Chapter 1: The Fourier Integral and Related Topics

The purpose of this chapter is to demonstrate the use of certain mathematical tools associated with the Fourier Integral transform. Along the way, we do many simple examples, with emphasis on the particular example of an isolated square pulse in the time domain. If we can make things work out for a square pulse, we can presumably go on to harder problems with the same tools.

One normally analyzes a square wave pulse train using a Fourier Series, since such a pulse train is a periodic function. In Chapter 2 below, we will make the connection between the conventional Fourier Series, and our Fourier Integral approach.

1. Fourier Integral Transform.

Here is a statement of the Fourier Integral transform. Let x(t) be some function of time. If you define X(ω) to be the "spectral components" of x(t) according to (1.1), then the claim is that you can recover x(t) from these spectral components according to (1.2):

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inversion} \qquad (1.2)$$

In equivalent language, (1.2) represents an expansion of x(t) in terms of the spectral components X(ω). Equation (1.1) shows how these components are "projected out" of the function x(t). The expansion (1.2) can be rewritten in terms of frequency f = ω/2π (so df = dω/2π) as follows:

$$X'(f) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i2\pi ft} \qquad \text{projection = transform} \qquad (1.1a)$$

$$x(t) = \int_{-\infty}^{\infty} df\, X'(f)\, e^{+i2\pi ft} \qquad \text{expansion = inversion} \qquad (1.2a)$$

where X'(f) = X(f/2π). This form gets rid of the 2π out front, but sticks you with 2π factors in the exponents.

2. Proof of the Fourier Integral Transform.

A simple proof of the Fourier Integral theorem follows from this fact:

$$\int_{-\infty}^{\infty} dx\, e^{\pm ikx} = 2\pi\delta(k) \qquad (2.1)$$

The proof of the Fourier Integral in one direction uses $x=\omega$ and $k = t'-t$. In the other direction, it uses $x = t$, and $k = \omega' - \omega$. That is, you plug one equation into the other, then both sides become equal based on (2.1).

Intuitively, one understands the above delta function integral (2.1) as saying this: if $k \neq 0$, the phases cancel perfectly in the integral to give 0. If $k=0$, the integrand is just 1, and obviously the integral is then infinite.

One can "prove" (2.1) as follows: expand the exponential as the usual $\cos(kx) \pm i \sin(kx)$. The $\sin(kx)$ part integrates to 0 since $\sin(kx)$ is an odd function of x. The $\cos(kx)$ term represents the area under a cosine curve, from $-\infty$ to ∞. As long as $k \neq 0$, it seems reasonable to assume this to be zero, which would imply that the average value of $\cos(kx)$ is zero over the infinite interval. For $k=0$, the integrand is 1, so you know you are getting something infinite, so the result must be proportional to $\delta(k)$.

To show the factor 2π, integrate both sides of (2.1) over dk from -a to +a. The RHS becomes 2π. On the left side do the dk integral first to get

$$\text{LHS} = 2\int_{-\infty}^{\infty} dx\, \frac{\sin(ax)}{x} = 2\pi \qquad (2.2)$$

Thus, we have verified that 2π is the correct constant on the right of (2.1), and we have therefore concluded our proof of (2.1) and also of the Fourier Integral Theorem.

3. Convolution Theorem.

We had better make a statement of this thing consistent with our conventions before going much farther. Suppose three functions of t are related as follows (the convolution integral):

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')\, c(t') \qquad \text{sometimes written} \qquad a = b * c \qquad (3.1)$$

(Note that the RHS is unchanged if you do $b \leftrightarrow c$. Just change variables to $t'' = t - t'$ to see this.)

Insert into this expansions of the form (1.2) in terms of $A(\omega)$, $B(\omega)$ and $C(\omega)$. The time integral on the right can then be done first, using (2.1) above. This leaves a single ω integration on the right, and there is also single ω integration on the left. In order for the resulting equation to be true for each spectral component, the integrands must be equal. This leads to the famous result:

$$A(\omega) = B(\omega)\, C(\omega) \qquad (3.2)$$

The significance of this result cannot be overstated. It says that, whereas things are intrinsically complicated in the time domain, things are simple in the ω domain. Here, a messy integral equation in t is reduced to simple multiplication in ω.

There is one small penalty we pay for our choice of the 2π factors in (1.1) and (1.2). Because the two equations are not perfectly symmetric, the convolution theorem obtained by $\omega \leftrightarrow t$ looks a little different. It is proved in the same way. Note the extra $1/2\pi$ in (3.3) which is not present in (3.1).

$$A(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, B(\omega-\omega')\, C(\omega') \qquad (3.3)$$

$$a(t) = b(t)\, c(t) \qquad (3.4)$$

Of course if you work with $df = d\omega/(2\pi)$, this asymmetry goes away.

4. Applications of the Convolution Theorem .

This section is included here because books often do not make the connection between the convolution theorem, Green's Functions, and the real world of everyday electronics. Often too, this discussion is presented in the language of Laplace Transforms, so here we will do it in terms of the above Fourier Transform. We will start general, and end up specific.

(a) general case.

The real world seems to be described by linear differential equations. Here is a general form:

$$L_t\, u(t) = f(t) \qquad (4.1)$$

where L_t contains perhaps first and second order differential operators like d/dt. You would like to solve this equation for u, given some driving function f. It would be nice if you could find some operator that is the inverse of L_t and apply it to both sides of (4.1). The problem would then be solved. This is exactly what we are going to do.

You can define a related equation as follows

$$L_t\, g(t-t') = \delta(t-t') \qquad (4.2)$$

If you can solve (4.2) for g, then you know the solution to (4.1) for u(t) in terms of f and g. It is this:

$$u(t) = \int_{-\infty}^{\infty} dt'\ g(t - t')\ f(t') \qquad (4.3)$$

To see that (4.3) is true, apply L_t to both sides, and make use of (4.2), you end up with (4.1). The function g is called the Green's Function, "propagator", or "kernel" of L_t. In (4.3) one is applying an integral operator $\int g$ to function f to get function u. Looking at (4.1), this integral operator must in some sense be the inverse of the differential operator L_t. This notion is supported by integrating (4.2) over dt so you get unity on the right.

