Chapter 5: Some Theoretical Topics PhL 6.22.91
last update: 3.16.03

5.1 Spectral Dispersion Relations

(a) dispersion relations for X(ω) : The Hilbert Transform

In Section 1 we defined the Fourier integral spectrum X(ω) of a function x(t).

Let us assume for the moment that the spectrum X(ω) has no singularities in the upper half plane, and that as you take ω to infinity along any ray in the upper half plane, the limit is a constant which we will call X(∞).

In filter theory, we usually mean "poles" by the term singularities, but in general there could be "branch cuts", which is what the function ln(ω) has on its negative real axis. What we are really saying about X(ω) is that it is "analytic" in the upper half plane. In practical terms, this means that X(ω) is any smooth and reasonable function.

We have seen an example already that fulfills these requirements. For our RC filter, we had

$$G(\omega) = 1/(i\omega RC + 1)$$

This has a pole in the *lower* half ω plane, and G(∞) = 0.

If you make the above assumption about X(ω), this fact follows, where ω = a real number:

$$\int_C d\omega' \frac{X(\omega')}{\omega' - \omega} = 0 \qquad (5.1)$$

Here, C is a counterclockwise contour which goes around the upper half ω plane, but which detours infinitesimally around and above the pole at ω' = ω. You can regard this integral as being made of three pieces.

(1) infinite semicircle; its contribution to the above is $i\pi X(\infty)$, just do it.

(2) tiny semicircular detour around the pole at ω = ω'. Basically, you pick up one half of the residue since you go half way around this pole, so you pick up a contribution $i\pi\, X(\omega)$.

(3) the two pieces along the real ω' axis. This is basically the integral along the real axis, but missing the single point ω = ω'. This is called a principle part integral, and sometimes people put a little tick mark through the integral, but we shall not do this.

Thus, we can rewrite (5.1) as follows:

$$X(\omega) = X(\infty) + (1/i\pi) \int_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega' - \omega} \qquad (5.2)$$

$X(\omega)$ has analytic in upper half plane, no poles.

Notice the very important factor of i. If you now break $X(\omega)$ into its real and imaginary parts, and then you write down the real and imaginary parts of the above single equation, you find that Re(X) and Im(X) are related to each other by the following two equations:

$$\mathrm{Re}[X(\omega)] = \mathrm{Re}[X(\infty)] + (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}[X(\omega')]}{\omega' - \omega} \qquad (5.3a)$$

Hilbert Transform

$$\mathrm{Im}[X(\omega)] = \mathrm{Im}[X(\infty)] - (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}[X(\omega')]}{\omega' - \omega} \qquad (5.3b)$$

$X(\omega)$ analytic in upper have ω plane;

Basically, this says that the real part of $X(\omega)$ along the real axis completely determines the imaginary part, and vice versa. You cannot arbitrarily set the real and imaginary parts independently. This is a general fact about smooth functions $X(\omega)$.

The above pair of equations is sometimes referred to as the Hilbert Transform. By doing parts integrations inside the integral, and making extra assumptions, you can recast this thing in other forms, which we will not do here.

Next, let us assume in addition that $X(\omega)$ is the spectrum of a *real* function x(t). As we saw in Section 7, this implies the reflection rule $X(-\omega) = X(\omega)^*$. Thus, we can fold the negative portions of the above integrations over to the positive side. For example,

$$\int_{-\infty}^{0} d\omega' \frac{\mathrm{Im}[X(\omega')]}{\omega' - \omega} = \int_{0}^{\infty} d\omega' \frac{\mathrm{Im}[X(\omega')]}{\omega' + \omega}$$

