Bateman Chapter VI: Confluent Hypergeometric Functions PhL 12.13.08

Contents of This Document

Comments on Bateman, A&S and Jahnke-Emde 1
Bateman 2
Abramowitz and Segun (AS) 2
Jahnke and Emde. 3
Magnus and Oberhettinger 3
List of Special Function books: 4

Notes on the Hypergeometric Functions F(a,b,c,z) 4
Loop Around Branch Point Theorem: 7
The Riemann Equation and P-function. (page 89) 8
Table of Formulas for HG: 9

Notes on the confluent hypergeometric functions 9
6.1 Orientation. 9
6.2 ODEs 10
6.3 Solution of CHG equation near x=0. 11
6.4 Relations for Φ(a,c;x) 11
6.5 Basic integral representations (255) 11
6.6 Relations of the Ψ function (257) 11
6.7 Fundamental solutions 11
6.7.1 The logarithmic case (260) 12
6.8 More on the Ψ function. (p 262) 12
6.9 Whittaker functions (264) 12
6.9.1 Bessel functions 12
6.9.2 What other functions can be expressed in terms of Φ and Ψ ? 12
6.10 Laplace Transforms which involve Φ and Ψ functions. 12
6.11 Fancy contour integral representations of Φ and Ψ and Whittakers (271) 12
6.12 Series of other functions which add up to Φ and Ψ functions. 13
6.13 Asymptotic behavior of Φ and Ψ in x and also in c and a 13
6.15.1 Series of Φ functions which add up to something expressible in closed form (283) 13
6.15.2 Integrals of Φ and Ψ functions 13
6.15.3 Products of Φ and Ψ functions and integrals of same 13
6.16 Zeros of Φ and Ψ. 13

How can we connect Messiah with Bateman on the confluent HG functions? 13
Connection with A&S page 504 16
Connection with GR 18

Comments on Bateman, A&S and Jahnke-Emde

Bateman

Harry Bateman (1882-1946) was a Caltech Prof in Math and Physics and Aeronautics who was very good at the math of special functions and their ODE's. He had a quest to gather this stuff up and publish it in a sort of encyclopedia , all of course pre-web. To this end, he accumulated shoe-boxes full of notes. Naturally, he died in 1946 before this plan could be implemented. However, funding was provided by the Office of Naval Research (he must have had ongoing grants with that entity) and Caltech to allow a crew of 4 top notch math folks to come in and get the job done, with a support staff of 6 younger math kids to do the grunt work no doubt. The entire project was called the Bateman Manuscript Project and lasted perhaps two years 1946-1948, just before my birth. The hot shots hired to finish the job were Erdelyi who was the editor, then Magnus, Oberhettinger and Tricomi. The result was 3 volumes on Special Functions and then 2 volumes of Integral Transforms. I own the first 2 volumes of the first series. The publication dates of my books are 1953 and have errata at the start. I don't know when I got these two books or why I failed to get volume 3, nor do I know the contents of volume 3. I see claims that Dover is republishing these books in 2007. (978-0-486-44616-5). But they don't really seem to exist, so I sent in a request to Dover for info. [publishing was cancelled they said]

Volume 1:

Chapter 1	**Gamma** and related
Chapter 2	**Hypergeometric** with separate formulas section
Chapter 3	**Legendre** functions
Chapter 4	Generalized Hypergeometric
Chapter 5	Further Generalized Hypergeometric: MacRoberts, Meijer, multiple variables
Chapter 6	**Confluent Hypergeometric**

Volume 2:

Chapter 7	**Bessel** with separate formulas section
Chapter 8	**Parabolic Cylinder** and related
Chapter 9	**Incomplete Gamma** and related
Chapter 10	**Orthogonal Polynomials**
Chapter 11	Spherical and Hyperspherical Harmonics (Gegenbauer polys)
Chapter 12	Orthogonal Polys in several variables
Chapter 13	**Elliptic** integrals and functions

The word coulomb does not even appear in the Bateman index, but AS have a whole chapter on same.

Abramowitz and Segun (AS)

Just read the preface. The emphasis of the preface is on "tables" and how they are still needed in 1954 despite computers. Plan was made for this collection at that conference with MIT and NBS and NSF funding. Abramowitz of the NBS was in charge but had a heart attack in 1958, Segun took over as editor. First printing was 1964. My copy is the "fifth Dover printing" dated 1968. In each printing, errors are corrected. AS was produced by the US gov and therefore has no copyright, so is on line at several places in scanned form (but with OC search), here is a good source

http://www.math.sfu.ca/~cbm/aands/

Oddly, the preface makes no mention of Bateman which came out in 1953, but Erdelyi was on the AS project. I presume the AS difference was "tables" and "quoted formulas", whereas Bateman has no tables and lots of descriptive text, some of which I have just been reading.

Here is a clip from my AS copyright page:

"Originally issued June 1964. Second printing, November 1964. Third printing, March 1965. Fourth printing, December 1965. Fifth printing, August 1966. Sixth printing, November 1967. Seventh printing, May 1968. Eighth printing, 1969. Ninth printing, November 1970."

Issued June 1964

Tenth Printing, December 1972, with corrections

So if you think there is an error, you can look at the scanned 10th printing which is pointed to at the above site. Amazon is now selling the following version: (corrected errors have * marking!)

This ninth Dover printing conforms to the tenth (December 1972) printing by the Government Printing Office, except that additional corrections have been made on pages 18, 79, 80, 82, 408, 450, 786, 825 and 934.

so the "ninth Dover printing" is the same as the "tenth GPO printing". Paper is about $24.

Jahnke and Emde.

The A&S preface says it was time to update Jahnke-Emde. This previous German project put out editions starting in 1909 and ending in 1945. The first author was 1863-1921. Third author Losch updated the thing in 1960. There was a 1933 edition and a 1938 as well.