Now we come to the main point. Equation (4.3) is a convolution equation of the form (3.1)! Therefore, we can write (4.3) in the ω-domain as follows:

$$U(\omega) = G(\omega)\ F(\omega) \qquad (4.4)$$

(b) a specific example: the RC filter section

Suppose you have a simple unloaded RC filter section (series R on input) with input voltage $v_i(t)$, and output voltage $v_o(t)$. Here is the differential equation:

$$[\ RC\ d/dt + 1]\ v_o(t) = v_i(t) \qquad (4.1a)$$

Define the Green's Function by

$$[\ RC\ d/dt + 1]\ g(t) = \delta(t) \qquad (4.2a)$$

Then the solution to (4.1a) is this:

$$v_o(t) = \int_{-\infty}^{\infty} dt'\ g(t - t')\ v_i(t') \qquad (4.3a)$$

This has the convolution form, so in the frequency domain we get

$$V_o(\omega) = G(\omega)\ V_i(\omega). \qquad (4.4a)$$

Sometimes this is called "filter theory", where G(ω) is the "transfer function" of the filter -- in our case a simple RC filter. The transfer function G(ω) and its time-domain Green's Function g(t) are:

$$G(\omega) = 1/(\ 1 + i\omega RC) = X_C\ /(\ R + X_C) \qquad (4.5)$$

$$g(t) = (1/RC)e^{-(t/RC)}\ \theta(t) \qquad (4.6)$$

In practice, of course, one computes G(ω) first by expanding both sides of (4.2a) using (1.2). In doing so, one considers a single spectral component exp(iωt), so time derivatives are replaced with factors of iω. This leads at once to G(ω) as shown above. It is even more obvious in terms of reactance X_C since this filter is just a voltage divider.

Then you can then compute g(t) from G(ω) using (1.2). Here, the dω integration contour is closed in the upper half ω plane if t>0, and in the lower half plane if t<0. Since G(ω) has a pole only in the upper half plane (at ω = i/RC) , g(t) is non-zero only for t>0, hence the θ(t). In this case, you pick up the pole residue and you get the g(t) shown above.

If $v_i(t) = \delta(t)$, then $V_i(\omega) = 1$. In this case, $V_o(\omega) = G(\omega)\ 1$ and it must be that $v_0(t) = g(t)$. Thus, one always interprets g(t) as the impulse response of the filter. In this case, it is of course a simple decaying exponential. One then interprets θ(t) as saying that the impulse response only propagates forward in time, never backward.

We conclude this section by writing out (4.3a), which shows the time domain solution of our simple RC filter:

$$v_o(t) = (1/RC) \int_{-\infty}^{\infty} dt'\, e^{-(t-t')/(RC)}\ \theta(t-t')\ )\ v_i(t')$$

$$= (1/RC) \int_{-\infty}^{t} dt'\, e^{-(t-t')/(RC)}\ \ v_i(t') \qquad (4.7)$$

This says that the present response of the system at time t is the cumulative result of the the impulse responses at all past times, weighted by the value of the input function.

(c) an even simpler example: $L_t = d/dt$

Let $L_t = d/dt$ and nothing else. Then we will mimic our equations above. We are saying that v_i is the derivative of v_o:

$$[\ d/dt\ ]\ v_o(t) = v_i(t) \qquad (4.1b)$$

Define the Green's Function by

$$[\ d/dt]\ g(t) = \delta(t) \qquad (4.2b)$$

Then the solution to (4.1a) is this:

$$v_o(t) = \int_{-\infty}^{\infty} dt'\, g(t - t')\ v_i(t') \qquad (4.3b)$$

This has the convolution form, so in the frequency domain we get

$$V_o(\omega) = G(\omega)\, V_i(\omega). \qquad (4.4b)$$

The Green's Function g which solves (4.2a), and its transform G are:

$$G(\omega) = 1/(i\omega) \qquad (4.5b)$$

$$g(t) = \theta(t) \qquad (4.6b)$$

And the integral becomes

$$v_o(t) = \int_{-\infty}^{t} dt' \quad v_i(t') \qquad (4.7b)$$

This result is rather encouraging. We have shown in a rather fancy fashion that differentiation in the time-domain is equivalent to multiplication by $+i\omega$ in the ω-domain. It follows that integration corresponds to dividing by $i\omega$.

5. Fourier Integral Conventions

(a) Sign The Fourier Transform (1.1) and (1.2) is also true if you replace i with -i in both equations. This follows trivially from (2.1). EE people usually think of the fundamental "spectral component" time dependence as $\exp(+i\omega t)$, whereas physics people think of $\exp(-i\omega t)$. We use the EE sign convention .

The use of $\exp(-i\omega t)$ in physics can be traced to the convention selected by Schrodinger in his basic equation describing quantum mechanics, and probably goes back beyond that. Physicists are used to seeing plane waves described by $\exp[+i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$.

If you use the physics convention (perhaps because you are reading a physics text), you must replace all our i by -i, and also Im[ ] by -Im[ ].

(b) j Versus i. EE texts favor j, physics texts always use i, which is of course the true historical symbol for $\sqrt{-1}$. I suspect the reason is that EE people deal with lumped circuits containing currents labeled "i", whereas physicists deal with Maxwell's equations which contain current density "j". Each profession chooses its symbol for $\sqrt{-1}$ to minimize confusion with these other symbols.

(c) Allocation of 2π. Our convention has been to put the factor of $(1/2\pi)$ into the inversion formula (1.2), and to have no factor at all in the transform formula (1.1). We shall describe our motivations for doing this below.

Sometimes books put a $1/\sqrt{2\pi}$ factor in the transform (1.1), which causes the appearance of an identical $1/\sqrt{2\pi}$ factor in equation (1.2). This has the advantage of making the two equations completely symmetrical, and reminds us that there is complete symmetry between the conjugate variables t and ω. We have chosen not to do this here.

And of course the world would not be complete if some people did not prefer to put a $1/2\pi$ into the expansion equation (1.1), and have none of it in (1.2).