There are a lot of minus signs you have to keep track of. If you combine this result with the positive portion of the integration in (5.3), you just have to combine the denominators to get a final result. The other equation is treated in a similar fashion, and here are the results:

$$\mathrm{Re}[X(\omega)] = \mathrm{Re}[X(\infty)] + (2/\pi) \int_0^\infty d\omega' \, \omega' \frac{\mathrm{Im}[X(\omega')]}{\omega'^2 - \omega^2} \qquad (5.4a)$$

$$\mathrm{Im}[X(\omega)] = \mathrm{Im}[X(\infty)] - (2\omega/\pi) \int_0^\infty d\omega' \, \frac{\mathrm{Re}[X(\omega')]}{\omega'^2 - \omega^2} \qquad (5.4b)$$

$X(\omega)$ analytic in upper have ω plane; reflection property $X(\omega) = X(-\omega)^*$.

These two equations are completely general, given the assumptions we have made. They are known as the Kramers-Kronig "dispersion relations" for reasons given in Section 29 below.

(b) dispersion relations for $\gamma(\omega)$

In filter theory, one thinks of $X(\omega)$ as the "transfer function" of a filter. It is usually easier to think in terms of the function $\gamma(\omega)$ which we define as

$$\gamma(\omega) + - \ln [X(\omega)] = \alpha(\omega) + i\beta(\omega) \qquad (5.5)$$

Then we get

$$X(\omega) = e^{-\gamma(\omega)} = e^{-\alpha(\omega)} \, e^{-i\beta(\omega)} \qquad (5.6)$$

Notice that we defined $\gamma(\omega)$ with a minus sign, so both exponents have minus signs. The real quantities $\alpha(\omega)$ and $\beta(\omega)$ are the attenuation and phase functions of our filter.

Can we apply our dispersion relations to the function $\gamma(\omega)$ instead of $X(\omega)$? Yes, provided $\gamma(\omega)$ meets the same specs we assumed for $X(\omega)$. If $X(\omega)$ has a pole in the lower half ω plane, then $\gamma(\omega)$ has a branch cut singularity in the lower half plane starting at the pole and going off to the left. No problem. If $X(\omega)$ has no poles in the upper half plane, then $\gamma(\omega)$ has no such cuts in the upper half plane. However, consider:

$$\gamma(\omega) + - \ln [X(\omega)] = + \ln[1/X(\omega)]$$

This says that a zero in $X(\omega)$ is just as bad as a pole from $\gamma(\omega)$'s point of view. A zero of $X(\omega)$ in the upper half plane means $\gamma(\omega)$ has a branch cut in the upper half plane starting at this zero location and going off to the left.

Thus, we must now assume that $X(\omega)$ has neither zeros nor poles in the upper half plane. Filter's having transfer functions of this type are sometimes called "minimum phase".

Since we have assumed $X(\omega)$ goes to $X(\infty)$ on the great circle at infinity, we know that $\gamma(\omega)$ goes to $\gamma(\infty) = -\ln[X(\infty)]$, so no extra assumption here. If $X(\infty) = 0$, then $\gamma(\infty) = -\infty$, which is a little inconvenient. It just says that the attenuation of our filter is infinite as $\omega \to \infty$.

So now we can write the dispersion relations for $\gamma(\omega)$ instead of $X(\omega)$, assuming now that $X(\omega)$ has neither poles nor zeros in the upper half ω plane. Note that $\mathrm{Re}[\gamma(\omega)] = \alpha(\omega)$ and $\mathrm{Im}\{\gamma(\omega)] = \beta(\omega)$, so here is the result:

$$\alpha(\omega) = \alpha(\infty) + (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{\beta(\omega')}{\omega' - \omega}) \qquad (5.7a)$$

$$\beta(\omega) = \beta(\infty) - (1/\pi) \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega}) \qquad (5.7b)$$

$X(\omega)$ has neither zeros nor poles in upper half ω plane ("minimum phase").