A predecessor to Abramowitz and Stegun is an amazing book by Jahnke and Emde, "Tables of Higher Functions", published in Germany in 1909. (Reprinted by Dover in 1945 and by McGraw Hill in 1960.) Jahnke and Emde, and their students, computed tables of dozens of different functions. They also created, again by hand, intricate 2- and 3-dimensional graphs showing the behavior of these functions in the complex plane. I've included two of my favorite graphs here. One shows the complex valued gamma function. The height of the surface is the modulus, or absolute value, and the contour lines are the modulus and phase. Since

$\Gamma(n) = \Gamma(n+1)/n$

you can see that this function has poles at the negative integers. The other graph shows the Hankel function, or complex valued Bessel function. I've made MATLAB versions of these graphs and added color and lighting, but it's not so easy to recreate the labeling and other nuances of the hand-drawn work.

Magnus and Oberhettinger, by the way, wrote their own German special functions books perhaps a little before the Bateman project which they were both principles in. You see many references MO in GR

for example. Most of GR's Whittaker references are from MO, some are from WH which is Whittaker and Watson, a few from Bateman which is their EH I.

I wonder if MO continued to think of their own books as the main act, and Bateman as just some project they were paid to work on for a while?

List of Special Function books:

http://www.ericweisstein.com/encyclopedias/books/SpecialFunctions.html

Notes on the Hypergeometric Functions F(a,b,c,z)

The **claim** is that there is **only one** 2nd order ODE **which has** regular singularities at 0,1,∞, and this equation is the "hypergeometric equation" (1) which I transform to

z(1-z) u" + [c - (a+b+1)z] u' -abu = 0 a,b,c = constants

u" + [c/ [z(1-z)] - (a+b+1)/ (1-z)] u' -ab/ [z(1-z)]u = 0

You can see there are simple poles at z = 0 and 1 in the p and q functions, so yes, these are "regular singular" points. Notice that the equation is symmetric in the appearance of constants a and b. The usual solution series also shows this **a↔b symmetry**. That is why careful people write F(a,b;c;z) to show that you can switch the positions of a and b and it makes no difference. Here is the series:

$$F(a,b;c;x) = 1+\frac{ab}{1!c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3+\ldots$$

Aside on singularity at z = ∞: Let's try w = 1/z so that dw = -dz/z^2 and $\partial_w = -z^2\partial_z$ and $\partial_z = (-w^2\partial_w)$. Then the first equation above becomes

z(1-z) = (1/w)(1-1/w) = $(1/w^2)$(w-1)

$\partial_z{}^2 = (-w^2\partial_w)(-w^2\partial_w) = w^2[2w\partial_w + w^2\partial_w{}^2]$

z(1-z) $\partial_z\,\partial_z$ u+ [c - (a+b+1)z] ∂_{zu}u -abu = 0

$(1/w^2)(w-1)w^2[2w\partial_w + w^2\partial_w{}^2]$ u+ [c - (a+b+1)/w] $(-w^2\partial_w)$u -abu = 0

(w-1)[$2w\partial_w + w^2\partial_w{}^2$] u + [w^2 - (a+b+1)w] $(-\partial_w)$u -abu = 0

(w-1) $w^2\partial_w{}^2$ u + {(w-1) 2w - [w^2 - (a+b+1)w]}(∂_w)u - abu = 0

$(w-1)\,\partial_w^2 u + \{(w-1)\,2/w - [\,1 - (a+b+1)/w]\}(\partial_w)u - abu/w^2 = 0$

$\partial_w^2 u + \{2/w - [\,1 - (a+b+1)/w]/(w-1)\}(\partial_w)u - abu/[w^2(w-1)] = 0$

That is not easy to do! There must be a faster trick $z = \infty$ nature. In any event, you can see that this transformed equation has regular singularities at w=0 and w=1 (second order pole OK in the u term). Thus, the HGE has a regular singular point at $z=\infty$. [Faster trick is stated W&W page 196 !!]

I have not read about this, but **I imagine that** at any kind of singular point, a particular solution of an ODE *may* have a singularity, either a single or multiple order pole, or a branch cut. When you can write an analytic power series solution centered on a singular point (like z = 0), that solution obviously has no singularity there, but probably has singularities at the other two points. [The last sentence is probably wrong.] It seems clear that a *solution* of an ODE cannot have singularities other than those which appear as singular points of the ODE coefficient functions. The argument is simply that away from a singular point of the ODE, everything is smooth and nothing violent can happen.

So again, the HGE is the most general form of a 2nd order ODE that has regular singulars at $0,1,\infty$ and there are three constants a,b,c and one solution is going to be the series F(a,b,c,z). This solution is said to be **"regular at z=0"**, so the word "regular" describes both ODE points and ODE solutions. As my other notes say, the words **regular, analytic and holomorphic** are roughly equivalent and when a solution is this at a point z, it means things are smooth in a disk around z.

You can see that there are **poles at c** = 0,-1,-2... so if c has one of these values, the u_1 solution is not F(a,b,c,z) but instead is z^{n+1} F(a+n+1,b+n+1,n+2,z) c = -n. Bateman spends lots of page inches on special cases like this.

In general, if a *or* b is 0,-1,-2..., the series **truncates** and you have a **polynomial**. Otherwise the series is infinite with convergence in $|z| < 1$ (since it is going to hit the singular point at z=1). Again from above, this is obvious from

$$F(a,b;c;x) = 1+\frac{ab}{1!c}x+\frac{a(a+1)b(b+1)}{2!c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3+\ldots$$

Bateman goes on to develop **relations** between different F's, **derivatives** of F, and then we get to the main integral representation on page 59 which provides analytic continuation and reveals the cut structure. In (12) we find that F(a,b,a+b+1-c,1-z) also solves the HGE, so this would be a power series centered at z = 1.

Here is that **main integral representation** which is very "typical" :

$F(a,b,c,z) = \Gamma(c)/[\Gamma(b)\Gamma(c-b)] * \int_0^1 dt\, t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}$

The integrand as g(t) has branch points at t=0,1 and t=1/z. You would run them off to infinity and clear the range 0,1 where the integral is done. Do not confuse the branch points of g(t) with those of F(z). Notice that the integral has all the a dependence, and the $a \leftrightarrow b$ symmetry is not very obvious just staring at the RHS. You could of course swap a and b everywhere on the RHS and result must be the same.