In general, the product of the two factors must be $1/2\pi$. This is simply due to the 2π factor sitting on the right of (2.1). The main reason we choose to put the factor entirely in (1.2) is the following. Suppose we have a constant k sitting in front of (1.1) other than k=1. Then the transform $X(\omega)$ so defined is scaled differently than our $X(\omega)$. If you rescale all terms in the ω-plane convolution result (3.2), you must end up with an extra factor of k hanging around in your new version of (3.2). The other alternative is to add a k factor into the definition of the convolution integral (3.1). Neither is very nice, and there is a lot of history behind (3.1) and (3.2) as written. This why we have done our 2π factors as shown above.

(d) Comments. The conventions discussed above have no real physical significance, they just lead to different definitions of $X(\omega)$, so there are slight variations in (1.1) and (1.2). It is important I think to at least adopt some convention so you know what you are talking about. A potential problem comes when you try to look up something in a handbook, then you may be off by factors of 2π if you are not clear on the conventions. The conventions we have adopted are consistent with my 1968 Shaum's Mathematical Handbook , and also with my 1967 printing of the Fourth Edition ITT Reference Data for Radio Engineers.

6. Relation between Fourier Transform and Laplace Transform.

There is one other motivation for our choice of convention for the exponential sign and the factor of 2π appearing in the Fourier Integral Transform pair (1.1) and (1.2). If you start with our inversion formula (1.2) and make the replacement $s = i\omega$, you end up with this result:

$$x(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds\, X(s/i)\, e^{+st} \qquad \text{expansion = inversion} \qquad (6.1)$$

This is almost the traditional Laplace Transform inversion formula -- there are no wrong signs, or extra factors of 2π. Normally the Laplace contour runs vertically in the s-plane from $c-i\infty$ to $c+i\infty$ to the right of all singularities. If there are no poles in the right half s-plane, the contour can be slid to the left to give the above. Thus, it appears that $L\{ x(t), s\} = X(s/i)$ in this case.

No poles in the right half s-plane means no poles in the bottom half ω plane, since $s = i\omega$. We have already seen in Section 4 (b) and 4 (c) above how this implies that x(t) must be a causal function, that is, x(t) will be proportional to $\theta(t)$ -- x(t) must vanish for negative time. For such functions, we can write (1.1) as follows, using $s = i\omega$:

$$X(s/i) = \int_{o}^{\infty} dt\, x(t)\, e^{-st} \qquad \text{projection = transform} \qquad (6.2)$$

And this is exactly the Laplace Transform.

Here then is our conclusion: for the set of functions x(t) which are "causal", like our Green's Function propagator g(t) discussed above, and which therefore vanish at negative time, we can make an exact

identification between the Laplace Transform L{x(t), s} and our Fourier Transform X(ω) evaluated at ω =s/i. Of course if we think of s = real, then we are "analytically continuing" our function X(ω) off its real ω axis. And if we think of ω as real, then we are analytically continuing the Laplace Transform to imaginary s. For all other functions x(t), the Fourier and Laplace Transforms are different.

Of course, when considering some general function x(t), we can easily make it causal "by fiat" by simply multiplying it by θ(t). In this case, our association holds all the time,

$$L\{\theta(t)x(t), s\} = X(s/i) \qquad X(\omega) = L\{\theta(t)x(t), i\omega\} \qquad (6.3)$$

This lets us make use of extensive tables of Laplace Transforms to look up our X(ω) for given x(t), and lets us also understand that the "general properties" of Laplace Transforms also apply to our Fourier transform, with the appropriate replacement s = iω.

For example, the Laplace Transform of $x(t) = e^{at}$ is 1/(s-a). And a trivial "property" of Laplace is that kx(t) maps into kL{x(t),s} (ie, the transform is linear). Thus, for our Green's Function of (4.6),

$$L\{g(t), s\} = L\{(1/RC)\, e^{-t/(RC)}, s\} = (1/RC) / (s + (1/RC)) = 1/(sRC + 1)$$

Thus we would conclude from (6.3) that

$$X(\omega) = 1/(i\omega RC + 1)$$

in agreement with (4.5) above.

7. Real x(t) and the Reflection Rule.

Nothing in our Section 2 proof of the Fourier Transform required that x(t) be real. However, if we do assume that x(t) is real (for example, a voltage or current in a real circuit), then from (1.1) the following fact follows at once:

$$X(-\omega) = [X(\omega)]^* \qquad (7.1)$$

Thus, one can think of the negative frequency spectral components of a real function x(t) as simply being defined in this manner in terms of the positive spectral components. Note that X(ω) is in general complex, even if x(t) is real, because exp(-iωt) is complex in (1.1).

8. Some very simple examples of spectra.

(a) The spectrum of x(t) = 1.

In this example, x(t) is a constant over all time. Using (1.1) and (2.1), we find:

$$x(t) = 1 \qquad X(\omega) = 2\pi\,\delta(\omega) \tag{8.1}$$

This comes as no surprise. For a DC function, all the energy is concentrated at zero frequency.

(b). The spectrum of $x(t) = \delta(t - t_1)$.

Here x(t) is an infinitely narrow pulse of area 1, positioned at $t=t_1$. Using (1.1), we get the following Fourier spectrum:

$$x(t) = \delta(t - t_1) \qquad X(\omega) = e^{-i\omega t_1} \tag{8.2}$$

The spectrum X(ω) has a constant magnitude for all ω, out to infinite frequency. For such a pulse at t=0,

$$x(t) = \delta(t) \qquad X(\omega) = 1 \tag{8.3}$$

and here the phase is also constant.

(c) The spectrum of the Heaviside step function $\theta(t - t_1)$.

$$x(t) = \theta(t - t_1) \qquad X(\omega) = (1/i\omega)e^{-i\omega t_1} \tag{8.4}$$

We did this directly from (1.1), setting the upper endpoint contribution to 0, which is the usual method of correctly dealing with singular functions ("distributions").

If we apply our upcoming differentiation rule (11.1) to (8.4), we reproduce (8.2).

Applying our Laplace equivalence notion (6.3), we would predict from (8.4) that $L\{\theta(t - t_1), s\} = (e^{-st_1})/s$, which is in fact correct.