Now what happens if we again apply the x(t) = real assumption:

$$X(-\omega) = X(\omega)^* \quad \Rightarrow \quad \gamma(-\omega) = \gamma(\omega)^* \qquad (5.8)$$

$$\Rightarrow \quad \alpha(-\omega) = \alpha(\omega) \quad \text{and} \quad \beta(-\omega) = -\beta(\omega)$$

Thus, we can fold the above integrals as before to get

$$\alpha(\omega) = \alpha(\infty) + (2/\pi) \int_{0}^{\infty} d\omega' \, \omega' \frac{\beta(\omega')}{\omega'^2 - \omega^2} \qquad (5.9a)$$

$$\beta(\omega) = \beta(\infty) - \omega (2/\pi) \int_{0}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'^2 - \omega^2} \qquad (5.9b)$$

$X(\omega)$ has no zeros of poles in upper half plane ("minimum phase")
reflection property $X(\omega) = X(-\omega)^*$.

The main point of the above is that the phase of a (minimum phase) filter is completely determined by its attenuation, and vice versa. Even for general filters there will be some relation like the above, but it will include terms to describe the zeros of $X(\omega)$ in the upper half plane. The conclusion that the phase and attenuation cannot be independently set is unavoidable.

(c) dispersion and attenuation

The group delay of a filter is given by

$$\tau(\omega) = d\beta(\omega) / d\omega$$

If we call the integral (5.9b) $K(\omega)$, including the $(2/\pi)$, then we find that

$$\tau(\omega) = d [\omega K(\omega)] / d\omega$$

If the integral $K(\omega)$ were somehow a constant κ, you would conclude that $\tau(\omega) = \kappa$, and the filter would be "non-dispersive". All frequency components of a pulse packet would then traverse the filter in the same time, so the pulse would not spread out (disperse) in time.

Obviously $K(\omega)$ cannot really be independent of ω, so a non-dispersive (min phase) filter does not exist. However, over certain ranges of ω where $K(\omega)$ is very slowly varying, such a filter can be reasonably non-dispersive. This would be a region of ω far away from any region where the attenuation $\alpha(\omega)$ is large. If $\alpha(\omega)$ were large and varying in the region of interest, the integral $K(\omega)$ would be strongly dependent on ω, because the denominator of the integral then vanishes there, and you would then have significant dispersion.

Attenuation and dispersion are intertwined. You can't have one without the other. This is a general fact one learns from the dispersion relations, without any specific filter in mind.

(d) application to coaxial cable

In the physics of a dielectric medium, the index of refraction is a function of frequency n(ω), a fact which causes light of different colors ω to be refracted by different angles. This effect is also known as "dispersion". It happens that $n(\omega) = \sqrt{\varepsilon(\omega)}$, where ε(ω) is called the dielectric constant of the medium, although it is sometimes not very "constant" as a function of ω.

It turns out that ε(ω) has the right properties to satisfy the above pair of equations (5.3). In this context, these equations applied to X(ω) = ε(ω) are called the "Kramers-Kronig dispersion relations".

In effect, a dielectric medium is a filter, and ε(ω) is the transfer function of the filter. The input and output of this "filter" are known as the "electric field" E and the "electric displacement" D.

$$D(\omega) = \varepsilon(\omega)\, E(\omega)$$

To say that there is significant "dispersion" means that Re[ε(ω)] has significant dependence on frequency, and this can happen -- according to (5.3a) -- only if ε(ω) has a significant imaginary part. With no imaginary part, the integral on the right is 0, and Re[ε(ω)] = Re[ε(ω)] = independent of ω.

Away from electromagnetic resonances of the medium, ε(ω) has a very small imaginary part for a non-polar dielectric like polyethylene or Teflon. Thus, if we could ignore ohmic losses in the conductors, coax cables using these materials as dielectrics would be non-dispersive up to infrared frequencies -- where vibrational and rotational resonances set in -- so the real part of the dielectric constant of such a coax cable is extremely constant below this range.

(e) Dispersion Relation as a Convolution.