On page 63 top Bateman writes $F(..;\, z) = K_a\,(-z)^{-a}\,F(.....;\, 1/z) + K_b\,(-z)^{-b}\,F(.....;\, 1/z)$ where you now have series centered at $z=\infty$, the other possibility I guess. Obviously you would use this kind of thing

to get the large z behavior, and notice that there are two terms. You can see that the leading term in the **large-z expansion** is going to look like this:

$$F(a,b,c,z) = K(a,b,c)\,(-z)^{-a} + K(b,a,c)\,(-z)^{-b} \qquad K(a,b,c) = \Gamma(c)\Gamma(b-a)/[\Gamma(b)\Gamma(c-a)]$$

The **Barnes integral representation** is quite fascinating, given in (15) with integral up the imaginary axis. Looking at that thing, you see that if you close to the right, (small z) you pick up the residues of the poles of Γ(-s) and this will give you the series F(a,b,c,z). But if you close to the left for large z, then you pick up the poles of both Γ(a+s) and Γ(b+s) and this is where the two terms in (17) come from, which I quoted above.

But what are the **two linearly independent solutions** of the HGE? Bateman doesn't get around to this question until page 74. Recall that the **Frobenius** method in general shows what the two solutions are, and here they are presented as

$u_1 = F(a,b,c,z)$
$u_2 = z^{1-c}\,F(a-c+1,b-c+1,2-c,z)$

so the Frobenius indicial equation solutions must have been r = 0 and r = 1-c. I can see that these two solutions have different behavior at z=0, one having a branch point there, so I guess this tells you they cannot be the same, and so must be linearly independent.

Confirmation of the above comment:

The Gaussian **Hypergeometric Differential Equation** is:

$$x(1-x)y'' + \{c-(a+b+1)x\}y' - aby = 0$$

where a, b, and c are constants. The *indicial equation* of the hypergeometric differential equation is:

$$r^2 - (1-c)r = 0$$

which has the roots $r_1 = 0$ and $r_2 = 1-c$. Using the *Frobenius method*, the series

Bateman then on page 68 discusses a certain **"degenerate case"** of the HGE. He first comments that the three singular points of the normal HGE "are branch points of the solutions". He then repeats this idea I saw earlier and which I need to understand: " If we have u_1 = a power series around some point z_o, and if we continue u_1 analytically along a closed curve L which encircles at least one of the three branch points and returns to the start (so you move to the same place on another sheet), then we shall obtain a solution which is $\mathbf{u} = \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2$, where the λ_i are constants and where u_1 and u_2 are any linearly independent solutions. In general λ_2 will not be zero, so all solutions of the ODE can be obtained from a single one u_1 by analytic continuation." However, there can be an exceptional situation where $\lambda_2 = 0$ and then this does not work and this is the "degenerate case" he is getting into here. "The effect of a simple loop L_0^+ or L_1^+ around the points z = 0 or 1 CCW amounts to multiplication of u_1 by a **phase factor** $e^{2\pi i\rho}$ or $e^{2\pi i\sigma}$ (for our two cases 0 and 1). Suppose then we define u* by saying $z^{-\rho}\,(1-z)^{-\sigma}\,u_1 = u^*(z)$, where we are trying to remove these two phases so u*(z) won't have them. Bateman quotes "the general theory of Fuchsian equations" to conclude that u*(z) therefore won't have a singular point at z=∞ and must therefore be a

rational function which therefore only has poles at the three singular points etc etc and therefore yada yada. Basically, he is finding the situation that causes the "degenerate case". Notice that this case is not the same as the situation which gives the "confluent" HGE which GR call "degenerate", so we have **two different uses of the word degenerate**. Right now, I don't care about this Bateman degenerate case, but I do care about what he says doing those continuations around loops around singularities.

Now **how do we know** the basic facts Bateman has just claimed above just in terms of "continuing in a loop around a singular point" ? First, I find this statement on the web: (Mathematical Physics by Sadri Hassani google books).

> **14.0.1. Proposition.** (continuation principle) *The function obtained by analytic continuation of any solution of an analytic differential equation along any path in the complex plane is a solution of the analytic continuation of the differential equation along the same path.*

This seems fairly reasonable and is part of what Bateman is saying above. Now here is an idea. Consider the result quoted above

$u_1 = F(a,b,c,z)$
$u_2 = z^{1-c}\, F(a-c+1,b-c+1,2-c,z)$

A *general* solution to our ODE can therefore be taken as

$$f(z) = A\, F(a,b,c,z) + B\, z^{1-c}\, F(a-c+1,b-c+1,2-c,z)$$

This general ODE solution has the singularity at z=0 *exposed* in the factor z^{1-c}. The F functions are analytic at z=0, but we see for general c that we have a branch point in f(z) at z=0. If we circle around this branch point in the positive sense, the second term gains a phase $e^{2\pi i(1-c)}$ and we then have

$$g(z) \equiv f(ze^{2\pi i} \text{ continued}) = A\, F(a,b,c,z) + B\, e^{2\pi i(1-c)}\, z^{1-c}\, F(a-c+1,b-c+1,2-c,z)$$

$$= A\, F(a,b,c,z) + B\, e^{2\pi i(1-c)}\, z^{1-c}\, F(a-c+1,b-c+1,2-c,z) + e^{2\pi i(1-c)}\, A\, F(a,b,c,z) - e^{2\pi i(1-c)}\, A\, F(a,b,c,z)$$

$$= e^{2\pi i(1-c)} \{ A\, F(a,b,c,z) + B\, z^{1-c}\, F(a-c+1,b-c+1,2-c,z) \} + A\, F(a,b,c,z) \{ 1 - e^{2\pi i(1-c)} \}$$

$$= e^{2\pi i(1-c)}\, f(z) + \lambda\, F(a,b,c,z) \quad \text{where} \quad \lambda = A \{ 1 - e^{2\pi i(1-c)} \}$$

where on the second line we added and subtracted $e^{2\pi i(1-c)}\, A\, F(a,b,c,z)$ to the RHS. We have just proven the following theorem:

Loop Around Branch Point Theorem: If we take any solution f(z) of our ODE that has a singularity at z=0, and if we "analytically continue" that solution on a loop around the singularity, we obtain a phase times the original function f(z) plus a second term which also solves the ODE. If we think of f(z) as our "first" solution to the ODE, then this second term will be a "second" solution.