9. Spectrum of an isolated square pulse.

Consider a positive pulse of amplitude A and width τ which is centered at t=0. We can represent this using the Heaviside step function,

$$x(t) = A\,[\,\theta(t + \tau/2) - \theta(t - \tau/2)\,]. \tag{9.1}$$

Apply (1.1) to get the spectrum of this pulse,

$$X(\omega) = A \int_{-\tau/2}^{\tau/2} dt\, e^{-i\omega t} = (A\tau)\,\mathrm{sinc}(\omega\tau/2) \tag{9.2}$$

where sinc(x) = sin(x)/x. There are now several observations we can make:

(a) The spectrum is a continuous function of ω.

(b) Because sinc(-x) = sinc(x), X is an even function of ω.

(c) $X(\omega)$ has the shape we are all familiar with, see any text or handbook. The positive zeros are at $(\omega\tau/2) = n\pi$ for n=1,2,3... The first zero is at $\omega = 2\pi/\tau$ $(f = 1/\tau)$. The central peak is height $(A\tau)$.

(d) Most of the spectral energy is in the central hump, and this represents positive frequency in the range f=0 to $f=1/\tau$.

(e) Quantity $(A\tau)$ is the Area under the t-domain pulse. If the pulse is made twice as narrow and twice as high, this area stays constant. But, the first zero of $X(\omega)$ moves out twice as far (as do all zeros), so the spectral width doubles.

(f) In the limit $\tau \rightarrow 0$ with $(A\tau)$ = (Area) = fixed, the square pulse x(t) approaches (Area) $\delta(t)$. Since $\text{sinc}(x) \rightarrow 1$ as $x \rightarrow 0$, we find that $X(\omega)$ = (Area) This is in agreement with (8.3) above. In this limit, the height of the central hump stays fixed, and the zeros move out to infinity, so it is as if a small flat portion of the central hump has been expanded to fill all ω.

(g) What about the limit $\tau \rightarrow \infty$? If we take this limit with A = fixed, we are converting our pulse to a constant DC signal x(t) = A. In this limit, as long as $\omega \neq 0$, the argument of the sinc function oscillates infinitely fast, giving a function that is zero when averaged over any finite interval. At $\omega = 0$, something singular happens. The result comes out $X(\omega) = 2\pi A\ \delta(\omega)$, in accordance with (8.1) above. In terms of (9.1), in this limit the central hump gets higher and higher, and the zeros all move in toward $\omega = 0$. As these zeros get closer together, the oscillation frequency of the tail of sinc(x) becomes infinite and washes out. To derive $X(\omega) = 2\pi A\ \delta(\omega)$ from (9.2), one needs this fact

$$\lim_{R\rightarrow\infty} \{\ \sin(Rx)/x\ \} = \pi\ \delta(x) \qquad (9.3)$$

This thing in turn can be verified by multiplying both sides by some function f(x) and integrating:

$$\int_{-\infty}^{\infty} dx\ f(x)\ \sin(Rx)/x = \int_{-\infty}^{\infty} dy\ f(y/R)\ \sin(y)/y = f(0) \int_{-\infty}^{\infty} dy\ \sin(y)/y = f(0)\ \pi$$

10. The Area Rules and Parseval's Theorem

If we set $\omega = 0$ in the Fourier Transform (1.1) and (1.2), we find

$$X(0) = \int_{-\infty}^{\infty} dt\ x(t) = [\text{area under } x(t)\ ] \qquad (10.1)$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega) = (1/2\pi) \text{ [area under } X(\omega) \text{]} \qquad (10.2)$$

For the square wave pulse example above, we saw that $X(0) = (A\tau)$ from (9.2). In light of (10.1), it is thus not a coincidence that this is the area under the time domain square pulse.

From (10.2), we may conclude that the total area under the $X(\omega)$ curve for our square pulse is $2\pi A$, since $x(0) = A$, the height of our pulse. This is consistent with this fact:

$$\int_{-\infty}^{\infty} dx\, \text{sinc}(x) = \pi \,. \qquad (10.3)$$

Another area rule involves the power spectrum. First, it is easy using (1.1), (1.2) and (2.1) to prove this identity (Parseval's Identity),

$$\int_{-\infty}^{\infty} dt\, a(t)\, b^*(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, A(\omega)\, B^*(\omega) \qquad (10.4)$$

The * means complex conjugation and is needed to make the thing work so you get $\delta(\omega - \omega')$ in the proof. Again, the 2π factor is missing if you use df in place of $d\omega$. In the case $a = b = x$, you get the power area rule (Parseval's Theorem) which says

$$\int_{-\infty}^{\infty} dt\, |x(t)|^2 = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |X(\omega)|^2 \qquad (10.5)$$

If x(t) is a voltage or current pulse, this says that the total energy contained in the pulse is the same no matter in which space you add it up. For our square pulse, the LHS is $A^2\tau$. The right hand side gives the same result if you use (9.2) and the following fact,

$$\int_{-\infty}^{\infty} dx\, \text{sinc}^2(x) = \pi \,. \qquad (10.6)$$

It is rather interesting that sinc(x) and $\text{sinc}^2(x)$ have the exact same area, see (10.3) and (10.6). I am not sure I know any other functions with this property! [Higher powers have different area.]

11. Differentiation and Integration Rules

In Section 4 (c) we gave an informal proof of what everyone knows about the effect of d/dt. Here we state the differentiation rule both ways:

$$dx(t)/dt \leftrightarrow [i\omega X(\omega)] \qquad (11.1)$$

$$dX(\omega)/d\omega \leftrightarrow [itx(t)] \qquad (11.2)$$

From (11.1), you get the spectrum of the derivative of a function by multiplying the original function's spectrum by iω. For integration, you must therefore just divide by iω.

(a) Let's apply (11.1) to our square pulse function. We have from (9.1) and (9.2),

$$x(t) = A [\theta(t + \tau/2) - \theta(t - \tau/2)]$$

$$X(\omega) = (A\tau) \text{ sinc}(\omega\tau/2) .$$

If we differentiate x(t), we get a pair of opposite signed delta functions separated by distance τ. This is some sort of doublet impulse.

$$x_1(t) = x'(t) = A [\delta(t + \tau/2) - \delta(t - \tau/2)]$$

The spectrum must be

$$X_1(\omega) = (A\tau) (i\omega)\text{sinc}(\omega\tau/2)$$

As expected, there is no DC component.

