Messiah Appendix B I.1: The $W_{1,2}$ Functions PhL 12.8.08

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Notes on my first reading

This little appendix is the only place I know that talks about these things.

1. Confluent HG functions. The ODE for the **confluent hypergeometric series** solution is given in Bateman vol 1 page 248 (left here):

$$x\,y'' + (c-x)y' - \alpha y = 0 \qquad\qquad z\,y'' + (\beta - z)y' - \alpha z = 0 \qquad\qquad \text{B.1}$$

Messiah puts it as shown on the right above. He refers to this as **"Laplace's Equation"** which is a far cry from E&M and "the Laplace Equation" we spent months messing with. I wonder who did the first work with this ODE? Where did it first arise? It appears in GR on page 1059 with no name. In A&S the ODE is called **Kummer's Equation**, and I suspect there might be some Germany/France going on here.

The usual series solution to this thing has various names

$F(\alpha|\beta|z)$ [Messiah p 480]
$M(\alpha,\beta,z)$ [AS p 504]
$\Phi(\alpha,\beta;\, z)$ [GR 1057]
${}_1F_1(\alpha;\, \beta;\, z)$ and $\Phi(\alpha,\beta;\, z)$ [Bateman p 248]

It appears as B.2 in Messiah. Notice on page 504 the A&S notation that $(a)_2 = a(a+1)$ and so on. We can note that

$$(a)_n = \Gamma(a+n)/\Gamma(a) = (a+n-1)!\,/\,(a-1)!$$

and this explains the four Γ functions in the Messiah B.2.

Messiah then lists off a number of properties for this function F. It is analytic everywhere, called therefore an "entire function". You can make the series truncate by setting $\alpha = -p$, negative integer. There is a **Kummer's Relation** which is shown as B.3 which one could probably prove just from the ODE. And "the second" solution to the ODE is stated as a power times a modified F.

2. Laplace's Method. We then get into the Laplace's Method business. I have written this up in a document **"laplaces method.doc"**, a very interesting way to solve ODE's with at-worst linear coefficients. The upshot is that the integral shown in B.4 solves the ODE, provided the integration path contributes zero "parts". If one uses a closed contour on a given Riemann sheet, this must be true, and a useful contour is suggested on the top of p 481.

This contour led me to a second document **"contours and cuts.doc"** where I learned why it is that the two cuts for the integrand of B.4 can be deformed into an isolated cut between 0 and 1. This is true *only* because of the form of the exponents in B.4 and the fact that $\beta = b$ is taken as an integer. This is another long subject, see that document for details.

So, we then know that B.4 solves our ODE. Looking at the way z appears in this thing, we know that B.4 will be analytic in the entire z plane (entire), and this tells us that it must be proportional to the F series. On page 481, the constant of proportionality is determined, and we arrive at B.6 for F. The details of this are described in "contours and cuts.doc" and I won't repeat them here. Red checks on everything so far.

3. Deformation. Now we deform the two branch cuts as shown. In the top picture, you can think of the two branch cuts going to the right and they "cancel" above t = 1 (when b = integer), and here we have just rotated the two branch cuts down to some negative angle called τ which I have drawn into the picture. The expression B.6 can now be decomposed into two separate integrals, one for each of the two contour paths shown bottom of page 481, and these two integrals are then **the W_1 and W_2 functions!** So this gives us then the definition **B.7** and the obvious fact **B.9** that they add up to F.

The integrals will converge if the expo makes them converge, so we can write

$$\exp(zt) = \exp(|z|\, e^{i\theta}\, |t|\, e^{i\tau}) \sim \exp[|z|\, |t| \cos(\theta+\tau)]$$

so we need $\cos(\theta + \tau) < 0$ which means for example $\pi/2 < \theta + \tau < 3\pi/2$ to which you can add $2n\pi$ as shown in page 482 A, which we have now verified.

He then makes some conventions for the sign of τ, namely, that $\text{sign}(\tau) = \text{sign}(\theta)$ where $\theta = \arg(z)$. He then claims that with these conventions, you get two symmetric Kummer-like relations. In the original Kummer, only F appears, but in these, each one has W_1 on one side and W_2 on the other side. Obviously if you were to add these B8a and b up, you would get B3. These Kummer's just relate -z to + z solutions and involve e^z as a factor. [I have some doubts about B.8, see Bateman notes elsewhere.]

4. Asymptotic Behavior. Now let's jump over to A&S page 504 where at the bottom of the left column, we see large z behavior for the F function. There are two limits, one for Rez > 0 (which involves e^z) and the other limit for Rez < 0 which does not involve the e^z factor. The W_1 and W_2 functions have just this kind of large-z behavior, as shown in B.10 and B.11, but here we have W_2 for normal z and W_1 for z' = -z.

The series itself probably does not converge for large z, but here we have an analytic continuation. I think with these W_1 and W_2 that provide the large z forms. For example, suppose you look at the normal z form which is B.11 for W_2 . You then also know what W_1 is doing as its last argument goes large negative, from B8a. So OK, I think all the asymptotic behavior is under control, and that concludes this little appendix. I did not do the details that are not checked off.

Notice, for example, that if Re(z) < 0

Later Notes where the Fog Lifts

The above notes were taken before I had read Bateman in detail about the confluent hypergeometric Φ and Ψ. In my Bateman notes doc, I finally figured out what these W_1 and W_2 functions really are. See there for details, but here is a summary:

In Bateman page 259 (7), we have this way to write the Φ function as a linear combination of Ψ functions

$$\Phi(a,c,x) = e^{+i\pi a}\ \Gamma(c)/\Gamma(c-a)\ \Psi(a,c,x) \quad + \quad \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ \Psi(c-a,c,-x)$$

These two terms are, in turns out, exactly the W_1 and W_2 functions.

$$W_1(a,c,x) = e^{+i\pi a}\ \Gamma(c)/\Gamma(c-a)\ \Psi(a,c,x)$$

$$W_2(a,c,x) = \Gamma(c)/\Gamma(a) * e^{i\pi(a-c)}\ e^{x}\ \Psi(c-a,c,-x)$$

We now have a handle on the properties of these functions. Bateman gives the large-x form of the ψ functions and from that, we can deduce the large-x behavior of the W_r functions. The results are exactly B.10 and B.11. And of course $\Phi = W_1 + W_2$. It is a pretty normal thing to do to take your z=0 convergent series solution and analytically continue it in terms of second-kind functions so you can get the large-z behavior, and of course in scattering that is going to be what we want to know about.

So in essence, we could just define W_1 and W_2 in terms of Ψ as above, then we don't really need this Messiah appendix with its lack of referencing!

I sometimes write this shorthand that $\Phi = \Psi + \Psi'$ for the above equation, This is $\Phi = W_1 + W_2$!