This theorem probably applies to *any* 2nd order ODE which has Frobenius solutions like the pair u_1 and u_2 shown above. In general we will have $u_1 = z^{r1}\,\psi_1$ and $u_1 = z^{r2}\,\psi_2$ where the ψ_i are analytic at z=0. Then we can repeat all the above as follows

$$f = Au_1 + Bu_2$$

$$g = e^{2\pi i r1}Au_1 + e^{2\pi i r2}Bu_2 = e^{2\pi i r1}\{\, Au_1 + e^{2\pi i (r2-r1)}\, Bu_2 + Bu_2 - Bu_2 \,\}$$

$$= e^{2\pi i r1}\{\, Au_1 + Bu_2 \,\} + e^{2\pi i r1}\{\, e^{2\pi i (r2-r1)} - 1 \,\}\, Bu_2$$

$$= e^{2\pi i r1} f + \lambda\, u_2 \qquad\qquad \lambda = e^{2\pi i r1}\{\, e^{2\pi i (r2-r1)} - 1 \,\}B$$

so again we get f times a phase plus the second solution. Since this is true for a z=0 singular point, it is probably true for any such singular point since there is nothing special about z = 0.

The Riemann Equation and P-function. (page 89)

In (1) we see a 2nd order ODE which involves $z_1\ z_2\ z_3$ and constants (called **exponents**) $a_1\ a_2\ a_3$ and $a_1'\ a_2'\ a_3'$. The claim is that this is the unique ODE which has regular singular points $z_1\ z_2\ z_3$! The quantities $a_i - a_i'$ are called **exponent differences**. Think of there being 9 parameters here.

We can write down the two linearly independent solutions Frobenius style as in (3) where we do it separately three times, each time expanding around one of the singular points. We see now that for each of the three cases, exponents a_i and a_i' are the just Frobenius indices that multiply a series.

Now that very fancy notation in (4) is just a way to write down the 9 parameters in a compact way. And it makes it easy to see which Frobenius indices go with which singularity.

We then get some properties of the P function from 1892 !

The first property (5) shows that you can extract simple rational function powers by shifting the parameters a_i and a_i' by certain simple linear combinations of those powers which are called ρ and σ . A special rule here is that if one of the $z_i = \infty$, then you replace $z-z_i$ with 1 in the powers.

The second property (6) keeps these 6 parameters fixed, but transforms z and the three z_i into ξ and ξ_i all by the same simple rational function (8) with parameters A,B,C,D. This gives you the freedom to move the z_i into any three new singular points you want.

Now let's apply this as follows. Try $\xi = k\,(z-z_1)/(z-z_2)$ in (6). That is, $A = k$, $B = -kz_1$ etc. We then find that $\xi_1 = 0$ and $\xi_2 = \infty$ and $\xi_3 = k\,(z_3-z_1)/(z_3-z_2) = 1$ if we select $k = (z_3-z_2)/(z_3-z_1)$. So doing this causes an argument shift as shown in (8) along with having the right three singularities that we want. Having done this, look at (5) but think of the three z_i there as being our three ξ_i with the special rule just noted. Then in our case, the exponent stuff out front is just $\xi^{\rho}\,(\xi-1)^{\sigma}$. We can select ρ and σ to clear out two of the six shifted parameters. Of particular interest is clearing out one item in the first column and one item in the third column, that is, in the 0 and 1 columns (not the ∞ column). There are four ways to pick ρ and σ to achieve this goal. If we do any one of these ways, then the P function can be written as a HG function! I think (9) is just one of these four ways. You see that the remaining 4 P parameters are functions of a,b,c. BUT, for each of these four ways, we can permute the points 1,0,∞ in 6 ways. Each of these permutations can be thought of as a version of (6). For example, we might have $\xi = 1/z$ which swaps the 0 and 1 points. The claim then is that we end up with 24 different functions of the form page 92A

which solve the HG equation! These 24 ways are listed on page 105-106, and are known as the famous **Kummer's 24 solutions**. It turns out that the 24 may be partitioned into 6 groups where in each group all 4 functions are equal, so there are really only 6 distinct functions called u_1 through u_6. Then, since we know there are only two linearly independent solutions of the ODE, if you pick any 3 of Kummer's solutions, there must be a linear relation between them. There are (6,3) = 6!/3!3! = 20 different ways to pick three of the six u_i , so there are 20 such triple relations and these are listed on pages 106 and 107.

Detail: Now consider this list of "homographic substitutions" shown in (10). Each one of these permutes the singularities in some manner, and there are 3*2*1 = 6 such permutations. For example

$\xi = 1 - 1/z$:	$0 \to \infty$	$1 \to 0$	$\infty \to 1$
$\xi = 1/z$:	$0 \to \infty$	$1 \to 1$	$\infty \to 0$

Detail: The P{...} notation does not represent a particular solution to the ODE, it represents all solutions, the complete set of solutions, as Bateman says. The matrix just displays the 9 parameters in an organized way, and then the solution is some function of z, so z is then shown. A relation like (5) then relates the set of solutions of one ODE to the set of solutions of another ODE with different parameters. In (3) we have 6 possible Frobenius power series of interest. All six of these are implied by the P notation in (4). The notation let's you quickly write down any of the 6 power series.

OK, I don't have this stuff nailed very well, but I see the basic theory.

Table of Formulas for HG: I don't think I was really aware this was present. The first part contains various oddments including the fact that $\sin^{-1}$ and $\tan^{-1}$ can be written as F's. The fifteen Gauss relations on page 103 show how you can shift the F(a,b,c,z) = F parameters by ±1 and you get relations with factors of z and (1-z) appearing in various places along with constants formed from a,b,c. Page 104 lists F with special values of parameters and argument. Then the Kummer's 24 + 20 mentioned above. On page 108 we get the form we would use to look at large z. Then comes a table of obscure higher order transformations more complicated than those 6 items above like 1/(z-1). There are some that result in either quadratic or even cubic arguments (cubic in z). Next comes the integral representations, and finally some references.