(b) Now lets apply (11.2) to the same square pulse. Let

$$x_1(t) = t\, x(t) = A\, t [\theta(t + \tau/2) - \theta(t - \tau/2)].$$

This represents a doublet sawtooth pulse centered at t=0. According to (11.2),

$$X_1(\omega) = -i\omega dX(\omega)/d\omega = (-i\omega A\tau^2/2) \text{ sinc}'(\omega\tau/2)$$

where $\text{sinc}'(x) = \cos(x)/x - \sin(x)/x^2$. Again, $X_1(\omega)$ has no DC component, as expected.

12. Time translation x(t) causes phase on X(ω).

Assume that some x(t) has a spectrum X(ω).

$$x(t) \leftrightarrow X(\omega)$$

Then it follows directly from (1.1) (just change variables to $t' = t - t_1$) that:

$$x(t - t_1) \quad \leftrightarrow \quad X(\omega)\, e^{-i\omega t_1} \qquad (12.1)$$

We saw this happening in the special case of (8.2), here we see that the result is completely general. Translation of a signal in time causes the spectrum to gain the phase shown.

A parallel result holds if you do $\omega \leftrightarrow t$, but the sign of the phase is opposite:

$$X(\omega - \omega_1) \quad \leftrightarrow \quad x(t)\, e^{+i\omega_1 t} \qquad (12.2)$$

13. Exponential Sum Rules.

In (2.1) proven above, the delta function is written as an infinite integral of an exponential. A similar result involves a summation of exponentials. For k in the range $-\pi$ to π we claim that:

$$\sum_{n=-\infty}^{\infty} e^{ink} = 2\pi\delta(k) \qquad -\pi < k < \pi \qquad (13.1)$$

To "prove" this, we argue similarly to (2.1) that for $k \neq 0$, the phases exactly cancel and you get zero. For $k = 0$, the thing being summed is 1, so the result is infinite, and thus the result is proportional to $\delta(k)$. As we did above, we can prove that the factor of 2π is correct by integrating both sides over k from -a to +a. The RHS gives 2π. On the LHS, do the dk integral to get

$$\text{LHS} = \sum_{n=-\infty}^{\infty} [2\sin(na)/n] = 2a + 4 \{ \sum_{n=1}^{\infty} [\sin(na)/n] \} = 2a + 4 \{ \frac{\pi - a}{2} \} = 2\pi$$

where the infinite sum appears as 1.441.1 in Gradshteyn-Ryzhik, and no doubt in other tables. The sum formula is restricted to $-\pi < a < \pi$.

For the reader unhappy with the vague statement that the "phases exactly cancel" away from k=0, here is a simple proof that they in fact do. Add and subtract 1 to get:

$$\sum_{n=-\infty}^{\infty} e^{ink} = (1 + e^{ik} + e^{i2k} + ...) + (1 + e^{-ik} + e^{-i2k} + ...) - 1$$

$$= \frac{1}{1 - e^{ik}} + \frac{1}{1 - e^{-ik}} - 1 .$$

In this last step, we used $1 + x + x^2 + ... = 1/(1-x)$, which is certainly *not* valid for x=1. Thus, our series summations above are not valid for k=0. So away from k=0, we go on to find a common denominator:

$$= \frac{(1 - e^{-ik}) + (1 - e^{+ik}) - (1 - e^{-ik})(1 - e^{+ik})}{-(1 - e^{-ik})(1 - e^{+ik})}$$

But the numerator is exactly 0! Just multiply it out. This concludes our more rigorous proof that the exponential sum exactly vanishes away from k = 0.

Now we want to generalize (13.1) for k in the range -∞ to +∞. The result is fairly obvious. Instead of just a delta function at k=0, you have delta function spikes at each place where the thing being summed equals 1. Thus, spikes will be at k = 0, ± 2π, ± 4π, and so on,

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

Again, one can prove that the constant is 2π at each spike by integrating over dk from 2πm-a to 2πm+a, for a small.

The exponential sum rule (13.2) plays a critical role in the analysis of periodic pulse trains in the Chapter 2 below.

13A. Delta Function Technology

The delta function is a mathematical tool that simplifies the description of mathematical relationships and allows one to consider certain idealized situations which cannot really exist in practice. The down side is that delta functions also serve to confuse readers who are not used to working with them. As later sections of this report were written, it became apparent that many things lean fairly heavily on "delta function technology", and one must understand them at a somewhat deeper level than the level used earlier in this Chapter. Of particular importance is the meaning of the symbol δ(0) which appears in many equations in later chapters.

(a) The Integral Form of the Exponential Sum Rule.

Consider the fundamental result (2.1) quoted and "proved" in Section 2 of this chapter.

$$\int_{-\infty}^{\infty} dx\, e^{\pm ikx} = 2\pi\delta(k) \qquad (13A.1)$$

The reader may have noticed that the proof had a certain "hokey" quality. One is never quite happy with such proofs because one must rely on vague statements like " the average of cos(x) over an infinite interval is zero".

The reason the proofs never seem quite right is that the delta function is not a "reasonable function". Instead, it is the limit of a reasonable function. Such limits are called "distributions". Here now is a

completely reasonable proof of (2.1). First, consider this definite integral which you can find in any list of integrals

$$\int_{-\infty}^{\infty} dx \;\; \exp[\, -(ax^2 + bx + c) \,] \;\; = \;\; \sqrt{\frac{\pi}{a}} \;\;\; \exp[\;\; (b^2/4a) - c \;\;] \qquad (13A.2)$$

We can use this general integral to derive the following result:

$$\int_{-\infty}^{\infty} dx \; e^{ikx} \exp(-Ax^2) \; = 2\pi \; \delta_1(k,A) \qquad (13A.3)$$

$$\text{where } \delta_1(k,A) = \; \{ \; \frac{1}{\sqrt{4\pi A}} \;\; \exp(\, -k^2/4A) \;\; \}$$

The number A > 0 is acting as a reasonable "cutoff" in the integration on the left, causing it to converge in a reasonable manner. If A is a small number, then $\delta_1(k,A)$ is a strongly peaked narrow function centered at k=0. Its width is $\sim \sqrt{A}$, and its height is $\sim 1/\sqrt{A}$. As you can check with the above general formula applied to a dk integration, the area under $\delta_1(k,A)$ is exactly 1, independent of A. In the limit that $A \rightarrow 0$, the $\delta_1(k,A)$ becomes $\delta(k)$ so (13A.3) becomes:

$$\int_{-\infty}^{\infty} dx \; e^{ikx} \; = 2\pi\delta(k) \qquad \delta(k) \;\equiv\; \mathrm{Lim}_{A \rightarrow 0} \; \{ \; \delta_1(k,A) \;\; \} \qquad (13A.4)$$

This then concludes our proof. Whenever one sees (13A.1), one understands that it is simply a compact *shorthand notation* for (13A.3) for very small A. If one is willing to overlook certain strange sounding handwavings, one can pretend that $\delta(k)$ is a normal function and work with it as such. Whenever there are doubts, one should back off and re-install the cutoff A to see what is really going on.

