta^2)\} \qquad \delta = (\pi/2)(m-n-1)$$

where now ± refer to the sign of ζ , before it was sign of Im(z). Now we can back up and write

$$(z^2-1)^{-m/2} = (1-z^2)^{-m/2} e^{\pm i\pi(-m/2)} \qquad \text{// page 123 (12) rule}$$

so that

$$(-\zeta^2-1)^{-m/2} = (1+\zeta^2)^{-m/2} e^{\pm i\pi(-m/2)} = (1+\zeta^2)^{-m/2} e^{\mp i\pi m/2} = (\mp i)^m (1+\zeta^2)^{-m/2}$$

and we get these improved results

$$P_n^m(i\zeta) = (\mp i)^m (1+\zeta^2)^{-m/2} \{ A F(-\zeta^2) + B i\zeta G(-\zeta^2)\}$$

$$e^{-i\pi m} Q_n^m(i\zeta) = (\mp i)^m (1+\zeta^2)^{-m/2} \{ e^{\pm i\delta} A' F'(-\zeta^2) + e^{\pm i\sigma} B' i\zeta G'(-\zeta^2)\} \qquad \delta = (\pi/2)(m-n-1)$$

or

$$(\pm i)^m P_n^m(i\zeta) = (1+\zeta^2)^{-m/2} \{ A F(-\zeta^2) + B i\zeta G(-\zeta^2)\} \qquad (**)$$
$$e^{-i\pi m} (\pm i)^m Q_n^m(i\zeta) = (1+\zeta^2)^{-m/2} \{ e^{\pm i\delta} A' F'(-\zeta^2) + e^{\pm i\sigma} B' i\zeta G'(-\zeta^2)\} \qquad \delta = (\pi/2)(m-n-1)$$

which the reader should compare to

$$P_n^m(z) = (z^2-1)^{-m/2} \{ A F(z^2) + B z G(z^2)\} \qquad (*)$$
$$e^{-i\pi m} Q_n^m(z) = (z^2-1)^{-m/2} \{ e^{\pm i\delta} A' F'(z^2) + e^{\pm i\sigma} B' z G'(z^2)\} \qquad \delta = (\pi/2)(m-n-1)$$

Now recall how from these two equations (*), we obtained the following four:

$$p_e(z) \equiv P_n^m(z) + P_n^m(-z) = (z^2-1)^{-m/2} \quad 2A F(z^2) \quad \text{even}$$
$$p_o(z) \equiv P_n^m(z) - P_n^m(-z) = (z^2-1)^{-m/2} \quad 2B z G(z^2) \quad \text{odd}$$
$$q_e(z) \equiv e^{-i\pi m} [Q_n^m(z) + Q_n^m(-z)] = (z^2-1)^{-m/2} \cos(\delta) \; 2A' F'(z^2) \quad \text{even}$$
$$q_o(z) \equiv (i) e^{-i\pi m} [e^{\mp i\delta} Q_n^m(z) - e^{\pm i\delta} Q_n^m(-z)] = (z^2-1)^{-m/2} \sin(\delta) \; B' z G'(z^2) \quad \text{odd}$$

In similar fashion, using the same steps, we start from (**) and create the following four:

$$p'_e(\zeta) \equiv (\pm i)^m P_n^m(i\zeta) + (\mp i)^m P_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \quad 2A F(-\zeta^2) \quad \text{even}$$
$$p'_o(\zeta) \equiv (\pm i)^m P_n^m(i\zeta) - (\mp i)^m P_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \quad 2B i\zeta G(-\zeta^2) \quad \text{odd}$$
$$q'_e(\zeta) \equiv e^{-i\pi m} [(\pm i)^m Q_n^m(i\zeta) + (\mp i)^m Q_n^m(-i\zeta)] = (1+\zeta^2)^{-m/2} \cos(\delta-\sigma) \; 2A' F'(-\zeta^2) \quad \text{even}$$
$$q'_o(\zeta) \equiv (i) e^{-i\pi m} [e^{\mp i\delta} (\pm i)^m Q_n^m(i\zeta) - e^{\pm i\delta} (\mp i)^m Q_n^m(-i\zeta)] = (1+\zeta^2)^{-m/2} \sin(\delta-\sigma) \; B' i\zeta G'(-\zeta^2) \quad \text{odd}$$

Notice that, when we take ζ to -ζ, the term $(\pm i)^m P_n^m(i\zeta)$ becomes $(\mp i)^m P_n^m(-i\zeta)$, and so on, because now the signs refer to sign of ζ, upper sign if ζ > 0. We are not quite done, however, because we need to divide the 2nd and 4th equations by i to real RHS's. So

$p'_e(\zeta) \equiv (\pm i)^m P_n^m(i\zeta) + (\mp i)^m P_n^m(-i\zeta)$ $= (1+\zeta^2)^{-m/2}$ $2A\ F(-\zeta^2)$ even

$p'_o(\zeta)/i \equiv (\pm i)^m [P_n^m(i\zeta) - (\mp i)^m P_n^m(-i\zeta)]/i$ $= (1+\zeta^2)^{-m/2}$ $2B\ i\zeta\ G(-\zeta^2)$ odd

$q'_e(\zeta) \equiv e^{-i\pi m} [(\pm i)^m Q_n^m(i\zeta) + (\mp i)^m Q_n^m(-i\zeta)]$ $= (1+\zeta^2)^{-m/2}$ $\cos(\delta-\sigma)\ 2A'\ F'(-\zeta^2)$ even

$q'_o(\zeta)/i \equiv e^{-i\pi m} [\ e^{\mp i\delta} (\pm i)^m Q_n^m(i\zeta) - e^{\pm i\delta} (\mp i)^m Q_n^m(-i\zeta)]$ $= (1+\zeta^2)^{-m/2}$ $\sin(\delta-\sigma)\ B'\ i\zeta\ G'(-\zeta^2)$ odd

So here our four real functions of ζ that satisfy the flipped Legendre equation! We can simplify a bit as follows:

$$\delta = (\pi/2)(m-n-1) \quad \Rightarrow \quad e^{\pm i\delta} = e^{\pm i(\pi/2)(2\delta/\pi)} = (\pm i)^{(2\delta/\pi)} = (\pm i)^{m-n-1}$$

Then we have

$$e^{\pm i\delta}(\mp i)^m = (\pm i)^{m-n-1}(\mp i)^m = (\pm i)^{-n-1}$$

But these functions are complex. Now just as before, we can construct four real functions of definite symmetry as follows:

$(\pm i)^m P_n^m(i\zeta) + (\mp i)^m P_n^m(-i\zeta) = (1+\zeta^2)^{-m/2}\ 2A\ F(-\zeta^2)$ even

$(1/i)[\ (\pm i)^m P_n^m(i\zeta) - (\mp i)^m P_n^m(-i\zeta)] = 2B\ \zeta\ G(-\zeta^2)$ odd

$(\pm i)^m Q_n^m(i\zeta) + (\mp i)^m Q_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \{ e^{\pm i\delta} + e^{\mp i\delta} \} A'\ F'(-\zeta^2)$

$= (1+\zeta^2)^{-m/2} \cos(\delta)\ 2\ A'\ F'(-\zeta^2)$

$(\pm i)^m e^{\mp i\delta} Q_n^m(i\zeta) - (\mp i)^m e^{\pm i\delta} Q_n^m(-i\zeta) = (1+\zeta^2)^{-m/2} \{ e^{\mp i\delta} + e^{\pm i\delta}\ A'\ F'(-\zeta^2)$