Notes on the confluent hypergeometric functions

6.1 Orientation.

Write down f(x) = F(a,b,c,x/b). This thing obviously has singularities at x = bz = b,0 and ∞. If you then take the limit b→0, the b singularity joins the one already at ∞, and this is called a **confluence** of the two singularities, like the Green and Colorado rivers, hence the famous "confluent name". It is pretty obvious what happens to the basic power series. For example, $b(b+1)(b+2)/b^3 \to 1$ so you just take the normal F series (the Gauss series) and delete all the factors involving b, and you get the confluent or **Kummer series** as shown in (1). This series is called ${}_1F_1(a,c;x) = \Phi(a,c,x) = M(a,c,x)$ and other names in other books.

If you write the confluent solution $y(x) = x^{-c/2}e^{x/2}g(x)$, and if you replace $a = 1/2 - \kappa + \mu$ and $c = 1+2\mu$, then the ODE for M(a,c,x) becomes another ODE for g(μ,κ,x) which is **Whittaker's equation**. This is just another convenient way to deal with this subject

6.2 ODEs

Equation (1) shows the **most general ODE you can write with linear coefficients**. Recall that this is also the general form to which one can apply Laplace's Method to get certain integral representations. Now Bateman tells us to replace $y(x) = e^{h\lambda\xi+h\mu}\, z(\xi)$ allowing three parameters h,μ and λ. This converts the ODE into another one of exactly the same form! But in this new form, if we can set $\beta_0 = 0$, $\alpha_o = 1$, $\alpha_1 = -1$, $\beta_1 = c$, $\alpha_2 = 0$, $\beta_2 = -a$, then this ξ equation has the confluent form (2). Let's look into this a tad, where I have k be an overall constant

$\alpha_o = k$ => $\lambda = a_o/k$

$\beta_o = 0$ => $a_0\mu + b_0 = 0$ => $\mu = -b_o/a_0$

$\alpha_2 = 0$ => $A(h) = 0$ => $a_o h^2 + a_1 h + a_2 = 0$

$\alpha_1 = -k$ => $A'(h) = -k$ => $2a_0h + a_1 = -k$ => $h = -(k+a_1)/(2a_0)$

$\beta_2 = -ak$ => $B(h) = -ak$ => $a = -B(h)/k$

$\beta_1 = ck$ => $(-\mu k + B'(h)) = ck\lambda$ => $B'(h) = ck\lambda + \mu k = 2b_0h + b_1$

Notice that $a_o + \lambda A'(h) = \lambda k + \lambda(-k) = 0$, so all three of his equations in 249 A are true. Let's compare this to the top row (6) in the page 250 table where we will replace k by -A'(h):

$\lambda = a_o/k = -a_o/A'(h)$ agrees

$\mu = -b_o/a_0$ agrees

$h = (-k-a_1)/(2a_0)$ table says $(D-a_1)/(2a_0)$

Now let's look at our A(h) = 0 condition. The quad formula says

$$h = \{ -a_1 \pm [\, a_1^2 - 4a_a a_2]^{1/2} \,\} / (2a_o) \Rightarrow 2a_o h + a_1 = \{ \pm [\, a_1^2 - 4a_a a_2]^{1/2} \,\} = D$$

$$\Rightarrow h = (D-a_1)/(2a_0)$$

So, we have to compute h this way, and then we find that k = -D = -A'(h)

$$ck\lambda + \mu k = 2b_0h + b_1 \Rightarrow ck\, a_o/k - kb_o/a_0 = -2b_0\,(k+a_1)/(2a_0) + b_1$$

$$c - kb_o/a_0^2 = -2b_0\,(k+a_1)/(2a_0^2) + b_1/a_0$$

$$\Rightarrow c = -2b_0\,(k+a_1)/(2a_0^2) + b_1/a_0 + kb_o/a_0^2$$

$$= \{-b_0\,(k+a_1) + b_1a_0 + kb_0\}/a_0^2 = \{-b_0\, a_1 + b_1a_0\}/a_0^2$$ agrees

$a = -B(h)/k = A'(h)B(h)$

So I have complete agreement on the first row of the page 250 table. The point is that any 2nd order ODE in x with *linear* coefficients can be converted to our CHG equation in ξ by doing $y(x) = e^{h\lambda\xi+h\mu}\, z(\xi)$ where we select h, λ and μ from the top row of the table.

Alternatively, you can convert this same linear coefficient equation into $F(a,1/2,k\xi^2)$ as in (7), or two ways to convert to Bessel's equation. Thus suggests to me a close linkage between the CHF and Bessel functions of a certain type.

I skip the remaining details of this section.

6.3 Solution of CHG equation near x=0.

Four solutions are stated as $y_{1,2,3,4}$ and the first two are linearly independent:

$y_1(x) = \Phi(a,c;x)$ // our basic series

$y_2(x) = x^{1-c}\,\Phi(a-c+1,2-c;x)$

The other two solutions are really the same as these, given the **Kummer Transformation** shown in (7), p 253, which says you can go to a -x argument extracting an expo and shifting the first parameter:

$\Phi(a,c;x) = e^x\Phi(c-a,c;-x)$

6.4 Relations for Φ(a,c;x)

Bateman goes on to give the various shifting ±1 formulas (Gauss) and the derivative formulas.

6.5 Basic integral representations (255)

Now we come to more important (for me) stuff. We have a little review on the Laplace Method, and we find that Φ(a,c,x) is given by our standard integral (1) with a contour C just going 0 to 1. It turns out that another useful contour runs 0 to $-\infty$ for the same integrand. Change to t = -u, this ray goes 0 to ∞ and we get the representation in (2) which is called **Ψ(a,c,x)**. As noted in my Laplace notes, same integrand but different contour gives in general a different ODE solution, so $\Psi \neq \Phi$. As it turns out, they are related as shown in (7). Here we see that Ψ is just a certain linear combination of our two basic independent solutions called y_1 and y_2 above.

Note: some people refer to Ψ as G. Tricomi introduced Ψ in 1927 and called it G.

6.6 Relations of the Ψ function (257)

We just get a list of corresponding shifting and derivative relations, similar to what we had for Φ.

6.7 Fundamental solutions

We start off with a list of four possible Ψ functions as we had earlier for Φ functions. As before, they are pairwise equal due to a relation (6) where you shift parameters and pull out a power. The phase shown page 259 (5) has $\varepsilon = \pm 1$ depending on Im(x).