There are many other functions besides $\delta_1(k,A)$ which have $\delta(k)$ as their limit. Here is another one that has a certain appeal. Consider this simple integral:

$$\int_{-B}^{B} dx \; e^{ikx} \; = 2\pi \; \delta_2(k,B) \qquad (13A.5)$$

$$\text{where} \quad \delta_2(k,B) \; = \; \{ \;\; \frac{\sin(kB)}{\pi k} \;\;\; \}$$

For large B, $\delta_2(k,B)$ has the same characteristics as $\delta_1(k,A)$ -- a tall, narrow function peaked at k=0. It differs from $\delta_1(k,A)$ in that it has the usual sinx/x oscillation tail. $\delta_2(k,B)$ also has the property that its area equals 1 regardless of B. If we now take the limit $B \rightarrow \infty$, we get

$$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k) \qquad \delta(k) \equiv \mathrm{Lim}_{B\to\infty}\{\ \delta_2(k,B)\ \} \qquad (13A.6)$$

(b) The Discrete Form of the Exponential Sum Rule

We present here a third form of the delta function as a limit, one that is more appropriate to sums than integrals. Consider this sum

$$\sum_{n=-N}^{N} e^{ink} = (1 + e^{ik} + e^{i2k} + ... + e^{iNk}) + (1 + e^{-ik} + e^{-i2k} + ... + e^{-iNk}) - 1 \qquad (13A.7)$$

Using the fact that

$$1 + x + x^2 + ... + x^N = (1 - x^{N+1})/(1-x) \qquad (13A.8)$$

we arrive at this result (valid for any positive integer N):

$$\sum_{n=-N}^{N} e^{ink} = 2\pi\ \{\ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)}\ \} \qquad (13A.9)$$

There is certainly a striking similarity between the RHS of this sum, and the RHS of (13A.5). The big difference is the extra sine in the denominator, which causes this thing to be large at places other than just k = 0. In fact, it is easy to show that the RHS of (13A.9) is a periodic function of k with period 2π. It thus has identical peaks at locations $k = 2\pi m$ for m integer, and it suffices to look at the peak at k=0.

Let us now assume that N is a very large but finite number. The numerator sine has its first zero at a value k such that $(N+1/2)k = \pi/2$ or roughly $k = \pi/(2N)$ for large N. Since this is a very small number, we can expand the denominator using sinx = x in this region. For larger values of k beyond the first 0, the function { } has negligible size, and we make negligible error by replacing sinx = x for larger k as well.

We now repeat this argument in the region of $k = 2\pi m$ for m = all integers. We then can approximate (valid for large N) the RHS of (13A.9) with a sum of functions each of which generates a particular peak,

$$\sum_{n=-N}^{N} e^{ink} \approx 2\pi \sum_{m=-\infty}^{\infty} \{\ \frac{\sin[(N+1/2)(k - 2\pi m)]}{\pi (k - 2\pi m)}\ \} \qquad (13A.10)$$

This leads us to consider the following definition of a delta function,

$$\delta_3(k,N) \equiv \{ \quad \frac{\sin[(N+1/2)(k)]}{\pi\, k} \quad \} \tag{13A.11}$$

$$\delta(k) = \lim_{N \to \infty} \{ \quad \delta_3(k,N) \quad \} \tag{13A.12}$$

and we then arrive at the following discrete sum rule, which appears as (13.2) in Section 13,

$$\sum_{n=-\infty}^{\infty} e^{ink} \quad = \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty \; < k < \infty \tag{13A.13}$$

When we use this sum rule, we use it as a symbolic limit. In practice, we always deal with some finite but large value of N. In this case, we understand that $\delta(k) = \delta_3(k,N)$. This gives us a working meaning for the symbol $\delta(0)$, namely

$$\delta(0) \; = \; \delta_3(0,N) \; = \; \frac{(2N + 1)}{2\pi} \tag{13A.14}$$

It is important for us to have a clear unambiguous notion of what we mean by $\delta(0)$ in practice, since this quantity will be appearing in equations in later sections of this report.

(c) Undoing the Limit for $\delta(0)$

The idealized delta function has the scaling property that $\delta(x/a) = a\,\delta(x)$. If we apply this property for some value a, then take the limit $x \to 0$, we end up with $\delta(0) = a\delta(0)$. For the idealized delta function, this is not a contradiction, since $\delta(0) = \infty$.

However, when we are interested in "practical situations", we want to have a particular candidate delta function in mind, so we can "undo" the limit if we wish. For such candidate "pre-limit" delta functions, the property $\delta(x/a) = a\,\delta(x)$ is no longer true. Thus, if we are going to have formulas in which the symbol $\delta(0)$ appears, and if we want to have an unambiguous notion of what $\delta(0)$ means -- for example $2\pi\delta(0) = (2N+1)$ -- then we must be sure not to rescale the delta function argument.

(d) Example 1.

When we deal with pulse trains, we will be adding exponentials of the form $\exp(in\omega T_1)$. From the above, we therefore have

$$\sum_{n=-\infty}^{\infty} e^{in\omega T_1} \quad = \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{13A.15}$$

If we evaluate the above formula at $\omega = 0$, we get

$$\sum_{n=-\infty}^{\infty} 1 \quad = \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(-2\pi m)$$

We now observe that $\delta(-2\pi m) = \delta_{m,0}\,\delta(0)$, so we then get:

$$\sum_{n=-\infty}^{\infty} 1 \quad = \quad 2\pi\,\delta(0) \qquad (13A.16)$$

We understand this as a symbolic limit. We can now undo the limit by replacing the sum endpoints with -N and N, and replace $2\pi\delta(0)$ with (2N+1), and the result is consistent. Had we rescaled the delta function by saying for example $\delta(-2\pi m) = (1/2\pi)\,\delta(-m)$, we would end up with a contradiction when we tried to "undo the limit".