Here is one of the dispersion relations from above,

$$(1/2)\{\, \mathrm{Re}[X(\omega)] - \mathrm{Re}[X(\infty)]\,\} \;=\; (1/2\pi) \int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Im}[X(\omega')]}{\omega' - \omega} \qquad (5.3a)$$

Notice that the integral has exactly the convolution form shown in Chapter 1 (3.3) where we make these identifications:

$$A(\omega) = (1/2)\{\, \mathrm{Re}[X(\omega)] - \mathrm{Re}[X(\infty)]\,\}$$
$$B(\omega) = -1/\omega \qquad \text{so that } B(\omega-\omega') = 1/(\omega'-\omega)$$
$$C(\omega) = \mathrm{Im}[X(\omega')]$$

Similarly, we could interpret the other dispersion relation in the same manner,

$$(1/2)\{\ \mathrm{Im}[X(\omega)] - \mathrm{Im}[X(\infty)]\} \ = \ -\ (1/2\pi) \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}[X(\omega')]}{\omega' - \omega} \qquad (5.3b)$$

$A'(\omega) = (1/2)\{\ \mathrm{Im}\ [X(\omega)] - \mathrm{Im}\ [X(\infty)]\}$
$B'(\omega) = +1/\omega$ so that $B'(\omega-\omega') = 1/(\omega-\omega')$
$C'(\omega) = \mathrm{Re}[X(\omega')]$

where we have accounted for the minus sign by in the factor B'. So the pole factor in the integration has the effect of a filter $H(\omega) = \pm 1/\omega$, so we can identify the Hilbert Transform itself with this filter transfer function.

(f) The Hilbert Transformer

Recall the complex form of the Hilbert Transform shown above

$$X(\omega) = X(\infty) + (1/i\pi) \int_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega' - \omega} \qquad (5.2)$$

Now consider the same integral *form* shown here, but in the time domain and make the following definition,

$$\hat{x}(t) \equiv \ -\ (1/\pi) \int_{-\infty}^{\infty} dt' \frac{x(t')}{t' - t} \quad = \quad (1/\pi) \int_{-\infty}^{\infty} dt' \frac{x(t')}{t - t'} \qquad (5.10)$$

We associate the name Hilbert Transform or Hilbert Transformer with this definition because the integral has the *form* of the usual Hilbert Transform integral. Now, as noted earlier, this is a convolution:

$$\hat{x}(t) \ = \ (1/\pi t) * x(t) \qquad \text{compare to:} \qquad a(t) \ = h(t) * c(t) \qquad (5.11)$$

The convolving function $h(t) = (1/\pi t)$ is the Hilbert Transformer, but in the time domain. What is it in the frequency domain?

$$H(\omega) \ = \int_{-\infty}^{\infty} dt\ h(t)\ e^{-i\omega t} \ = \ (1/\pi) \int_{-\infty}^{\infty} dt\ e^{-i\omega t}/t = (-i/\pi) \int_{-\infty}^{\infty} dt\ \sin(\omega t)/t \ = (-2i/\pi) \int_{0}^{\infty} dt\ \sin(\omega t)/t$$

$$= \ (-2i/\pi)\ \mathrm{sign}(\omega) \int_{0}^{\infty} dx\ \sin(x)/x \quad \text{where we have used } x = \omega t\text{, think about that upper endpoint!}$$

$= (-2i/\pi)\ \mathrm{sign}(\omega)\ \mathrm{Si}(\infty)$, where Si(x) is the Sine Integral of AS page 231 in 5.2.1,
and where $\mathrm{Si}(\infty) = \pi/2$ as in 5.2.25 on page 232

$$= (-2i/\pi)\,\mathrm{sign}(\omega)(2/\pi)$$

$$= -i\,\mathrm{sign}(\omega) \qquad (5.12)$$

Let's put these facts in a little box:

The Hilbert Transformer

Define $\hat{x}(t)$ as the convolution of x(t) with the transformer function

$$\hat{x}(t) \equiv (1/\pi)\int_{-\infty}^{\infty} dt' \frac{x(t')}{t - t'} = \frac{1}{\pi t} * x(t) = h(t) * x(t)$$

where we define the Hilbert transformer function as (5.13)

$$h(t) = \frac{1}{\pi t}$$

The Fourier Transform of h(t) is given by

$$H(\omega) = -i\,\mathrm{sign}(\omega)$$

In the frequency domain, the diagonalized relation is then

$$\hat{X}(\omega) = [-i\,\mathrm{sign}(\omega)]\,X(\omega) = H(\omega)X(\omega)$$

The effect of the Transformer is to apply phase shift of $\pm\pi/2$ based on the sign of ω,

$$H(\omega)\,X(\omega) = \begin{cases} e^{-i\pi/2}\,X(\omega) & \omega > 0 \\ e^{+i\pi/2}\,X(\omega) & \omega < 0 \end{cases}$$

In a classic audio amplifier, before the output push-pull stage you often had a stage called a phase splitter which put out two different polarities of the signal to drive the push-pull output stages. If that signal was a sine wave, then the two polarities created by the phase-splitter could be thought of as shifted by π relative to each other. Here, we have a circuit that puts out two different phase shifts of a spectral signal depending on the sign of ω, and the phase difference is π just as in the audio case. I think $H(\omega)$ came to be *associated with* the name **phase splitter** for this reason. The actual phase splitter name applies to the combination involving $H(\omega)$ shown in the following section.

(g) The Phase Splitter and Modulation

Define the phase splitter filter as follows

$$P(\omega) = (1/2) \{ \ 1 \ + \ j \, H(\omega) \ \} \ = (1/2) \{ \ 1 \ + \text{sign}(\omega) \ \} \qquad (5.14)$$

where $H(\omega)$ is the Hilbert Transformer defined in the previous section. Notice that

$$P(\omega) \, X(\omega) \ = \ \begin{cases} X(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases} \qquad (5.15)$$

so the phase splitter is really a projection operator the projects out the only the positive frequency components of a spectrum. When $P(\omega)$ operates on a signal as shown above, the result is called an **analytic signal**. Note also that

$$P(\omega)^2 \ = P(\omega) \qquad (5.16)$$

which follows since it is a projector, or by direct trivial calculation. The time domain version of the splitter is this,

$$p(t) = (1/2) \, [\, \delta(t) + j/(\pi t) \,] \qquad (5.17)$$

Now, it is useful to describe the modulation process as in L&M (2.23) [and as in Chen with extra factor],

$$y(t) = \sqrt{2} \ \text{Re}[\, u(t) \exp(j\omega_c t)] \qquad (5.18)$$

For such a real function, it is useful to define the following associated real function

$$\hat{y}(t) \equiv h(t) * y(t) = 1/\pi \int_{-\infty}^{\infty} dt' \, \frac{y(t')}{t-t'} \ = (\sqrt{2} \, /\pi) \int_{-\infty}^{\infty} dt' \, \frac{\text{Re}[\, u(t') \exp(j\omega_c t')]}{t-t'} \qquad (5.19)$$

We then bring to bear the Hilbert Transform (5.3b),

$$\text{Im}[\, u(t) \exp(j\omega_c t)] = \text{Im}[\, u(\infty) \exp(j\omega_c \infty)] + (1/\pi) \int_{-\infty}^{\infty} dt' \, \frac{\text{Re}[\, u(t') \exp(j\omega_c t')]}{t-t'}$$