So now we have four useful solutions of type Φ and Ψ (8 but they are pairwise equal). For general parameters, you can pick any two of these, but our Coulomb scattering application implies that c = positive integer (called b in Messiah), so this is going to require special treatment, see log section below.

Then in (7) we have Φ expressed as a linear combination of y_5 and y_7, which two exist for all parameter cases.

6.7.1 The logarithmic case (260)

I think this is the situation that arises in Coulomb scattering, so pay attention. We have c = integer. For positive n, y_2 does not exist because Φ has poles there, and for negative n, y_1 does not exist, same reason. So where is your second solution going to come from? The claim is that this second solution is going to involve logarithms somehow.

A good method here is to start with Ψ in its arbitrary ray contour form (3) page 256. This integral works for all values of c. For non-integral c, you can use p 257 (7) to write Ψ as a power series in x. But this does not work for integral c because of those Φ poles in c. Bateman considers c = 1+n n=0,1,2. He then uses the Mellin-Barnes form page 256 (5). Somehow when you do the pole residues to compute the integral, log(x) appears, as well as ψ(x) which is Γ'(x)/Γ(x). Instead of getting pure power series in x, you get the huge mess shown in (13) for Ψ. You see that log x factor sitting there in the first term with Φ, then all the rest is a normal power series in x with very messy coefficients.

6.8 More on the Ψ function. (p 262)

We have those power series just discussed, so they can be used for small x limit.

Ψ has a cut going from x=0 to the left, and (15) gives the discontinuity across this cut and is valid even if c = integer.

6.9 Whittaker functions (264)

We see that $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ are just variations of the Φ and Ψ functions. (Watson and Whittaker 1927 book)

6.9.1 Bessel functions

In the special case κ= 0 (which means a = c/2), you get Z_ν type Bessel functions as your solutions.

6.9.2 What other functions can be expressed in terms of Φ and Ψ ?

The list is very impressive!! I underlined all the functions and a table on page 268 correlates the function name with the value of the parameters

Other topics in this chapter:

6.10 Laplace Transforms which involve Φ and Ψ functions.

6.11 Fancy contour integral representations of Φ and Ψ and Whittakers (271)

An interesting compact notation like (0+) used to described the contours.

6.12 Series of other functions which add up to Φ and Ψ functions.

6.13 Asymptotic behavior of Φ and Ψ in x and also in c and a

This section is important for me, but I think we already saw the results above.

6.15.1 Series of Φ functions which add up to something expressible in closed form (283)

6.15.2 Integrals of Φ and Ψ functions

6.15.3 Products of Φ and Ψ functions and integrals of same

6.16 Zeros of Φ and Ψ.

How can we connect Messiah with Bateman on the confluent HG functions?

Messiah page 480 writes a solution of the CHGE as an integral over a path Γ. In B.6, this path is shown in the figure which surrounds the isolated cut between 0 and 1. This cut is only isolated like this if b = integer. I think we can identify the integral on the RHS of B.6 *without the phase* as integral 0 to 1. This integral is the function Φ(a,b,z) as seen on page 255 of Bateman. So at least we have some connection. So the Messiah thing is an unusual contour.

I think this Messiah integral with this interesting contour exactly agrees with Bateman's strange integral shown on page 272 top, the path called 0 to (1+) where we start at 0 and circle 1 and return. Bateman agrees that this contour does give Φ(a,b,z). But Bateman does not do anything with this integral, so not much use to me.

Messiah then deforms the contour into two separate loops reaching out to infinity as shown bottom page 481, putting them at an angle τ.

Question: We know that the sum of both these contours gives a solution to the ODE. How do we know that the individual contours which then make the W_r functions provide solutions to the ODE? Maybe only the sum $W_1 + W_2$ is an ODE solution. The question would then be: for each contour, do the "parts" vanish? Since each contour is a closed loop (albeit infinite), the answer is that yes, each W_r is a solution of the ODE. No such contour as this appears in the Bateman chapter, so Bateman is not going to relate this W_r to anything he talks about. For example,

But wait! Page 273 has an interesting integral rep for the Ψ function that is ∞ to (0+) which I think is exactly the W_1 contour, and he even has it tipped at phase φ. But the t is not quite right. What happens in a contour integral of you change the sign of the variable? I think you just "do it" and the contour becomes a parity reflection of the original contour. So, the page 273 form has the contour at angle φ = + 45 degrees, say and does CCW around the origin. The parity contour has the same rotation sense, but the long part goes off at φ = 45-180 degrees. Now, Bateman's version converges only for φ going off in the right half plane, so the parity one will converge only for the rubber band in the left half plane, exactly as Messiah has drawn it.

So here is my parity-converted form of Bateman page 273 (9):

$$\Psi(a,c,x) = (2\pi i)^{-1}\, e^{-i\pi a}\, \Gamma(1-a) \int_{\infty e^{i\varphi}}^{(0+)} dt\, e^{-xt}\, t^{a-1}(1+t)^{c-a-1} \qquad \text{// as in Bateman}$$

$$= (2\pi i)^{-1}\, e^{-i\pi a}\, \Gamma(1-a) \int_{\infty e^{i(\varphi-\pi)}}^{(0+)} dt'\, e^{xt'}(-t')^{a-1}(1-t')^{c-a-1} \qquad \text{// parity reflected}$$

Now just replace t' with t and replace φ-π with angle τ to match Messiah. Also, (-t') = (t' $e^{+i\pi}$) so we obtain a phase factor $e^{+i\pi(a-1)}$ from this factor, and then $e^{-i\pi a}\, e^{+i\pi(a-1)} = e^{-i\pi} = -1$. Subtle assumptions are made here about why I set $-1 = e^{+i\pi}$, but that was the direction in which I did my analytic cancellation. Then we get [practice this with residue integral of a simple pole!]

$$= (2\pi i)^{-1}\, \Gamma(1-a) \int_{\infty e^{i\tau}}^{(0+)} dt\, e^{xt}(-t)^{a-1}(1-t)^{c-a-1} \qquad \text{// parity reflected}$$