(e) Example 2.

Consider the *square* of the sum shown in Example 1,

$$\{\ \sum_{n=-\infty}^{\infty} e^{in\omega T_1}\ \}^2 \quad = \{\ \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\ \}^2 \qquad (13A.17)$$

We write the RHS using m and n for our two summation indices, then we move both summations to the left. Inside this double sum we get

$$\delta(\omega T_1 - 2\pi m)\delta(\omega T_1 - 2\pi n)$$

Since the first delta function will "pin" ωT_1 to the values $2\pi m$, we can replace ωT_1 with $2\pi m$ in the second delta function and not change a thing, so we get

$$\delta(\omega T_1 - 2\pi m)\delta(2\pi m - 2\pi n) = \delta(\omega T_1 - 2\pi m)\quad \delta_{m,n}\,\delta(0) \qquad (13A.18)$$

The Kronecker delta now removes one of the summations, and we end up with this result:

$$\{\ \sum_{n=-\infty}^{\infty} e^{in\omega T_1}\ \}^2 \quad = [\ 2\pi\delta(0)\ ]\ \{\ \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\ \} \qquad (13A.19)$$

Evaluating the above at $\omega = 0$ yields

$$\{ \sum_{n=-\infty}^{\infty} 1 \}^2 = [\, 2\pi\delta(0) \,]\ [\, 2\pi\delta(0) \,] \qquad (13A.20)$$

And if we now undo the limit, we get this self-consistent result,

$$\{ \sum_{n=-N}^{N} 1 \}^2 = [\, 2\pi\delta(0) \,]\ [\, 2\pi\delta(0) \,] = (2N+1)^2 .$$

14. About the sinc function.

The fact is that both definitions of this function are in use even currently. I used the following definition,

$\mathrm{sinc}(x) = \sin(x)/x$

The other definition we will call

$\mathrm{sinc}_\pi(x) = \sin(\pi x)/(\pi x)$

It is thought that the first user of the function name "sinc" was Woodward in a 1953 book.

[1] P. M. Woodward, *Probability and Information Theory, With Applications to Radar.* New York: Pergamon, 1953, pp. 40–42.

and he used the second definition. A recent authoritative $100 Fourier book by **Bracewell** has this to say on the subject, which I quote from a news group,

In the introduction to his second edition of "The Fourier Transform and its Applications", Ron Bracewell remarks: "Notation is a vital adjunct to thinking and I am happy to report that the _sinc function, which we learned from P.M. Woodward's book, is alive and well and surviving erosion by ***occasional authors who do not know that `sine x over x' is not the sinc function."*** *sin(x)/x is usually denoted as Sa(x) in EE books, Sa standing for "sampling function".*

I do think most modern texts such as Goodman use the second definition, and so should I, but I have never seen Sa(x) used. Some web commenters thought maybe si(x) should be the non-π sinc function, but that is then confused with the sine integral which is pretty well installed in function books.

Here is another comment,

yes, sinc means "sinus cardinal". sinc(x) = sin(pi.x)/(pi.x). It is "cardinal" because sinc(n) = 0 if n is a cardinal number (i.e. an integer, except 0).

which seems pretty reasonable. And here is a reference

J. McNamee, F. Stenger, E.L. Whitney: "Whittaker's Cardinal Function in Retrospect", Math. Comput., 25, 113 (1971), pp. 141-54.

The Whittaker reference is this

> [Wh1] E.T. Whittaker, *On the Functions which are Represented by Expansion of the Interpolation Theory*, Proc. Roy. Soc. Edinburgh, **35** (1915) 181–194.

and people speak of Whittaker's Cardinal Function or "sine cardinal".

15. The question of "negative frequencies" and the real world.

When a real function x(t) is Fourier projected onto ω, the spectrum X(ω) has the property X(-ω) = X(ω)*. Thus, there are always both positive and negative frequency contributions in the spectrum, you cannot have one without the other.

How do we relate the frequency ω to a "real thing" in the real world? In Fourier theory, ω is just a parameter of an expansion theorem.

The problem of course is that, although x(t) might be real, we are still expanding onto a set of complex basis functions $e^{i\omega t}$ and the result is a spectrum X(ω) that is in general complex. X(ω) is only real in fact if real x(t) is symmetric in t, x(t) = x(-t), so the person concerned about physical things should be worried about the fact that X(ω) has negative frequency components, and about the fact that it is complex.

It is possible to recast the Fourier expansion in a way that involves only real functions and positive frequencies. For example, start with

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad (15.1)$$

and decompose X(ω) into parts that are even and odd in ω,

$$X(\omega) = 1/2[\, X(\omega) + X(-\omega)] + 1/2[\, X(\omega) - X(-\omega)] = X_{even}(\omega) + X_{odd}(\omega) \qquad (15.2)$$

Then we can write the above as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, [X_{even}(\omega) + X_{odd}(\omega)\,]\ [\ \cos(\omega t) + i \sin(\omega t)\,]$$

We can then throw out the two totally odd terms to get this result

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, [X_{even}(\omega)\cos(\omega t) + i\, X_{odd}(\omega)\sin(\omega t)\,]$$

and at this point we can fold the integral so only positive frequencies occur

$$x(t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega\, [X_{even}(\omega)\cos(\omega t) + i\, X_{odd}(\omega)\sin(\omega t)\,] \qquad (15.3)$$

This gives us an integral only over positive frequencies, but we still have complex spectra in there. Let's therefore go back and re-examine the even and odd spectral pieces in the case that x(t) is real,

$$X_{even} = 1/2[\, X(\omega) + X(-\omega)] = 1/2[\, X(\omega) + X(\omega)^*] = \mathrm{Re}\, X(\omega)$$

$$X_{odd} = 1/2[\, X(\omega) - X(-\omega)] = 1/2[\, X(\omega) - X(\omega)^*] = i\, \mathrm{Im}\, X(\omega)$$

