An Overview of Erdelyi's Chapter on Orthogonal Polynomials PhL 1.14.05

Overview of the entire Chapter. The first five sections are completely general, and the results apply to all orthogonal polynomial systems. Remember that all you need is an interval (a,b) and a weight function w(x) (non neg) and boom, you have an orthog poly system! The next two sections give general results for a particular class of orthog polys known as the "classical" orthog polys. Then we have specific discussions of the classical polys: Jacobi = Hypergeometric(Gegenbauer, Legendre, Chebyshev), generalized Laguerre, and Hermite. Then come two sections on asymptotic behaviors, then sections on the zeros of our functions. After a section on inequalities, he discusses the notion of expanding functions as lincoms of ortho polys. There are many amazing "sum rules" for simple functions which involve our fancy ortho polys. Then comes a section on obscure ortho poly systems, followed by a section or two on "discrete variable" versions of this stuff. Then the final sections are on certain ortho polys which have the "difference" version of a Rodriguez formula. The only sections I "studied" were the general sections, so I stopped after taking a brief look at the Jacobi section.

10.1 Systems of Ortho Polys. Things are based on a scalar product hence a Hilbert Space, and it can be continuous or discrete. **Gram Schmidt** can be used to get things orthogonal. Page 154 shows how lin indep but non-orthogonal ψ can be made into orthogonals φ using G-S.

Abruptly the method changes, and we are given an alternative (and I think preferred) method which takes these same ψ and creates orthog ϕ functions using a determinant (10). The upper left part of this thing is G_{n-1}, the **Gram Determinant**. The ϕ are only orthogonal, but you can use (11) to make orthonormal φ. Notice that the determinant size scales up as the index n on ϕ_n gets larger.

We are reminded of the fundamental fact: any interval and weight gives an orthogonal system. This is not exactly proven, but once we set $\psi_n = x^n$ in the next section, the determinant/matrix method (10) gives you your system, so it must exist! It is also more or less unique.

10.2 The Approximation Problem. The term Hilbert Space does not appear in this book really. The point made here is that the usual **Fourier coefficients** $a_n = (f,\varphi_n)$ for some arbitrary function f(x) corresponding to your ortho poly system φ_n are those that minimize the least squares error of a fit. Recall that as you add more terms to get a better fit, you don't have to change the earlier coefficients -- an effect that results from using an orthonormal set. Otherwise you have to deal with normal equations as in Scheid. In the Scheid Hilbert Space discussion, we know that the Fourier coefficients take us to the closest (think metric) point to f within the subspace to which we are restricted, ie, a sum of polys to some degree. If you take a finite number of terms, or an infinite number and you are unlucky and your space is non-Hilbert (as discussed in the **Riesz-Fischer** theorem), then the sum of the squares of the Fourier coefficients will be $< \| f \|^2$ by L_2, and this is called **Bessel's Inequality**. In any reasonable system, if you take the infinite sum, you get $= \| f \|^2$ and this is called **Parseval's Formula**. The **Weierstrass Theorem** idea is mentioned that you can converge to a perfect fit by increasing the degree of your polynomial fit. There can be problems with infinite intervals. So all these comments do in fact relate to "the approximation problem" where we try to "fit" some arbitrary function with our orthog polys.

10.3 General Properties of Ortho Polys. We now dispense with the general ψ idea, and work specifically with $\psi_n(x) = x^n$ which we know are lin indep. We define c_n as $(1,x^n)$, these are the **moments**. Then at once we can draw our Gram determinant and our matrix/det form for what we now call the $\mathbf{p_n(x)}$. The item $\mathbf{k_n}$ is the coefficient of the leading power in p_n -- this fact follows from (4) since the subdet for x^n is exactly G_{n-1}. These p_n are NOT normalized and have some $\mathbf{h_n} = (p_n, p_n)$ = as shown. You are allowed

to set k_n however you like (but it always retains its interpretation just stated). If you select k_n as shown in mid page, you can make the p_n be orthonormal. It is stated (with a proof I did not study) that the $p_n(x)$ has all it's n real **zeros** inside the interval (a,b). Some other facts on zeros are quoted.

The first major item is the **recurrence formula** (7). This is derived and I wrote it up, and it is certainly not obvious! There is one typo here that is minor on page 159. We now have new constants called $\mathbf{r_n}$, $\mathbf{A_n}$, $\mathbf{B_n}$ **and** $\mathbf{C_n}$ in addition to our k_n and h_n, and we also add $\mathbf{k_n'}$ for the second leading coefficient in $p_n(x)$.

The next big enchiladas are the two **Christoffel-Darboux formulas** which have finite sums of p_n's on the left and p_n and p_n' on the right. Again, I did these in detail and they are written up. Proofs are moderate, not one-liners. I skipped **Christoffel's formula** which shows how you can start with the ortho system $\{w,p_n\}$ and create a new otho system $\{w\rho, q_n\}$, ρ must be a poly.

10.4 Mechanical Quadrature. We are given a very brief overview of Gaussian quadrature, and the main act here is the formula for the **Gaussian Weights** (8). Christoffel generalized Gauss's work here and so the weights are called **Christoffel Numbers** in this book. I skipped the rest. This formula has a denominator of the form $p_n'(x_\upsilon)\, p_{n+1}(x_\upsilon)$. We can use the recursion (7) to get a similar formula with p_{n+1} replaced by p_{n-1} but we don't yet have any way to replace the derivative $p_n'(x_\upsilon)$. It is only when we get to the classical ortho polys that we get a $p_n'(x_\upsilon)$ formula! I then skipped the end of this section (I usually do!).