So I conclude that

$$\int_{\infty e^{i\tau}}^{(0+)} dt\, e^{xt}(-t)^{a-1}(1-t)^{c-a-1} = 2\pi i \,/\, \Gamma(1-a) * \Psi(a,c,x)$$

and therefore that (from Messiah B.7)

$$W_1(a,c,x) = 2\pi i\, (1 - e^{-2\pi i\alpha})^{-1} * \Gamma(c) \,/[\, \Gamma(1-a)\Gamma(a)\Gamma(b-a)] \; * \; \Psi(a,c,x)$$

But we know that $\Gamma(1-a)\Gamma(a) = \pi/\sin(\pi a)$ so this becomes

$$W_1(a,c,x) = 2\pi i\, (1 - e^{-2\pi i\alpha})^{-1} \sin(\pi a) * \Gamma(c) \,/[\pi]\Gamma(b-a)] \; * \; \Psi(a,c,x)$$

But then $(1 - e^{-2\pi i\alpha}) = e^{-i\pi a}(\, e^{+i\pi a} - e^{-\pi i a}) = e^{-i\pi a}\, 2i \sin(\pi a)$

$$(1 - e^{-2\pi i\alpha})^{-1} = e^{+i\pi a}/[2i\sin(\pi a)]$$

$$W_1(a,c,x) = 2\pi i\, (1 - e^{-2\pi i\alpha})^{-1} \sin(\pi a) * \Gamma(c) \,/[\pi]\Gamma(b-a)] \; * \; \Psi(a,c,x)$$

$$= 2\pi i\, e^{+i\pi a}/[2i\sin(\pi a)] \;\; \sin(\pi a) * \Gamma(c) \,/[\pi]\Gamma(b-a)] \; * \; \Psi(a,c,x)$$

$$= e^{+i\pi a}\, \Gamma(c)/\, \Gamma(c-a)\, \Psi(a,c,x)$$

But now look at Bateman page 259 (7). He writes phase $e^{i\pi a\varepsilon}$ where ε = + for just above the left side cut, so I would equate that with phase $e^{+i\pi a}$. Therefore, $W_1(a,c,x)$ is exactly the first term in (7), which was my original hope. We then of course know that $W_2(a,c,x)$ is that second term! Therefore

$$W_2(a,c,x) = \Gamma(c)/\, \Gamma(a) * e^{i\pi(a-c)}\; e^{x}\, \Psi(c-a,c,-x)$$

If this is all really true, then we would expect the large x limit of Ψ to give that for W_1 in Messiah.

Now the large-x limit of $\Psi(a,c,x)$ is given Bateman page 278 top, I quote

$$\Psi(a,c,x) = (x)^{-a} \Sigma_n [\ \Gamma(a+n)\Gamma(a-c+1+n)/\Gamma(a)\Gamma(a-c+1)\ ]\ (-x)^{-n} / n!$$

$$\Psi(a,b,x) = (x)^{-a} \Sigma_{n=0} [\ \Gamma(n+a)\Gamma(n+a-b+1)/\Gamma(a)\Gamma(a-b+1)\ ]\ (-x)^{-n} / n!$$

$$= (x)^{-a}\ \{sum\} \qquad \text{// leading term in } \{sum\} \text{ is just "1" for n=0}$$

This looks very good comparing to Messiah B.10 because the {sum} part exactly matches. Now lets guess that $(-x)^{-a} = (xe^{-\pi i})^{-a} = e^{+\pi i a}(x)^{-a}$. Then we can rewrite this as

$$\Psi(a,b,x) = e^{-\pi i a} [\ (-x)^{-a}\ \{sum\}]$$

Then our prediction for W_1's behavior would be

$$W_1(a,b,x\ ) = e^{+i\pi a}\ \Gamma(b)/\ \Gamma(b-a)\ \psi(a,c,x)$$

$$= e^{+i\pi a}\ \Gamma(c)/\ \Gamma(b-a)\ e^{-\pi i a} [\ (-x)^{-a}\ \{sum\}]$$

$$= \Gamma(b)/\ \Gamma(b-a) [\ (-x)^{-a}\ \{sum\}]$$

and this is in exact agreement with B.10 !!!

I would presume from the above that

$$\Psi(a,c,x) = (x)^{-a} \Sigma_n [\ \Gamma(a+n)\Gamma(a-c+1+n)/\Gamma(a)\Gamma(a-c+1)\ ]\ (-x)^{-n} / n!$$

$$\Rightarrow \quad \Psi(c-a,c,-x) = (-x)^{-[c-a]} \Sigma_n [\ \Gamma(c-a+n)\Gamma([c-a]-c+1+n)/\Gamma(c-a)\Gamma([c-a]-c+1)\ ]\ (x)^{-n} / n!$$

$$= (-x)^{-[c-a]} \Sigma_n [\ \Gamma(c-a+n)\Gamma(-a+1+n)/\Gamma(c-a)\Gamma(-a+1)\ ]\ (x)^{-n} / n!$$

$$= (-x)^{-[c-a]}\ \{sum2\}$$

and as before we say $(-x)^{-[c-a]} = (xe^{+\pi i})^{-[c-a]} = e^{+\pi i[c-a]}\ (x)^{-[c-a]}$ and therefore

$$W_2(a,c,x\ ) = \Gamma(c)/\ \Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ \Psi(c-a,c,-x)$$

$$= \Gamma(c)/\ \Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ [e^{+\pi i[c-a]}\ (x)^{-[c-a]}\ \{sum2\}$$

$$= \Gamma(c)/\ \Gamma(a)\ e^{x}\ (x)^{a-c}\ \{sum2\}$$

and this exactly agrees with B.11. Therefore, we don't have to rely on Messiah, we get our asymptotic limits now directly from Bateman. So I think I will "red check" B.10 and B.11.