The above then becomes

$$x(t) = \frac{1}{\pi} \int_0^\infty d\omega\, [\mathrm{Re}\, X(\omega) \cos(\omega t) - \mathrm{Im}\, X(\omega) \sin(\omega t)] \qquad \text{// real } x(t) \qquad (15.4)$$

Here then is a "physical" version of the expansion where everything is real and has positive frequency. You can see that x(t) has even and odd (in t) contributions. The spectrum of the even part is Re X(ω), while the spectrum of the odd part is - Im X(ω). Symmetric functions like box(t) have ImX(ω) = 0. Another way to write the above is this

$$x(t) = \frac{1}{\pi} \int_0^\infty d\omega\, \mathrm{Re}\, [X(\omega)\, e^{i\omega t}] \qquad \text{// real } x(t) \qquad (15.5)$$

which of course suggests a much faster derivation of this result as follows, where we assume real x(t). Start with,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, X(\omega)\, e^{+i\omega t} = \mathrm{Re}\, \{\frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, X(\omega)\, e^{+i\omega t}\} = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, \mathrm{Re}\, [X(\omega)\, e^{i\omega t}]$$

But the integrand in the rightmost form is an even function of ω, as follows:

$$f(-\omega) = \mathrm{Re}\, [X(-\omega)\, e^{-i\omega t}] = \mathrm{Re}\, [X(\omega)^*\, e^{-i\omega t}] = \mathrm{Re}\, [\, X(\omega) e^{+i\omega t}]^* = \mathrm{Re}\, [\, X(\omega) e^{+i\omega t}] = f(\omega)$$

allowing us to fold over the integral.

Now, let's go back to our physical-looking result,

$$x(t) = \frac{1}{\pi} \int_0^\infty d\omega\, [\mathrm{Re}\, X(\omega) \cos(\omega t) - \mathrm{Im}\, X(\omega) \sin(\omega t)] \qquad \text{// real } x(t)$$

We can combine this with projections as follows,

$$X(\omega) = \int_{-\infty}^\infty dt\, x(t)\, e^{-i\omega t}$$

$$\mathrm{Re}\, X(\omega) = \int_{-\infty}^\infty dt\, x(t) \cos(\omega t)$$

$$\mathrm{Im}\, X(\omega) = \int_{-\infty}^\infty dt\, x(t) \sin(\omega t)$$

So let's combine all this thinking in terms of separate sine and cosine transforms:

Expansion formula for real functions x(t)

$$C(\omega) = \int_{-\infty}^{\infty} dt\, x(t) \cos(\omega t)$$

$$S(\omega) = \int_{-\infty}^{\infty} dt\, x(t) \sin(\omega t) \qquad (15.6)$$

$$x(t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega\, [C(\omega) \cos(\omega t) - S(\omega) \sin(\omega t)]$$

$$\text{with} \quad \delta(t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \cos(\omega t)$$

This last result is just (2.1) folded up. The above expansion is the continuous version of the Fourier Series that we are quite used to. If you want to think of the Fourier Transform of real functions x(t) in terms of real projections and positive-only physical frequencies, then look no further than the above box. The content, however, is exactly the same as the general Fourier Transform, which is certainly easier to write.

If the results in the above box are extended to complex x(t), that is fine, but the expansion is not the same as the Fourier Transform expansion.

The text Lee & Messerschmitt offers an alternative to the above discussion using two transforms called the phase splitter transform and the Hilbert transform, see their page 18.

16. The Final Value Theorem

Consider the Fourier transform of $\dot{x}(t)$:

$$\dot{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, i\omega\, X(\omega)\, e^{+i\omega t}$$

$$i\omega X(\omega) = \int_{-\infty}^{\infty} dt\, \dot{x}(t)\, e^{-i\omega t}$$

Now we take the limit $\omega \to 0$ on both sides to get

$$\lim_{\omega \to 0} [\; i\omega X(\omega)\;] = \int_{-\infty}^{\infty} dt\, \dot{x}(t) = x(+\infty) - x(-\infty).$$

If we assume that our signal x(t) started at some finite time (perhaps t=0) and was zero before that time, we get this result,

$$\lim_{\omega \to 0} [\; i\omega X(\omega)\;] = x(+\infty)$$

which is the Final Value Theorem.

16. The Initial Value Theorem

The Initial Value Theorem requires more assumptions in the Fourier world. Start with

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t}$$

Now we want a limit as $\omega \to \infty$ Here we are going to assume that $x(t) = 0$ for $t < 0$, but has some finite value at $t = 0$, so there is a discontinuity at t=0. Now rewrite the above as

$$X(\omega) = (1/-i\omega) \int_{0}^{\infty} dt\; x(t)\, d/dt\, e^{-i\omega t}$$

then do parts integration to get

$$X(\omega) = (1/-i\omega) \{ \quad [x(t)\, e^{-i\omega t}] \Big|_0^{\infty} - \int_{0}^{\infty} dt\, \dot{x}(t)\, e^{-i\omega t} \}$$

The same parts could then be applied to this second integral, and assuming $\dot{x}(t)$ is reasonable and finite at t=0, we get a similar result which will have a $1/\omega$ behavior, so the above can be written as

$$X(\omega) = (1/-i\omega) \{ \quad [x(t)\, e^{-i\omega t}] \Big|_0^{\infty} - \text{order}(1/\omega) \}$$

where we are interested in large ω. Taking $\omega \to \infty$, we ignore the second term. We also assume that x(t) somehow tapers off at the large t limit, or make some other arm-waving argument to dispense with the parts contribution at $t = \infty$, then we get this result

$$\lim_{\omega \to 0} [\quad i\omega X(\omega) \,] = x(0+)$$

This last derivation is admittedly wobbly.

I think both these "theorems" are really more appropriate to the Laplace Transform, but Sklar's guest writer pulls the Fourier version of the FVT out of his hat on page 437, so at least here we have a dim idea of where this comes from. The guest writer does not really have Laplace Transform support in Sklar's book, and is trying to do something in control theory with a PLL, so he is forced to this compromise.