Since the ∞ is on the upper great circle, the exp(..) kills the constant term, and we end up with

$$\hat{y}(t) = \sqrt{2} \ \ \text{Im}[\, u(t) \exp(j\omega_c t)] \qquad (5.20)$$

Thus, we see that $y(t)$ and $\hat{y}(t)$ are in fact the real and imaginary parts of $\sqrt{2} \, u(t) \exp(j\omega_c t)$. If follows then that

$$y(t) + j \, \hat{y}(t) \ = \sqrt{2} \, u(t) \exp(j\omega_c t) \qquad (5.21)$$

from which we conclude that

$$u(t) = (1/\sqrt{2}\,)\,[y(t) + j\,\hat{y}(t)\,]\,\exp(-j\omega_c t) \qquad (5.22)$$

as shown in L&M (2.22). Another way to write this is

$$\begin{aligned} u(t) &= \sqrt{2}\ \{\ [(1/2)\,[\,\delta(t) + j/(\pi t)\,] * y(t)\ \}\,\exp(-j\omega_c t) \\ &= \{\,\sqrt{2}\,p(t) * y(t)\,\}\,\exp(-j\omega_c t) \end{aligned} \qquad (5.23)$$

which shows a series of operations we can apply to a real passband signal y(t) to obtain the (possibly complex) baseband signal u(t): (1) pass y(t) through the phase-splitter filter $\sqrt{2}\,p(t)$; (2) multiply by the carrier phasor shown.

We can write (5.22) above in the frequency domain. To do so, move the phasor to the left hand side, then apply the shift rule Chapter 1 (12.2) to get

$$\begin{aligned} U(\omega-\omega_c) &= (1/\sqrt{2}\,)[\,Y(\omega) + j\,\hat{Y}(\omega)] = (1/\sqrt{2}\,)[\,Y(\omega) + j\,H(\omega)Y(\omega)] \\ &= \sqrt{2}\ (1/2)[\,1 + j\,H(\omega)]Y(\omega) \\ &= \sqrt{2}\,P(\omega)Y(\omega) \end{aligned} \qquad (5.24)$$

where we have made use of the Fourier Transform of (5.19) above,

$$\hat{Y}(\omega) \equiv H(\omega)\,Y(\omega) \qquad (5.25)$$

Equation (5.24) states that 2-sided and generally asymmetric baseband spectrum U(ω) is a simple frequency downshift of the passband positive side spectrum. This is what modulation is all about.

Finally, we can prove that u(t) and y(t) have the same Parseval Energy as follows. From (5.24),

$$|\,U(\omega-\omega_c)\,|^2 = 2\,P(\omega)\,|\,Y(\omega)\,|^2 \qquad \text{since } P(\omega)^2 = P(\omega) = \text{real} \qquad (5.26)$$

Now integrate both sides over all frequency ω to get

$$\int_{-\infty}^{\infty} d\omega\,|\,U(\omega-\omega_c)\,|^2 = \int_{-\infty}^{\infty} d\omega\ \ 2\,P(\omega)\,|\,Y(\omega)\,|^2 \qquad (5.27)$$

But we know that the phase splitter P(ω) is 0 for negative frequencies, and at the same time we can shift the integration variable on the RHS, so we then get:

$$\int_{-\infty}^{\infty} d\omega\,|\,U(\omega)\,|^2 = 2\int_{0}^{\infty} d\omega\ \ |\,Y(\omega)\,|^2 \qquad (5.28)$$

Now since y(t) is real, we know that we get an equal contribution from both frequency sides, so we alter the RHS to get

$$\int_{-\infty}^{\infty} d\omega\,|\,U(\omega)\,|^2 = \int_{-\infty}^{\infty} d\omega\ \ |\,Y(\omega)\,|^2 \qquad (5.29)$$

Using Parseval's Theorem on both sides, we then conclude that

$$\int_{-\infty}^{\infty} dt \mid u(t) \mid^2 \;=\; \int_{-\infty}^{\infty} dt \;\; y(t)^2 \qquad (5.30)$$

and we finally arrive that the conclusion that the "energy" in signal u(t) is the same as that in y(t), and this is why L&M chose that factor of $\sqrt{2}$ in (2.23), and why we use that factor in (5.18) above.