Now what about things like Messiah B8a ?

$$W_1(c\text{-}a,c,\text{-}x) = e^{-x} W_2(a,c,x) \qquad ?$$

where we know that

$$W_1(a,c,x) = e^{+i\pi a}\ \Gamma(c)/\Gamma(c\text{-}a)\ \Psi(a,c,x)$$

$$W_2(a,c,x) = \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ \Psi(c\text{-}a,c,\text{-}x)$$

so the claim is this,

$$e^{+i\pi a}\ \Gamma(c)/\Gamma(c\text{-}a)\ \Psi(a,c,x) = e^{-x}\ \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ \Psi(c\text{-}a,c,\text{-}x)$$

$$1/\Gamma(c\text{-}a)\ \Psi(a,c,x) = e^{-i\pi c}\ 1/\Gamma(a) * \Psi(c\text{-}a,c,\text{-}x)$$

$$\Gamma(a)\ \Psi(a,c,x) = e^{-i\pi c}\ \Gamma(c\text{-}a)\ \Psi(c\text{-}a,c,\text{-}x)$$

Is this in Bateman somewhere? I don't see it, so hold on that. [then later I don't see it in AS either]

Connection with A&S page 504

Clearly, M(a,b,z) = Φ(a,b,z) but not clear on U(a,b,z). So look at that definition and replace

$$\pi/\sin(\pi b) = \Gamma(b)\Gamma(1\text{-}b)$$

so we have

$$U(a,b,z) = \Gamma(b)\Gamma(1\text{-}b)/[\ \Gamma(b)\Gamma(1+a\text{-}b)]\ \Phi(a,b,z) \ -\ \Gamma(b)\Gamma(1\text{-}b)/\ [\ \Gamma(a)\Gamma(2\text{-}b)]\ z^{1-b}\Phi(1+a\text{-}b,\ 2\text{-}b,z)$$

$$= \Gamma(1\text{-}b)/\Gamma(1+a\text{-}b)]\ \Phi(a,b,z) \ -\ \Gamma(b)\Gamma(1\text{-}b)/\ [\ \Gamma(a)\Gamma(2\text{-}b)]\ z^{1-b}\Phi(1+a\text{-}b,\ 2\text{-}b,z)$$

The first term agrees with Bateman p 257 (7) assuming U = Ψ , but second looks wrong. To make the second term agree, we need to show that

$$-\ \Gamma(b)\Gamma(1\text{-}b)/\ \Gamma(2\text{-}b) = \Gamma(b\text{-}1)$$

or

$$-\ \Gamma(b)\Gamma(1\text{-}b) = \Gamma(b\text{-}1)\ \Gamma(2\text{-}b)$$

or

$$-\ \pi/\sin(\pi b) = +\pi/\sin(\pi[b\text{-}1])$$

or

$$-\sin(\pi b) = \sin(\pi b\text{-}\pi)$$

which is of course true, shifting sine by π changes its sign.

So this clinches the connection between A&S and Bateman: We have

$M = \Phi$ and $U = \Psi$ exactly.

And I now see this is confirmed by A&S directly on page 504 above 13.1.11.

So now we can add the A&S repository of information to Bateman's. I was just wondering about this Messiah result which I above translated into

$$\Gamma(a)\,\Psi(a,c,x) = e^{-i\pi c}\,\Gamma(c-a)\,\Psi(c-a,c,-x)$$

I think I can prove this is in fact wrong as follows:

$$\Psi(a,c,x) = \Gamma(1-c)/\Gamma(1+a-c)]\,\Phi(a,c,x) + \Gamma(c-1)/\,\Gamma(a)\,x^{1-c}\Phi(1+a-c,\,2-c,x)$$

$$\Psi(a,c,-x) = \Gamma(1-c)/\Gamma(1+a-c)]\,\Phi(a,c,-x) + \Gamma(c-1)/\,\Gamma(a)\,(-x)^{1-c}\Phi(1+a-c,\,2-c,-x)$$

$$\Psi([c-a],c,-x) = \Gamma(1-c)/\Gamma(1+[c-a]-c)]\,\Phi([c-a],c,-x) + \Gamma(c-1)/\,\Gamma([c-a])\,(-x)^{1-c}\Phi(1+[c-a]-c,\,2-c,-x)$$

$$\Psi(c-a,c,-x) = \Gamma(1-c)/\Gamma(1-a)\,\Phi(c-a,c,-x) + \Gamma(c-1)/\,\Gamma(c-a)\,(-x)^{1-c}\Phi(1-a,\,2-c,-x)$$

Now use the Kummer transformation on page 253 (7) to replace both Φ functions as follows

$$\Phi(c-a,c,-x) = e^{-x}\,\Phi(a,c,x)$$

$$\Phi(C-A,C,-x) = e^{-x}\,\Phi(A,C,x)$$

let $C-A = 1-a$ and $C = 2-c \Rightarrow A = C+a-1 = 2-c+a-1 = 1+a-c$ so that

$$\Phi(1-a,\,2-c,-x) = e^{-x}\,\Phi(1+a-c,\,2-c,x)$$

Then we get

$$\Psi(c-a,c,-x) = \Gamma(1-c)/\Gamma(1-a)\,\Phi(c-a,c,-x) + \Gamma(c-1)/\,\Gamma(c-a)\,(-x)^{1-c}\Phi(1-a,\,2-c,-x)$$

$$= e^{-x}\,\{\,\Gamma(1-c)/\Gamma(1-a)\,\Phi(a,c,x) + \Gamma(c-1)/\,\Gamma(c-a)\,(-x)^{1-c}\,\Phi(1+a-c,\,2-c,x)\,\}$$

This is not proportional to $\Psi(a,c,x)$, so I claim that Messiah B8a is not true. Now maybe somehow the claim is true for $c = 1,2,3....$ From our expansion above for Ψ,

$$\Psi(a,c,x) = \Gamma(1-c)/\Gamma(1+a-c)]\,\Phi(a,c,x) + \Gamma(c-1)/\,\Gamma(a)\,x^{1-c}\Phi(1+a-c,\,2-c,x)$$

it seems clear that $\Psi(a,c,x)$ has poles at $c = 1,2,3...$ because we know that Φ does not have poles there and we see that $\Gamma(1-c)$ sitting there which does have poles. So maybe the claim that

$$\Gamma(a)\,\Psi(a,c,x) = e^{-i\pi c}\,\Gamma(c-a)\,\Psi(c-a,c,-x) \qquad c = 1,2,3....$$

is really the claim that the pole residues at these poles are the same. I guess I will just let this ride. I don't need these results.

Connection with GR

Exact same notation as Bateman, see page 1058. GR has only 1.5 pages of info