10.5 Continued Fractions. I skipped this entire section, but something called an **associated polynomial** is defined , ie, there is some $q_n(x)$ that you can associate with $p_n(x)$.

10.6. The Classical Polynomials. (p163) This is a very interesting section. We are given three alternative "definitions" called (i), (ii) and (iii) on page 164. It is stated that each one implies the other two. The first one (i) is the most obscure and the least used in this text. It just says that $\{ p_n(x) \}$ are classical if $\{ p_n'(x) \}$ *also* form an ortho poly system. Erdelyi comments on this just in passing in the Jacobi section, but we never really do anything with this fact. The second one (ii) states that the $p_n(x)$ satisfy a second order ODE with A(x) and B(x) and λ_n. Soon we learn that only a small number of forms for A(x) really appear, called X(x), so I am a little unclear on (ii). They say that solutions of the ODE can be "reduced" to an ortho system. For the purposes of this book, the main "definition" is (iii) which says that there exists a poly X(x) and a **Rodriguez formula** such that $\mathbf{K_n w\ p_n = D^n[wX^n]}$ where w of course is the weight function, and X is some polynomial. Here we introduce the constants $\mathbf{K_n}$. If you were to insist on some h_n, then selecting K_n forces a value for k_n. This relation appears later as (6) on page 167.

Now on page 165 we have a fascinating fact. The claim is that poly X(x) can have degree k = 0, 1 or 2 and nothing else. Higher degrees cause contradictions. Without loss of generality, we can take the three cases to be X(x) = 1, x and either $(1-x)^2$ or $1 - x^2$. This last choice is a **cause of confusion** because this section of the text uses $(1-x)^2$, but the Jacobi section uses $1 - x^2$. In one case we have X" = 2 and in the other we have X" = -2, for example.

So the upshot here is that the classical ortho polys now fall into **three distinct groups**, which we will for the moment call X = 1, x or $1-x^2$. It is not quite clear how they arrive at the interval (a,b) for these three groups, but the intervals are all different. The cases are finite, semi-finite and double infinite and probably anything else can be mapped onto one of these three cases. The table is shown on page 164. Here are the names: (in order k = 2, k=0 and then k=1).

1) for $X = 1-x^2$ we have on (-1,1) the most complex polys which are the **Jacobis** called $P^{(\alpha,\beta)}_n(x)$ where α and β are arbitrary parameters you get to pick (they appear in the weight function). Notice that the Jacobis have three indices, like the hypergeometric function does. Page 170 shows that in fact the Jacobis are just an alternative form to the **Hypergeometric Polynomials**. The hypergeometric function F(a,b;c;z) is a polynomial when the first argument is a negative integer, and we see on page 170 how the Jacobis are alternate most-general forms of the hypergeometric polys.

If you simplify the choice of α,β so that $\alpha = \beta$, then these reduce to the **Gegenbauers** $C^{\lambda}_n(x)$, aka **ultrasphericals.** A further simplification so $\alpha = \beta = 0$ gives $w(x) = 1$ and you have the **Legendres** aka **sphericals**. Another special case is when $\alpha = \beta = -1/2$, you get the **Chebyshev** polys.

2) for X = 1 you get the **Hermites** $H_n(x)$, interval is $(-\infty,\infty)$ and $w = e^{-xx}$

3) for X = x you get the **generalized Laguerres** $L^{\alpha}_n(x)$, interval $(0,\infty)$ and $w = x^{\alpha}e^{-x}$

Notice that the **associated Legendre functions** which appear in spherical harmonics are not even polynomials in general, so they are not an ortho poly system. Similarly, Bessel functions are not polynomials, nor are any of the other famous special functions that we deal with.

10.7 General Properties of the classical ortho polys. This is a meat and potatoes section, short but dense. The opener on page 166 verifies the fact that $(f,p_n) = 0$ when f is a poly of degree < n. This uses a **parts integration** idea with Rodriguez that we find ourselves using many times in proofs.

Now the first Whopper of this section: the **2nd order ODE is derived** from scratch from the Rodriguez formula. This required a simply massive amount of algebra, including lots of uses of the **Leibniz' formula** for $D^n(fg)$. In the end, I was able to produce results (1) which is the ODE. I once thought you could do this trivially on an envelope, ha! Notice the form

$$[\, X(x)\, D^2 + K_1 p_1(x)\, D + \lambda_n \,]\, p_n(x) = 0$$

The D^2 function is our Rodriguez function X. The D function is of the form (ax + b), linear. And finally, the "eigenvalue" λ_n is given by a fairly simple formula in (2), all derived by me in the companion document. Remember, a general ortho poly has no Rodriguez and no ODE ! Only the classicals have these things.

Whopper #2 is **the differentiation formula** (4) on page 167. This took me probably 12 pages of algebra in the companion document! Luckily I got it to work without having to dig up Tricomi's Italian paper of 1948. Notice the appearance here of two more "constants" called α_n and β_n.

Smaller result (6) gives the connection between h_n, k_n and K_n and an integral of X^n w.

Whopper #3 (for me at least) are the **Gaussian Weight** formulas (7) converted to the forms I like. Now we can use our diff formula to get rid of the $p_n'(x)$, for example.

We conclude on page 168 with an outline of how Erdelyi is going to do each specific section. He seems to not show k_n' in favor of showing $r_n = k_n'/ k_n$. His list has 10 constants, and they are now all very familiar to me!